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THREE NOTES ON SER'S AND HASSE'S REPRESENTATIONS  
FOR THE ZETA-FUNCTIONS

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**Abstract**

This paper is devoted to Ser's and Hasse's series representations for the zeta-functions, as well as to several closely related results. The notes concerning Ser's and Hasse's representations are given as theorems, while the related expansions are given either as separate theorems or as formulæ inside the remarks and corollaries. In the first theorem, we show that the famous Hasse's series for the zeta-function, obtained in 1930 and named after the German mathematician Helmut Hasse, is equivalent to an earlier expression given by a little-known French mathematician Joseph Ser in 1926. In the second theorem, we derive a similar series representation for the zeta-function involving the Cauchy numbers of the second kind (Nørlund numbers). In the third theorem, with the aid of some special polynomials, we generalize the previous results to the Hurwitz zeta-function. In the fourth theorem, we obtain a similar series with Gregory's coefficients of higher order, which may also be regarded as a functional equation for the zeta-functions. In the fifth theorem, we extend the results of the third theorem to a class of Dirichlet series. As a consequence, we obtain several globally convergent series for the zeta-functions. They are complementary to Hasse's series, contain the same finite differences and also generalize Ser's results. In the paper, we also show that Hasse's series may be obtained much more easily by using the theory of finite differences, and we demonstrate that there exist numerous series of the same nature. In the sixth theorem, we show that Hasse's series is a simple particular case of a more general class of series involving the Stirling numbers of the first kind. All the expansions derived in the paper lead, in turn, to the series expansions for the Stieltjes constants, including new series with rational terms only for Euler's constant, for the Maclaurin coefficients of the regularized Hurwitz zeta-function, for the logarithm of the gamma-function, for the digamma and trigamma functions. Throughout the paper, we also mention several "unpublished" contributions of Charles Hermite, which were very close to the results of Hasse and Ser. Finally, in the Appendix, we prove an interesting integral representation for the Bernoulli polynomials of the second kind, formerly known as the Fontana–Bessel polynomials.

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**1. Introduction**

The Euler–Riemann zeta-function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \prod_{n=1}^{\infty} (1 - p_n^{-s})^{-1}, \quad \begin{matrix} \operatorname{Re} s > 1 \\ p_n \in \mathbb{P}, \end{matrix}$$

and its most common generalization, the Hurwitz zeta-function

$$\zeta(s, v) \equiv \sum_{n=0}^{\infty} (n + v)^{-s}, \quad \begin{matrix} \operatorname{Re} s > 1 \\ v \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \end{matrix}$$

$\zeta(s) = \zeta(s, 1)$ , are some of the most important special functions in analysis and number theory. They were studied by many famous mathematicians, including Stirling, Euler, Malmsten, Clausen, Kinkelin, Riemann, Hurwitz, Lerch, Landau, and continue to receive considerable attention from modern researchers. In 1930, the German mathematician Helmut Hasse published a paper [37], in which he obtained and studied these globally convergent series for the  $\zeta$ -functions

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{1-s}, \tag{1}$$

$$\zeta(s, v) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{1-s}, \tag{2}$$

containing finite differences  $\Delta^n 1^{1-s}$  and  $\Delta^n v^{1-s}$  respectively.<sup>2</sup> Hasse also remarked<sup>3</sup> that the first series is quite similar to the Euler transformation of the  $\eta$ -function series  $\eta(s) \equiv \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2^{1-s}) \zeta(s)$ ,  $\operatorname{Re} s > 0$ , i.e.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s}, \tag{3}$$

the expression which, by the way, was earlier given by Mathias Lerch [48]<sup>4</sup> and later rediscovered by Jonathan Sondow [70], [26]. Formulæ (1)–(2) have become widely known, and in the literature they are often referred to as *Hasse’s formulæ* for the  $\zeta$ -functions. At the same time, it is not so well known that 4 years earlier, a little-known French mathematician Joseph Ser published a paper [66] containing very similar results.<sup>5</sup> In particular, he showed that

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s}, \tag{4}$$

<sup>2</sup>For the definition of the finite difference operator, see (32).

<sup>3</sup>Strictly speaking, the remark was communicated to him by Konrad Knopp.

<sup>4</sup>Interestingly, Lerch did not notice that his result is simply Euler’s transformation of  $\eta(s)$ .

