NON-INTERSECTIVITY OF PAPERFOLDING DRAGON CURVES AND OF CURVES GENERATED BY AUTOMATIC SEQUENCES

J.-P. Allouche
jean-paul.allouche@imj-prg.fr

M. Mendès France[1]
michel.mendes-france@wanadoo.fr

G. Skordev
skordev@gmail.com

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Abstract

We begin by surveying recent and less recent results about the (non-)intersectivity of curves generated by generalized Weyl sums associated with real sequences, in particular when these sequences are morphic or automatic. Then we turn to the case of paperfolding dragon curves, pointing in particular to the unpublished works of Spiros Michaelis and of Reimund Albers.

– To Jeff Shallit on the occasion of his 60th birthday

1. Introduction

Having recently read a paper by Tabachnikov entitled "The dragon curves revisited" [48] the authors remembered numerous discussions on this subject, which convinced them to revisit dragon curves, aka paperfolding curves. One of the open questions in Tabachnikov's paper is the (non-)interesectivity of dragon curves obtained by unfolding a regularly folded strip of paper at an angle different from 90 degrees. Recall that for 90 degrees the curve does not cross itself: see the papers of Davis and Knuth [10, 11]; also see Figure 1 below. Tabachnikov's paper cites in particular an observation of Knuth [23] who noted in 1969 that unfolding at 95 degrees would "lead to paths that cross themselves." The question is then: for which unfolding angles does the curve intersect itself?

More generally, what kind of curves are "non-intersective"? Non-intersective curves, also called "self-avoiding curves," or "simple curves," are curves – usually

[1] Michel Mendès France passed away on January 20, 2018.

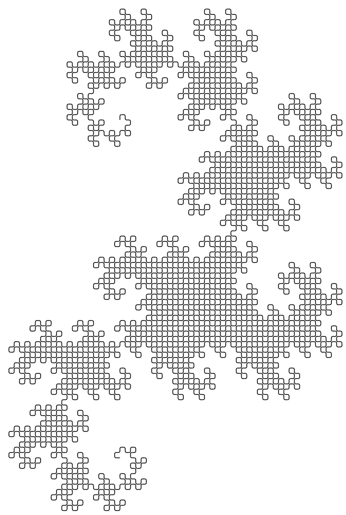

Figure 1: Dragon curve for an unfolding angle of 90 degrees

two-dimensional – that do not intersect themselves. Of course there are trivial examples of such curves: think of a straight line. Thus non-intersecting plane curves are interesting if they are somehow "complicated". In particular one can think of curves that fill the plane or a plane region of positive Lebesgue measure – such curves are called "plane-filling curves" – and/or of curves having a fractal dimension larger than 1.

Another class of curves, which are actually broken lines, consists of curves associated with a sequence taking its value in a finite set. For example, with a ± 1 sequence one can associate the broken line composed of segments of length 1, that starts from the origin and is such that, at the n-th step, the next segment is concatenated to the previous after a 90-degree right or left turn according to the sign of the ± 1 sequence. (A more precise definition can be found in [35] or [48]; also see below.) A natural generalization consists of replacing 90 degrees with other values: it can be expected that the non-intersectivity is true if the angle is large, say near 180 degrees, and false if the angle is very small. But what happens in between?

The present paper surveys the (non-)intersectivity of curves associated with sequences taking their values in a finite set, in particular in the case of morphic or automatic sequences. The last part of the paper will start from a remark/question of the second named author which appears to be not true, and survey the unpublished works of Spiros Michaelis (whose “Diplomarbeit” was supervised by the third named author) and of Reimund Albers (whose doctoral thesis was based upon a question of the third named author).

2. A Quick View on Automatic and Morphic Sequences

A way of generating sequences that are somehow regular without necessarily being “too simple” is to use morphisms of the free monoid and their iterations. We give precise definitions below. For more on the subject, the readers can look, e.g., at the books [3, 18, 19].

Definition 1.

- Given an *alphabet* (i.e., a finite set) $\mathcal{A}$, the *free monoid* generated by $\mathcal{A}$ is denoted by $\mathcal{A}^*$: this is the set of all *words* (i.e., finite sequences) on $\mathcal{A}$, including the empty word, equipped with the *concatenation* of words. In particular the empty word is the unit of this monoid. The elements of $\mathcal{A}$ are called *letters*. The *length* of a word is the number of its letters.
- If $\mathcal{A}$ and $\mathcal{B}$ are two alphabets, a *morphism* from $\mathcal{A}^*$ to $\mathcal{B}^*$ is a homomorphism for the concatenation. Note that a morphism from $\mathcal{A}^*$ to $\mathcal{B}^*$ is well-defined as soon as it is defined on $\mathcal{A}$.
- A morphism on $\mathcal{A}^*$ is called *uniform* if the images of all letters in $\mathcal{A}$ have the same length. If this length is ℓ, the morphism is called a *(uniform) morphism of length* ℓ or an ℓ*-morphism*.

Definition 2. The set of all sequences (finite and infinite) on an alphabet $\mathcal{A}$, i.e., the set $\mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, is equipped with the product topology (meaning that we consider pointwise convergence, or equivalently that two sequences are “close” if they have a long common prefix). Let φ be a morphism from $\mathcal{A}^*$ to itself. A sequence $\mathbf{U} = (u_k)_{k\geq 0}$ on $\mathcal{A}$ is called an *iterative fixed point of* φ if there exists a letter $a \in \mathcal{A}$ such that the sequence of words $(\varphi^k(a))_{k\geq 0}$ converges to $\mathbf{U}$. This is sometimes written as $\mathbf{U} = \varphi^{\infty}(a)$.

- A sequence $\mathbf{S}$ on $\mathcal{A}^*$ is called *morphic* if there exists an alphabet $\mathcal{B}$, a morphism φ on $\mathcal{B}^*$, an iterative fixed point $\mathbf{U}$ of φ, and a map (i.e., a 1-morphism) g from $\mathcal{B}$ to $\mathcal{A}$ such that $g(\mathbf{U}) = \mathbf{S}$.
- If furthermore the morphism φ is ℓ-uniform, the sequence $\mathbf{S}$ is said to be ℓ-automatic.

Examples 1. We give three examples of 2-automatic sequences and an example of a morphic sequence (see, e.g., [3] for these and other examples).

- The iterative fixed point of the morphism defined on $\{0,1\}^*$ by $0 \to 01$, $1 \to 10$ is the celebrated Prouhet-Thue-Morse sequence (see, e.g., [2]). It begins

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ \ldots$$

  This sequence is 2-automatic.

- The Golay-Shapiro-Rudin sequence is the 2-automatic sequence $\mathbf{V}$ defined as follows. Let λ be the 2-morphism defined on the alphabet $\{a,b,c,d\}$ by $a \to ab$, $b \to ac$, $c \to db$, $d \to dc$. The iterative fixed point of λ beginning with a is the sequence $\lambda^\infty(a) = a\ b\ a\ c\ a\ b\ d\ b\ \ldots$. We then define φ to be the map $a \to +1$, $b \to +1$, $c \to -1$, $d \to -1$. Then

$$\mathbf{V} = \varphi((\lambda^\infty(a)) = +1\ \ +1\ \ +1\ \ -1\ \ +1\ \ +1\ \ -1\ \ +1\ \ \ldots$$

- The (regular) paperfolding sequence is the 2-automatic sequence $\psi(\mu^\infty(a))$, where μ is the 2-morphism defined on the alphabet $\{a,b,c,d\}$ by $a \to ab$, $b \to cb$, $c \to ad$, $d \to cd$, and ψ is the map $a \to +1$, $b \to +1$, $c \to -1$, $d \to -1$. Thus the (regular) paperfolding sequence begins

$$+1\ \ +1\ \ -1\ \ +1\ \ +1\ \ -1\ \ -1\ \ \ldots$$

- The Fibonacci binary sequence is the iterative fixed point of the (non-uniform) morphism $0 \to 01$, $1 \to 0$. This is thus a morphic sequence, which begins