<sup>5</sup>Moreover, as we come to show in Remark 2, series (2) is also a particular case of a more general formula obtained by Nørlund in 1923 (although the formula may, of course, be much older).

and also gave this curious series

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s} = \\ &= \frac{1}{s-1} + \frac{1}{2} + \frac{1}{12}(1-2^{-s}) + \frac{1}{24}(1-2 \cdot 2^{-s} + 3^{-s}) + \dots \end{aligned} \tag{5}$$

[66, Eq. (4), p. 1076]<sup>6</sup>, [9, p. 382], to which Charles Hermite was also very close already in 1900.<sup>7</sup> The numbers  $G_n$  appearing in the latter expansion are known as *Gregory’s coefficients* and may also be called by some authors (*reciprocal*) *logarithmic numbers*, *Bernoulli numbers of the second kind*, normalized *generalized Bernoulli numbers*  $B_n^{(n-1)}$  and  $B_n^{(n)}(1)$ ,<sup>8</sup> and normalized *Cauchy numbers of the first kind*  $C_{1,n}$ . They are rational and may be defined either via their generating function

$$\frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n z^n, \quad |z| < 1, \tag{6}$$

or explicitly via the recurrence relation

$$G_n = \frac{(-1)^{n-1}}{n+1} + \sum_{k=1}^{n-1} \frac{(-1)^{n+1-k} G_k}{n+1-k}, \quad G_1 = \frac{1}{2}, \quad n = 2, 3, 4, \dots \tag{7}$$

see, e.g., [69, p. 10], [6], [54, pp. 261–262, § 61], [44, p. 143], [45, p. 423, Eq. (30)], or via

$$\begin{aligned} G_n &= \frac{C_{1,n}}{n!} = -\frac{B_n^{(n-1)}}{(n-1)n!} = \frac{B_n^{(n)}(1)}{n!} = \int_0^1 \binom{x}{n} dx \\ &= \frac{1}{n!} \int_0^1 (x-n+1)_n dx = \frac{1}{n!} \sum_{l=1}^n \frac{S_1(n,l)}{l+1}, \end{aligned} \tag{8}$$

where  $n$  is a natural number,  $(x)_n \equiv x(x+1)(x+2)\cdots(x+n-1)$  stands for the Pochhammer symbol (also known as the rising factorial) and  $S_1(n,l)$  are the

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<sup>6</sup>Our formula (5) is a corrected version of the original Ser’s formula (4) [66, p. 1076] (we also corrected this formula in our previous work [9, p. 382]). In Ser’s formula (4), the corrections which need to be done are the following. First, in the second line of (2) the last term should be  $(-1)^n(n+1)^{-s}$  and not  $(-1)^n n^{-s}$ . Second, in equation (3), “ $(1-x)((2-x))$ ” should read “ $(1-x)(2-x)$ ”. Third, the region of convergence of formula (4), p. 1076, should be  $s \in \mathbb{C} \setminus \{1\}$  and not  $s < 1$ . Note also that Ser’s  $p_{n+1}$  are equal to our  $|G_n|$ . We, actually, carefully examined 5 different hard copies of [66] (from the *Institut Henri Poincaré*, from the *École Normale Supérieure Paris*, from the *Université Pierre-et-Marie-Curie*, from the *Université de Strasbourg* and from the *Bibliothèque nationale de France*), and all of them contained the same misprints.

<sup>7</sup>Charles Hermite even aimed to obtain a more general expression (see, for more details, Theorem 3 and footnote 19).

<sup>8</sup>See (52) hereafter and also [57, pp. 145–147, 462], [53, pp. 127, 129, 135, 182]. Note also that  $B_n^{(n-1)} = B_n^{(n-1)}(0)$  from (52).

Stirling numbers of the first kind, which are the coefficients in the expansion of the falling factorial, and hence of the binomial coefficient,<sup>9</sup>

$$(x - n + 1)_n = n! \binom{x}{n} = x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{l=1}^n S_1(n, l) x^l. \quad (9)$$

Gregory’s coefficients are alternating  $G_n = (-1)^{n-1}|G_n|$  and decreasing in absolute value; they behave as  $(n \ln^2 n)^{-1}$  at  $n \rightarrow \infty$  and may be bounded from below and from above accordingly to formulæ (55)–(56) from [10].<sup>10</sup> The first few coefficients are:  $G_1 = +1/2$ ,  $G_2 = -1/12$ ,  $G_3 = +1/24$ ,  $G_4 = -19/720$ ,  $G_5 = +3/160$ ,  $G_6 = -863/60480, \dots$ <sup>11</sup> For more information about these important numbers, see [10, pp. 410–415], [9, p. 379], and the literature given therein (nearly 50 references).