$$0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ \ldots$$

3. A General Framework

Two papers that appeared the same year, one by Dekking and Mendès France [15], the other by Loxton [29], revisited a general idea that consists of associating with each sequence $(a_n)_{n\geq 0}$ of real numbers the polygonal line whose vertices are the points $(z_n)_{n\geq 0}$ of the complex plane given by $z_n = \sum_{k\leq n} e^{2i\pi a_k}$. The "complexity" of the pictures that are obtained is related to the behavior of the sequence they come from. Of course similar ideas can be found earlier in the literature: we do not resist citing the nice papers of Lehmer and Lehmer [27, 28]. Several other papers exploited the same idea in order to visualize either sums $z_n = \sum_{k\leq n} e^{2i\pi a_k}$ as above (including complete or incomplete *Gauss sums* where $a_k = \alpha k^2$ and *higher-order Gauss sums* also called *Weyl sums* where $a_k = P(k)$ for some polynomial P), or variations, e.g., for $(a_n)_n$ a ± 1 sequence, the sum that is considered is $z_n = \sum_{k\leq n} e^{2i\pi \sum_{j\leq k} a_j}$. One can call *generalized Weyl sums* these sums and their variations, when $(a_n)_{n\geq 0}$ is any real sequence.

It appears that the literature on similar questions is vast. We have selected below, in chronological order, papers addressing questions related to fractal curves, plane-filling curves, self-affine curves, or tilings, and showing spectacular pictures of curves obtained by drawing sums of exponentials as above: the question of non-intersectivity of such curves is either explicitly addressed or conjectured, or naturally questionable in view of these figures or of some unexpectedly large parts of them. More references, devoted to paperfolding and analogous curves, are given in Section 4.

- In 1976 Lehmer [26] studied incomplete Gauss sums, i.e., sums of the type $G_N(m) = \sum_{0 \leq \nu \leq m-1} e^{2i\pi\nu^2/N}$ (the complete Gauss sum is obtained for $m = N$). The paper shows in particular for some values of N the directed graphs with vertices $G_0(N), G_1(N), \ldots G_N(N)$ and with edges the segments of length 1 directed from $G_N(m)$ to $G_N(m+1)$, and a relation with the *Cornu spiral* (also known as *clothoid* or *Euler spiral*) for $m = O(N^{1/2})$.

- In 1979 and 1980 Lehmer and Lehmer published their papers [27] and [28] where they studied respectively the sums $\sum_{0 \leq n \leq m} e^{2i\pi(b_k(n)+nj)/k}$ and the sums $\sum_{0 \leq n \leq m} e^{2i\pi(e_k(n)+nj)/k}$ where $b_k(n)$ is the sum of the digits of n in base k and $e_k(n) = \sum n_\ell n_{\ell+1}$ if the base-k expansion of n is $n = \sum n_\ell k^\ell$. Note that for $k = 2$ and $j = 0$ these sums are respectively the summatory function of the ± 1 Prouhet-Thue-Morse sequence and the summatory function of the ± 1 Golay–Shapiro–Rudin sequence.

- In 1981, Dekking and Mendès France studied in [15] sums $\sum_{0 \leq k \leq n-1} e^{2i\pi q u_k}$ with q an integer, in relation with the distribution modulo 1 of the sequence of real numbers $(u_k)_k$. They obtained, in particular, surprising pictures, e.g., for $u_k = \sqrt{17}k$, $u_k = \sqrt{2}k^2$, $u_k = ek^2$, $u_k = \pi k^2$, $u_k = s(k)/4$ and $u_k = \sqrt{3}s(k)$ (where $s(k) = b_2(k)$ is the sum of the binary digits of the integer k), $u_k = k^{2/5}$, and $u_k = (k+1)\log(k+1)$.

- In 1981 Loxton [29] looked at sums $\sum_{1 \leq n \leq N} e^{2i\pi u_n}$ for sequences with $u_n = \log^k n$. Also see Loxton's 1983 paper [30], where there is a detailed study of the sums $\sum_{0 \leq n \leq N} e^{2i\pi t n^{1/2}}$.

- In 1983, Coquet [8] studied the sums $\sum(-1)^{s(3n)}$, where $(s(n))_{n \geq 0}$ is the Prouhet-Thue-Morse sequence. He found a link between these sums and the von Koch curve (also see [31] and [4]).

- A paper by Siromoney and Subramanian in 1983 [47] studies constructions of the Peano and Hilbert space-filling curves: one of these constructions uses morphic sequences.