## 2. On the Equivalence Between Ser’s and Hasse’s Representations for the Euler–Riemann Zeta-function

One may immediately see that Hasse’s representation (1) and Ser’s representation (4) are very similar, so one may question whether these expressions are equivalent or not. The paper written by Ser [66] is much less cited than that by Hasse [37], and in the few works in which both of them are cited, these series are treated as different and with no connection between them.<sup>12</sup> However, as we shall show later, this is not true.

In one of our previous works [9, p. 382], we already noticed that these two series are, in fact, equivalent, but this was stated in a footnote and without a proof.<sup>13</sup> Below, we provide a rigorous proof of this statement.

**Theorem 1.** *Ser’s representation for the  $\zeta$ -function [66, p. 1076, Eq. (7)]*

$$\zeta(s) = \frac{1}{s - 1} \sum_{n=0}^{\infty} \frac{1}{n + 2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k + 1)^{-s}, \quad s \in \mathbb{C} \setminus \{1\}, \quad (10)$$

<sup>9</sup>For more information about  $S_1(n, l)$ , see [10, Sect. 2] and numerous references given therein. Here we use exactly the same definition for  $S_1(n, l)$  as in the cited reference.

<sup>10</sup>For the asymptotics of  $G_n$  and their history, see [11, Sect. 3].

<sup>11</sup>Numerators and denominators of  $G_n$  may also be found in OEIS A002206 and A002207 respectively.

<sup>12</sup>A simple internet search with “Google scholar” indicates that Hasse’s paper [37] is cited more than 50 times, while Ser’s paper [66] is cited only 12 times (including several on-line resources such as [75], as well as some incorrect items), and all the citations are very recent. These citations mainly regard an infinite product for  $e^\gamma$ , e.g., [71], [12], [21], [20], and we found only two works [26], [29], where both articles [66] and [37] were cited simultaneously and in the context of series representations for  $\zeta(s)$ .

<sup>13</sup>We, however, indicated that a recurrence relation for the binomial coefficients should be used for the proof.

and Hasse's representation for the  $\zeta$ -function [37, pp. 460–461]

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{1-s}, \quad s \in \mathbb{C} \setminus \{1\}, \quad (11)$$

are equivalent in the sense that one series is a rearranged version of the other.

*Proof.* In view of the fact that

$$\frac{1}{k+1} \binom{n}{k} = \frac{1}{n+1} \binom{n+1}{k+1} \quad \text{and that} \quad \frac{1}{(n+2)(n+1)} = \frac{1}{n+1} - \frac{1}{n+2},$$

Ser's formula (10), multiplied by the factor  $s - 1$ , may be written as:

$$\begin{aligned} (s-1)\zeta(s) &= \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)} \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} (k+1)^{1-s} \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} (k+1)^{1-s} - \\ &\quad - \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} (k+1)^{1-s} = \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} (k+1)^{1-s} - \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} (k+1)^{1-s} = \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^s} + \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^{s-1}} \left\{ \binom{n+1}{k+1} - \binom{n}{k+1} \right\} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^s} + \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^{s-1}} \binom{n}{k} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^n \frac{(-1)^k}{(k+1)^{s-1}} \binom{n}{k} = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n \frac{(-1)^k}{(k+1)^{s-1}} \binom{n}{k}, \end{aligned}$$

where, between the seventh and eighth lines, we resorted to the recurrence relation for the binomial coefficients. The last line is identical with Hasse's formula (11).  $\square$

Thus, series (10) and (11) are equivalent in the sense that (11) may be obtained by an appropriate rearrangement of terms in (10) and vice versa. It is also interesting that Hasse’s series for  $\zeta(s)$  may be obtained from that for  $\zeta(s, v)$  by simply setting  $v = 1$ , while Ser’s series for  $\zeta(s)$ , as we come to see later, is obtained from a much more complicated formula for  $\zeta(s, v)$ ; see (129)–(128) hereafter.

**Corollary 1.** *The Stieltjes constants  $\gamma_m$  may be given by the following series*

$$\gamma_m = -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^{m+1}(k+1)}{k+1}, \quad m \in \mathbb{N}_0, \quad (12)$$

$$\gamma_m = -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^{m+1}(k+1), \quad m \in \mathbb{N}_0, \quad (13)$$

$$\gamma_m = \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^m(k+1)}{k+1}, \quad m \in \mathbb{N}. \quad (14)$$

*Proof.* The Stieltjes constants  $\gamma_m$ ,  $m \in \mathbb{N}_0$ , are the coefficients appearing in the regular part of the Laurent series expansion of  $\zeta(s)$  about its unique pole  $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{m=1}^{\infty} \frac{(-1)^m \gamma_m}{m!} (s-1)^m, \quad s \in \mathbb{C} \setminus \{1\}, \quad (15)$$

and  $\gamma_0 = \gamma$ .<sup>14</sup> Since the function  $(s-1)\zeta(s)$  is holomorphic on the entire complex  $s$ -plane, it may be expanded into the Taylor series. The latter expansion, applied to (10) and (11) in a neighborhood of  $s = 1$ , produces formulæ (12)–(13). Proceeding similarly with  $\zeta(s) - (s-1)^{-1}$  and Ser’s formula (5) yields (14).  $\square$

**Corollary 2.** *The normalized Maclaurin coefficients  $\delta_m$  of the regular function  $\zeta(s) - (s-1)^{-1}$  admit the following series representation*

$$\delta_m = \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^m(k+1), \quad m \in \mathbb{N}. \quad (16)$$

*Proof.* Analogously to the Stieltjes constants  $\gamma_m$ , may be introduced the normalized Maclaurin coefficients  $\delta_m$  of the regular function  $\zeta(s) - (s-1)^{-1}$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^m \delta_m}{m!} s^m, \quad s \in \mathbb{C} \setminus \{1\}. \quad (17)$$

They are important in the study of the  $\zeta$ -function and are polynomially related to the Stieltjes constants:  $\delta_1 = \frac{1}{2} \ln 2\pi - 1$ ,  $\delta_2 = \gamma_1 + \frac{1}{2} \gamma^2 - \frac{1}{2} \ln^2 2\pi - \frac{1}{24} \pi^2 + 2, \dots$ <sup>15</sup>

<sup>14</sup>For more information on  $\gamma_m$ , see [30, p. 166 *et seq.*], [10], [8], and the literature given therein.

<sup>15</sup>For more information on  $\delta_m$ , see, e.g., [47], [67], [27], [30, p. 168 *et seq.*].

Now, from Ser’s formula (5), it follows that

$$\sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s} = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^m \delta_m}{m!} s^m.$$

Expanding the left-hand side into the Maclaurin series and equating the coefficients of  $s^m$  yield (16). □

**3. A Series for the Zeta-function With the Cauchy Numbers of the Second Kind (Nørlund Numbers)**

An appropriate rearrangement of terms in another series of Ser, formula (5), also leads to an interesting result. In particular, we may deduce a series very similar to (5), but containing the normalized Cauchy numbers of the second kind  $C_n$  instead of  $G_n$ .

The normalized Cauchy numbers of the second kind  $C_n$ , related to the ordinary Cauchy numbers of the second kind  $C_{2,n}$  as  $C_n \equiv C_{2,n}/n!$  (numbers  $C_{2,n}$  are also known as signless *generalized Bernoulli numbers* and signless *Nørlund numbers*  $|B_n^{(n)}| = (-1)^n B_n^{(n)}$ ), appear in the power series expansion of  $[(1 \pm z) \ln(1 \pm z)]^{-1}$  and of  $\ln \ln(1 \pm z)$  in a neighborhood of zero

$$\begin{cases} \frac{z}{(1+z)\ln(1+z)} = 1 + \sum_{n=1}^{\infty} C_n (-z)^n, & |z| < 1, \\ \ln \ln(1+z) = \ln z + \sum_{n=1}^{\infty} \frac{C_n}{n} (-z)^n, & |z| < 1, \end{cases} \tag{18}$$

and may also be defined explicitly:

$$\begin{aligned} C_n &\equiv \frac{C_{2,n}}{n!} = \frac{(-1)^n B_n^{(n)}}{n!} = (-1)^n \int_0^1 \binom{x-1}{n} dx = \\ &= \frac{1}{n!} \int_0^1 (x)_n dx = \frac{1}{n!} \sum_{l=1}^n \frac{|S_1(n,l)|}{l+1}, \quad n \in \mathbb{N}. \end{aligned} \tag{19}$$

They also are linked to Gregory’s coefficients via the recurrence relation  $C_{n-1} - C_n = |G_n|$ , which is sometimes written as

$$C_n = 1 - \sum_{k=1}^n |G_k|, \quad n \in \mathbb{N}. \tag{20}$$

The numbers  $C_n$  are positive rational and always decrease with  $n$ ; they behave as  $\ln^{-1} n$  at  $n \rightarrow \infty$  and may be bounded from below and from above accordingly to formulæ (53)–(54) from [10]. The first few values are:  $C_1 = 1/2$ ,  $C_2 = 5/12$ ,  $C_3 = 3/8$ ,  $C_4 = 251/720$ ,  $C_5 = 95/288$ ,  $C_6 = 19087/60480$ , ...<sup>17</sup> For more information on the Cauchy numbers of the second kind, see [57, pp. 150–151], [28, p. 12], [58], [53, pp. 127–136], [5, vol. III, pp. 257–259], [24, pp. 293–294, n° 13], [39, 1, 80, 61], [10, pp. 406, 410, 414–415, 428–430], [9].