- In a paper which appeared in 1985 [17] Deshouillers made a fine study of sums $\sum e^{2i\pi u_k}$, with $u_k = \alpha k^{3/2}$ for some values of α, where the starting idea is to replace this sum by $\sum(e^{2i\pi u_k} + e^{2i\pi u_{k-1}})/2$ and to use the Poisson summation formula.

- In 1988, Berry and Goldberg [5] worked on the patterns obtained from the sums $\sum_{1 \leq n \leq L} e^{i\pi\tau n^2}$: they used the word *curlicues* for these patterns, recalling the definition of the Oxford English Dictionary (*curlicue*: a fantastic curl or twist).

- In 2008 Sinaĭ [46] revisited the curlicues associated with the sums $\sum e^{i\pi a n^2}$ and the link with the continued fraction expansion of a.

- A paper posted on the server HAL in 2009 by Monnerot-Dumaine [36] contains several interesting pictures linked to the Fibonacci fractal. The curves are obtained through morphic (or through Sturmian) sequences, in particular the fixed point beginning with 1 of the Fibonacci related morphism: $0 \to 10221$, $1 \to 1022$, $2 \to 1021$ (also see [37, Sequence A143667]), with various drawing angles, and there is an open problem about non-intersectivity.

- In 2012 Dekking published a nice paper on paperfolding morphisms, plane-filling curves and fractal tiles [14] where he gave in particular criteria for both the plane-filling and the non-intersecting properties of curves associated with paperfolding sequences.

- A paper of Ramírez and Rubiano in 2012 [40] contains many pictures of curves associated with the binary Fibonnaci sequence or with some other morphic sequences, but it does not systematically study the non-intersectivity of all the curves described there. [Also see the 2013 paper of Ramírez and Ribiano [41], the 2014 paper [43] by Ramírez, Rubiano and De Castro where the authors describe in particular the Fibonacci snowflake, and the 2015 paper [42] by Ramírez and Rubiano.]

- Optical illusions appear for curves associated with the sums $\sum_{1 \leq n \leq N} e^{2i\pi\alpha n^{1/2}}$ (where α is a fixed positive real): this is described and studied in a 2015 paper [7] by Chamizo and Raboso.

4. Paper Folding and Unfolding

Start from a strip of paper and iteratively fold it in half lengthwise, all the folds being in the same direction. Then unfold this folded piece of paper at a same given angle. This provides a nice "dragon" curve also called the Heighway curve. Two values of the unfolding angle were particularly studied, namely 90 and 60 degrees.

A nice, somehow unexpected, result for the 90-degree angle is that the curve does not cross itself except maybe at angles. In other words edges are never visited twice [10, 11]. We call *intersective* a dragon curve that crosses itself at points *that are not angles*. Thus a *non-intersective* dragon curve does not cross itself, *except possibly at angles*. So that the result of Davis and Knuth [10, 11] can be formulated as follows.

Theorem 1. *The* 90*-degree dragon curve is non-intersective.*

5. Changing the Angle of Unfolding

5.1. A Plausible But Incorrect Intuition

The paperfolding curve with angle θ degrees is clearly intersective[2] for θ strictly less than 90 degrees and clearly non-intersective for θ equal to 180 degrees. Since it is non-intersective for θ equal to 90 degrees, the following question is natural [33, p. 200–201]:

Question. Is it true that the dragon curve with unfolding angle θ is non-intersecting if and only if θ is larger than 90 degrees?

The authors were not aware of Knuth's observation[3] (intersectivity for 95 degrees) at that time, while the third named author tried – unsuccessfully – to prove that the answer to the question was yes. Note that Tabachnikov's paper [48, Figure 8] contains two pictures which show intersectivity for θ equal to 94 degrees and (visual) non-intersectivity for θ for an angle θ of about 100 degrees. So that the question above could be reformulated as:

Modified Question. Are there two intervals $[90, \theta_0) \cup (\theta_1, 180]$, with $90 < \theta_0 < \theta_1 < 180$, for which the corresponding paperfolding curves are non-intersecting?