**Theorem 2.** *The  $\zeta$ -function may be represented by the following globally convergent series*

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + 1 - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+2)^{-s} = \\ &= \frac{1}{s-1} + 1 - 2^{-s-1} - \frac{5}{12}(2^{-s} - 3^{-s}) - \frac{3}{8}(2^{-s} - 2 \cdot 3^{-s} + 4^{-s}) - \dots \end{aligned} \tag{21}$$

$s \in \mathbb{C} \setminus \{1\}$ , where  $C_n$  are the normalized Cauchy numbers of the second kind.

*Proof.* Using Fontana’s identity  $\sum |G_n| = 1$ , where the summation extends over positive integers  $n$ , see, e.g., [10, p. 410, Eq. (20)], Ser’s formula (5) takes the form

$$\zeta(s) = \frac{1}{s-1} + 1 + \sum_{n=1}^{\infty} |G_{n+1}| \sum_{k=1}^n (-1)^k \binom{n}{k} (k+1)^{-s}. \tag{22}$$

Now, by taking into account that  $C_{n-1} - C_n = |G_n|$ , and by employing the recurrence property of the binomial coefficients

$$\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1},$$

we find that

$$\zeta(s) - \frac{1}{s-1} - 1 =$$

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<sup>16</sup>This formula incorrectly appears in [42, p. 267, 269]: in several places, the formulæ for  $\psi_n(-1)$  should contain the sum with the upper bound  $n$  instead of  $n+1$ .

<sup>17</sup>See also OEIS A002657 and A002790, which are the numerators and denominators respectively of  $C_{2,n} = n! C_n$ .

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} C_n \sum_{k=1}^n (-1)^k \binom{n}{k} (k+1)^{-s} - \sum_{n=1}^{\infty} C_{n+1} \sum_{k=1}^n (-1)^k \binom{n}{k} (k+1)^{-s} \\
 &= \sum_{n=0}^{\infty} C_{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} (k+1)^{-s} - \sum_{n=1}^{\infty} C_{n+1} \sum_{k=1}^n (-1)^k \binom{n}{k} (k+1)^{-s} \\
 &= -C_1 2^{-s} + \sum_{n=1}^{\infty} C_{n+1} \left\{ \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} (k+1)^{-s} - \sum_{k=1}^n (-1)^k \binom{n}{k} (k+1)^{-s} \right\} \\
 &= -C_1 2^{-s} + \sum_{n=1}^{\infty} C_{n+1} \left\{ \frac{(-1)^{n+1}}{(n+2)^s} + \sum_{k=1}^n (-1)^k \binom{n}{k-1} (k+1)^{-s} \right\} \\
 &= -C_1 2^{-s} - \sum_{n=1}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+2)^{-s}, \tag{23}
 \end{aligned}$$

which is the same as (21), since  $C_1 2^{-s}$  is the zeroth term of the last sum. The global convergence of (21) follows from that of (5).  $\square$

**Corollary 3.** *The Stieltjes constants may be represented by the following series*

$$\gamma_m = - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^m(k+2)}{k+2}, \quad m \in \mathbb{N}_0. \tag{24}$$

*Proof.* Proceeding analogously to the proof of Corollary 1, we obtain the required result. Moreover, for Euler’s constant  $\gamma = \gamma_0$ , we have an expression which may be simplified thanks to the fact  $(-1)^n \Delta^n x^{-1} \Big|_{x=2} = [(n+1)(n+2)]^{-1}$ , so that

$$\begin{aligned}
 \gamma &= 1 - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n \frac{(-1)^k}{k+2} \binom{n}{k} = 1 - \sum_{n=0}^{\infty} \frac{C_{n+1}}{(n+1)(n+2)} = \\
 &= 1 - \frac{1}{4} - \frac{5}{72} - \frac{1}{32} - \frac{251}{14\,400} - \frac{19}{1728} - \frac{19\,087}{2\,540\,160} - \dots
 \end{aligned}$$

This series was already encountered in earlier works [10, p. 380, Eq. (34)], [9, p. 429, Eq. (95)].  $\square$

**Corollary 4.** *The coefficients  $\delta_m$  admit the following series representation*

$$\delta_m = - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^m(k+2), \quad m \in \mathbb{N}. \tag{25}$$

*Proof.* Analogous to Corollary 2, except that we replace (5) by (21).  $\square$

**4. Generalizations of Series With Gregory’s Coefficients and Cauchy Numbers of the Second Kind to the Hurwitz Zeta-function and to Some Dirichlet Series**

**Theorem 3.** *The Hurwitz zeta-function  $\zeta(s, v)$  may be represented by the following globally series  $s \in \mathbb{C} \setminus \{1\}$  with the finite difference  $\Delta^n v^{-s}$*

$$\zeta(s, v) = \frac{v^{1-s}}{s-1} + \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{-s} = \frac{v^{1-s}}{s-1} + \frac{1}{2}v^{-s} + \frac{1}{12}[v^{-s} - (1+v)^{-s}] + \frac{1}{24}[v^{-s} - 2(1+v)^{-s} + (2+v)^{-s}] + \dots, \tag{26}$$

where  $\operatorname{Re} v > 0$ ,

$$\zeta(s, v) = \frac{(v-1)^{1-s}}{s-1} - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{-s} = \frac{(v-1)^{1-s}}{s-1} - \frac{1}{2}v^{-s} - \frac{5}{12}[v^{-s} - (1+v)^{-s}] - \frac{3}{8}[v^{-s} - 2(1+v)^{-s} + (2+v)^{-s}] - \dots, \tag{27}$$

where  $\operatorname{Re} v > 1$ , and

$$\zeta(s, v) = \frac{1}{m(s-1)} \sum_{n=0}^{m-1} (v+a+n)^{1-s} + \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+1,m}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{-s} \tag{28}$$

where  $\operatorname{Re} v > -\operatorname{Re} a$ ,  $\operatorname{Re} a \geq -1$  and  $N_{n,m}(a)$  are the polynomials defined by

$$N_{n,m}(a) \equiv \frac{1}{n!} \int_a^{a+m} (x-n+1)_n dx = \int_a^{a+m} \binom{x}{n} dx = \psi_{n+1}(a+m) - \psi_{n+1}(a), \tag{29}$$

or equivalently by their generating function

$$\frac{(1+z)^{a+m} - (1+z)^a}{\ln(1+z)} = \sum_{n=0}^{\infty} N_{n,m}(a) z^n, \quad |z| < 1. \tag{30}$$

The function  $\psi_n(x)$  is the antiderivative of the binomial coefficient and is also known as the Bernoulli polynomials of the second kind and the Fontana–Bessel polynomials. All these series are similar to Hasse’s series (2) and contain the same finite difference  $\Delta^n v^{-s}$ .

*Proof.* First variant of proof of (26): Expanding the function  $\zeta(s, x)$  into the Gregory–Newton interpolation series (also known as the forward difference formula)

in a neighborhood of  $x = v$ , yields

$$\zeta(s, x + v) = \zeta(s, v) + \sum_{n=1}^{\infty} \binom{x}{n} \Delta^n \zeta(s, v) = \zeta(s, v) + \sum_{n=1}^{\infty} \frac{(x - n + 1)_n}{n!} \Delta^n \zeta(s, v), \tag{31}$$

where  $\Delta^n f(v)$  is the  $n$ th *finite forward difference* of  $f(x)$  at point  $v$

$$\Delta f(v) \equiv \Delta f(x)|_{x=v} = \{f(x + 1) - f(x)\}|_{x=v} = f(v + 1) - f(v),$$

.....

$$\Delta^n f(v) = \Delta^{n-1} f(v + 1) - \Delta^{n-1} f(v) = \dots \tag{32}$$

$$\dots = \sum_{k=0}^n (-1)^k \binom{n}{k} f(n - k + v) = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} f(v + k)$$

with  $\Delta^0 f(v) \equiv f(v)$  by convention<sup>18</sup>. Since the operator of finite difference  $\Delta^n$  is linear and because  $\zeta(s, v + 1) = \zeta(s, v) - v^{-s}$ , it follows from (32) that  $\Delta^n \zeta(s, v) = -\Delta^{n-1} v^{-s}$ . Formula (31), therefore, becomes

$$\zeta(s, v) = \sum_{n=0}^{\infty} (x + v + n)^{-s} + \sum_{n=1}^{\infty} \frac{(x - n + 1)_n}{n!} \Delta^{n-1} v^{-s}. \tag{33}$$