5.2. The Unpublished Work of Spiros Michaelis

Finally doubting that the answer to the second question was affirmative, the second named author proposed to Spiros Michaelis, a student preparing a "Diplomarbeit" at Bremen University, to make computer experiments. Indeed Michaelis found intersections, which were really calculated afterwards. It is not clear whether it is still possible to find a copy of this Diplomarbeit.

[2] If the unfolded strip is coded by R(ight) and L(eft) when passing from an edge to the next one, the sequence of R's and L's obtained after unfolding a strip folded N times contains LLL and RRR for N large enough. For an unfolding angle strictly less than 90 degrees, both LLL and RRR provide an intersection, as noted for example in [1].

[3] As indicated in [48] the observation of Knuth is a previously unpublished addendum in [23].

5.3. The Unpublished Doctoral Thesis of Reimund Albers

Gencho Skordev asked then some colleagues about the more general question:

General Question. Find intervals for θ for which the corresponding paperfolding curves are (non-)intersective.

This question inspired both H.-O. Peitgen –who made some computations (1995) showing intersectivity for some value > 90 degrees of the unfolding angle (as indicated in [1, p. 131])– and Reimund Albers who proved nice results that were not published, but can be found in his doctoral thesis [1, Chapter 9]. In particular he gave a partial answer to the general question above.

Theorem 2 (Albers). *The paperfolding curve is intersective if the unfolding angle θ satisfies $90 < \theta < 95.126$.*

Remark 1. Albers writes "94.126 degrees" on Page 142 of [1] and "95.126 degrees" on Page 143. The computations in the Table of Page 143 show that the "true" value is 95.126.

Proof. (Sketch) Very roughly speaking the proof is a careful geometrical study of the polygon $Q_{10}(\theta)$ obtained when unfolding at angle θ a strip of paper that has been folded 10 times. (Interestingly enough the self-intersection already occurs after ten folds.) The author proves that a self-intersection occurs if the inequality $f(\beta) < 0$ holds, where $\beta = \frac{\pi-\alpha}{2}$ and f is defined by

$$f(x) = 64\cos^5 x(\cos x + \cos(3x)) - \frac{\sin x}{2\cos x \sin(4x) - \sin(5x)} - 1.$$

The computations then show that $f(\beta) < 0$ for $\beta = 95.126$, but that $f(\beta) > 0$ for $\beta = 95.127$. Note that the condition on β is only a sufficient condition, so that this says nothing about $\beta \geq 95.127$. □

6. Conclusion

There are quite a few papers about (non-)intersectivity of dragon curves corresponding to unfolding at an angle θ, or of (non-)intersectivity of curves obtained by iterating some kind of algorithm (finite automata, L-systems, etc.). To mention just a few of them, let us cite the papers of Dekking [12, 13, 14], the paper of Dekking, Mendès France and van der Poorten [16], and the references therein. Other works deal with similar "machine generated" curves. We will cite for example self-affine maps with the papers of Kôno and of Kamae [24, 20], and sequential machines with the papers of Rodenhausen [44], and Peitgen, Rodenhausen, and Skordev [38] (also see the references given in [38] and the extra references cited in the review

MR1460970 on that paper), and self-similar functions and cellular automata with the paper of Peitgen, Rodenhausen, and Skordev [39]. Several books mention curves that are constructed algorithmically but have quite unusual properties (plane-filling, tiling, self-similar, non-intersective, etc.). We only mention here three of them: first the book of Mandelbrot [32] of course; then the book of Darst, Palagallo and Price [9] that studies "curious curves" in a self-contained way that can be accessible to a large audience; then, for the programming point of view, Chapter 11 (*Programming Examples of Space-Filling Curves*), by Szilard, of the book [25]. Of course several websites show all kinds of plane-filling and/or non-intersective curves, sometimes from an artistic point of view; among them we only cite two possibly less known sites, namely `http://www.fractalcurves.com` and `http://robertfathauer.com`.