Integrating termwise the latter equality over  $x \in [0, 1]$  and accounting for the fact that

$$\int_0^1 (x + v)^{-s} dx = \frac{1}{s - 1} \{v^{1-s} - (v + 1)^{1-s}\}, \tag{34}$$

as well as using (8), we have

$$\begin{aligned} \zeta(s, v) &= \frac{1}{s - 1} \left\{ \sum_{n=0}^{\infty} (v + n)^{1-s} - \sum_{n=0}^{\infty} (v + 1 + n)^{1-s} \right\} + \sum_{n=1}^{\infty} G_n \Delta^{n-1} v^{-s} \\ &= \frac{v^{1-s}}{s - 1} + \sum_{n=0}^{\infty} G_{n+1} \Delta^n v^{-s}, \end{aligned} \tag{35}$$

which is identical with (26), because  $G_{n+1} = (-1)^n |G_{n+1}|$  and

$$\Delta^n v^{-s} \equiv \Delta^n x^{-s}|_{x=v} = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} (k + v)^{-s}. \tag{36}$$

---

<sup>18</sup>Note, however, that due to the fact that finite differences may be defined in slightly different ways and that there also exist *forward*, *central*, *backward* and other finite differences, our definition for  $\Delta^n f(v)$  may not be shared by others. Thus, some authors call the quantity  $(-1)^n \Delta^n f(v)$  the  $n$ th finite difference, see, e.g., [74, p. 270, Eq. (14.17)] (we also employed the latter definition in [10, p. 413, Eq. (39)]). For more details on the Gregory–Newton interpolation formula, see, e.g., [40, § 9.02], [53, pp. 57–59], [42, pp. 184, 219 *et seq.*, 357], [13, Ch. III], [36, Ch. 1 & 9], [57], [76], [46, Ch. 3], [33, p. 192], [50, Ch. V], [43, Ch. III, pp. 184–185], [52], [68], [60, p. 31].

The reader may also note that if we put  $v = 1$  in (33) and (36), then we obtain Ser's formulæ (3) and (2) respectively.  $\square$

*Second variant of proof of (26):* Consider the generating equation for the numbers  $G_n$ , formula (6). Dividing it by  $z$  and then putting  $z = e^{-x} - 1$  yields

$$\begin{aligned} \frac{1}{1 - e^{-x}} &= \frac{1}{x} + |G_1| + \sum_{n=1}^{\infty} |G_{n+1}| (1 - e^{-x})^n \\ &= \frac{1}{x} + |G_1| + \sum_{n=1}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} e^{-kx}, \quad x > 0, \end{aligned} \tag{37}$$

since  $|G_{n+1}| = (-1)^n G_{n+1}$ . Now, using the well-known integral representation of the Hurwitz  $\zeta$ -function

$$\zeta(s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-vx} x^{s-1}}{1 - e^{-x}} dx$$

and Euler's formulæ

$$\frac{v^{1-s}}{s-1} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-vx} x^{s-2} dx, \quad v^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-vx} x^{s-1} dx,$$

we obtain

$$\begin{aligned} \zeta(s, v) - \frac{v^{1-s}}{s-1} - |G_1| v^{-s} &= \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-vx} x^{s-1} \left\{ \frac{1}{1 - e^{-x}} - \frac{1}{x} - |G_1| \right\} dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^{\infty} e^{-(k+v)x} x^{s-1} dx \\ &= \sum_{n=1}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{-s}. \end{aligned}$$

Remarking that  $|G_1| v^{-s}$  is actually the term corresponding to  $n = 0$  in the sum on the right yields (26).<sup>19</sup>  $\square$

<sup>19</sup>It seems appropriate to note here that Charles Hermite in 1900 tried to use a similar method to derive a series with Gregory's coefficients for  $\zeta(s, v)$ , but his attempt was not succesfull. A careful analysis of his derivations [38, p. 69], [59, vol. IV, p. 540], reveals that Hermite's errors is due to the incorrect expansion of  $(1 - e^{-x})^{-1}$  into the series with  $\omega_n$ , which, in turn, led him to an incorrect formula for  $R(a, s) \equiv \zeta(s, a)$ .<sup>20</sup> These results were never published during Hermite's lifetime and appeared only in epistolary exchanges with the Italian mathematician Salvatore Pincherle, who published them in [38] several months after Hermite's death. Later, these letters were reprinted in [59].