Examples of two-dimensional foldings (handkerchief folding) and three-dimensional foldings (wire-bending) can be found respectively in a paper of Salon [45, Sections III and IV] and in a paper of Mendès France and Shallit [34]. Note that, curiously enough, the wire-bending curves studied in [34] are all bounded, thus far from being non-intersective

Actually the question of (non-)intersectivity of a dragon curve holds for general paperfolding: fold a strip of paper where, at each step the fold is arbitrarily either in the positive or in the negative direction; unfolding at 90 degrees yields a "general" dragon curve which is known to be non-intersecting (this can be deduced from [10, 11] as indicated, e.g., in [35]). What happens if the unfolding angle is larger than 90 degrees? Another related question is the study of the (non-)intersectivity of modified classical curves, where the modification changes an angle that plays the role of an unfolding angle: for example Keleti [21], Keleti and Paquette [22], and Cantrell and Palagallo [6] studied the (non-)intersectivity of generalized von Koch curves.

Coming back to the general question of the (non-)intersectivity of generalized Weyl sums associated with morphic or automatic sequences, no complete study has been done yet. It would be nice to have a purely combinatorial condition on morphic or automatic sequences that is equivalent to non-intersectivity.

References

[1] R. Albers, *Papierfalten*, Doctoral thesis, Bremen University, 2006. Available at `http://www.math.uni-bremen.de/didaktik/ma/ralbers/Publikationen/Dissertation/index.html`

[2] J.-P. Allouche, J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in *Sequences and their Applications* (Singapore, 1998), Springer Ser. Discrete Math. Theor. Comput. Sci., Springer, 1999, pp. 1–16.

[3] J.-P. Allouche, J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, 2003.

[4] J.-P. Allouche, G. Skordev, Von Koch and Thue-Morse revisited, *Fractals* **15** (2007), 405–409.

[5] M. V. Berry, J. Goldberg, Renormalisation of curlicues, *Nonlinearity* **1** (1988), 1–26.

[6] M. Cantrell, J. Palagallo, Self-intersection points of generalized Koch curves, *Fractals* **19** (2011), 213–220.

[7] F. Chamizo, D. Raboso, Van der Corput method and optical illusions, *Indag. Math. (N.S.)* **26** (2015), 723–735.

[8] J. Coquet, A summation formula related to the binary digits, *Invent. Math.* **73** (1983), 107–115.

[9] R. B. Darst, J. A. Palagallo, T. E. Price, *Curious Curves*, World Scientific, 2010.

[10] C. Davis, D. Knuth, Number representations and dragon curves-1, *J. Recreational Mathematics* **3** (1970), 66–81.

[11] C. Davis, D. Knuth, Number representations and dragon curves-2, *J. Recreational Mathematics* **3** (1970), 133–149.

[12] F. M. Dekking, Recurrent sets, *Adv. Math.* **44** (1982), 78–104.

[13] F. M. Dekking, Replicating superfigures and endomorphisms of free groups, *J. Comb. Theory, Ser. A* **32** (1982), 315–320.

[14] F. M. Dekking, Paperfolding morphisms, planefilling curves, and fractal tiles, *Theoret. Comput. Sci.* **414** (2012), 20–37.

[15] F. M. Dekking, M. Mendès France, Uniform distribution modulo one: a geometrical viewpoint, *J. Reine Angew. Math.* **329** (1981), 143–153.

[16] F. M. Dekking, M. Mendès France, A. van der Poorten, Folds!, *Math. Intelligencer* **4** (1982), 130–138, 173–181, 190–195.

[17] J.-M. Deshouillers, Geometric aspect of Weyl sums, in *Elementary and Analytic Theory of Numbers* (Warsaw, 1982), Banach Center Publ., 17, PWN, Warsaw, 1985, pp. 75–82.

[18] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Mathematics **1794**, Springer-Verlag, 2002.

[19] F. von Haeseler, *Automatic Sequences*, de Gruyter Expositions in Mathematics **36**, de Gruyter, 2003.

[20] T. Kamae, A characterization of self-affine functions, *Japan J. Appl. Math.* **3** (1986), 271–280.

[21] T. Keleti, When is the modified von Koch snowflake non-self-intersecting?, *Fractals* **14** (2006), 245–249.

[22] T. Keleti, E. Paquette, The trouble with von Koch curves built from n-gons, *Amer. Math. Monthly* **117** (2010), 124–137.

[23] D. Knuth, *Selected Papers on Fun and Games*, CSLI Publications, 2011, pp. 571–614.

[24] N. Kôno, On self-affine functions, *Japan J. Appl. Math.* **3** (1986), 259–269.

[25] S. Konstantinidis (ed.), N. Moreira (ed.), R. Reis (ed.), J. Shallit (ed.), *The role of Theory in Computer Science. Essays dedicated to Janusz Brzozowski*, World Scientific, 2017.

[26] D. H. Lehmer, Incomplete Gauss sums, *Mathematika* **23** (1976), 125–135.

[27] D. H. Lehmer, E. Lehmer, Picturesque exponential sums. I, *Amer. Math. Monthly* **86** (1979), 725-733.

[28] D. H. Lehmer, E. Lehmer, Picturesque exponential sums. II, *J. Reine Angew. Math.* **318** (1980), 1–19.

[29] J. H. Loxton, Captain Cook and the Loch Ness monster, *James Cook Mathematical Notes* **3** (1981), 3060–3064 (available at `http://www.maths.ed.ac.uk/cook/iss_no27.PDF`).

[30] J. H. Loxton, The graphs of exponential sums, *Mathematika* **30** (1983), 153–163.

[31] J. Ma, J. Holdener, When Thue-Morse meets Koch, *Fractals* **13** (2005), 191–206.

[32] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman and Co., 1982.

[33] M. Mendès France, The inhomogeneous Ising chain and paperfolding, in *Number Theory and Physics (Les Houches, 1989)*, Springer Proc. Phys. **47**, Springer, 1990, pp. 195–202.

[34] M. Mendès France, J. Shallit, Wire bending, *J. Combin. Theory Ser. A* **50** (1989), 1–23.

[35] M. Mendès France, G. Tenenbaum, Dimension des courbes planes, papiers pliés et suites de Rudin-Shapiro, *Bull. Soc. Math. France* **109** (1981), 207–215.

[36] A. Monnerot-Dumaine, The Fibonacci word fractal, Peprint, HAL, 2009, available at `https://hal.archives-ouvertes.fr/hal-00367972`.

[37] *On-Line Encyclopedia of Integer Sequences*, founded by N. J. A. Sloane. Available at `http://oeis.org`.

[38] H.-O. Peitgen, A. Rodenhausen, G. Skordev, Self-affine curves and sequential machines, *Real Anal. Exchange* **22** (1996/97), 446–491.

[39] H.-O. Peitgen, A. Rodenhausen, G. Skordev, Self-similar functions generated by cellular automata, *Fractals* **6** (1998), 371–394.

[40] J. L. Ramírez, G. N. Rubiano, Generación de curvas fractales a partir de homomorfismos entre lenguajes [con Mathematica®], *Rev. Integr. Temas Mat.* **30** (2012), 129–150.

[41] J. L. Ramírez, G. N. Rubiano, On the k-Fibonacci words, *Acta Univ. Sapientiae, Inform.* **5** (2013), 212–226.

[42] J. L. Ramírez, G. N. Rubiano, Biperiodic Fibonacci word and its fractal curve, *Acta Polytech.* **55** (2015), 50–58.

[43] J. L. Ramírez, G. N. Rubiano, R. De Castro, A generalization of the Fibonacci word fractal and the Fibonacci snowflake, *Theoret. Comput. Sci.* **528** (2014), 40–56.

[44] A. Rodenhausen, Planefilling curves and sequential functions, Peprint, Inst. für Dynamische Systeme, Bremen, 1995.

[45] O. Salon, Quelles tuiles ! (Pavages apériodiques du plan et automates bidimensionnels), *J. Théorie Nombres, Bordeaux* **1** (1989), 1–26.

[46] Ya G. Sinaĭ, A limit theorem for trigonometric sums: theory of curlicues, (in Russian), *Uspekhi Mat. Nauk* **63** (2008), 31–38; translated in *Russian Math. Surveys* **63** (2008), 1023–1029.

[47] R. Siromoney, K. G. Subramanian, Space-filling curves and infinite graphs in *Graph-grammars and their Application to Computer Science*, 2nd int. Workshop, Haus Ohrbeck/Ger. 1982, Lect. Notes Comput. Sci. **153** (1983), pp. 380–391.

[48] S. Tabachnikov, Dragon curves revisited, *Math. Intelligencer* **36** (2014), 13–17.