<sup>20</sup>On p. 69 in [38] and p. 540 in [59, vol. IV] in the expansion for  $(1 - e^{-x})^{-1}$  the term  $\omega_1$  should be replaced by  $\omega_2$  and  $\omega_n$  by  $\omega_{n+1}$ . Note that Hermite's  $\omega_n = |G_n|$ .

*First variant of proof of (27).* Integrating term-by-term the right-hand side of (33) over  $x \in [-1, 0]$  and remarking that

$$\frac{1}{n!} \int_{-1}^0 (x - n + 1)_n dx = \int_{-1}^0 \binom{x}{n} dx = \int_0^1 \binom{x-1}{n} dx = (-1)^n C_n, \quad (38)$$

we have

$$\begin{aligned} \zeta(s, v) &= \frac{1}{s-1} \left\{ \sum_{n=0}^{\infty} (v-1+n)^{1-s} - \sum_{n=0}^{\infty} (v+n)^{1-s} \right\} + \sum_{n=1}^{\infty} (-1)^n C_n \Delta^{n-1} v^{-s} \\ &= \frac{(v-1)^{1-s}}{s-1} - \sum_{n=0}^{\infty} C_{n+1} (-1)^n \Delta^n v^{-s}, \end{aligned} \quad (39)$$

which coincides with (27) by virtue of (36). □

*Second variant of proof of (27).* In order to obtain (27), we also may proceed analogously to the demonstration of Theorem 2, in which we replace (5) by (26). The unity appearing from Fontana’s series in (21) becomes  $v^{-s}$  and the term  $(k+2)^{-s}$  becomes  $(k+1+v)^{-s}$ , that is to say

$$\zeta(s, v) = \frac{v^{1-s}}{s-1} + v^{-s} - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v+1)^{-s}, \quad (40)$$

Re  $v > 0$ . Using the recurrence relation  $\zeta(s, v) = \zeta(s, v+1) + v^{-s}$  and rewriting the final result for  $v$  instead of  $v+1$ , we immediately obtain (27). □

*Proof of (28).* Our method of proof, which uses the Gregory–Newton interpolation formula, may be further generalized. By introducing polynomials  $N_{n,m}(a)$  according to (29), and by integrating (33) over  $x \in [a, a+m]$ , we have

$$\zeta(s, v) = \frac{1}{m(s-1)} \left\{ \zeta(s-1, v+a) - \zeta(s-1, v+a+m) \right\} + \frac{1}{m} \sum_{n=1}^{\infty} N_{n,m}(a) \Delta^{n-1} v^{-s}. \quad (41)$$

Simplifying the expression in curly brackets and reindexing the latter sum immediately yields

$$\zeta(s, v) = \frac{1}{m(s-1)} \sum_{n=0}^{m-1} (v+a+n)^{1-s} + \frac{1}{m} \sum_{n=0}^{\infty} N_{n+1,m}(a) \Delta^n v^{-s}, \quad (42)$$

which is identical with (28). Note that expansions (26)–(27) are both particular cases of (28) at  $m = 1$ . Formula (26) is obtained by setting  $a = 0$ , while (27) corresponds to  $a = -1$ . □

**Corollary 5.** *The Euler–Riemann  $\zeta$ -function admits the following general expansions*

$$\zeta(s) = \frac{1}{m(s-1)} \sum_{n=1}^m (a+n)^{1-s} + \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+1,m}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s} \tag{43}$$

$a > -1$ ,  $m \in \mathbb{N}$ , and

$$\zeta(s) = 1 + \frac{1}{m(s-1)} \sum_{n=1}^m (a+1+n)^{1-s} + \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+1,m}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k+2)^{-s} \tag{44}$$

$a > -2$ ,  $m \in \mathbb{N}$ , containing finite differences  $\Delta^n 1^{-s}$  and  $\Delta^n 2^{-s}$  respectively. Ser’s series (5) and our series from Theorem 2 are particular cases of the above expansions.<sup>21</sup>

*Proof.* On the one hand, setting  $v = 1$  in (42) we immediately obtain (43). On the other hand, putting  $v + 1$  instead of  $v$  and using the relation  $\zeta(s, v + 1) = \zeta(s, v) - v^{-s}$ , equality (42) takes the form

$$\zeta(s, v) = v^{-s} + \frac{1}{m(s-1)} \sum_{n=1}^m (v+a+n)^{1-s} + \frac{1}{m} \sum_{n=0}^{\infty} N_{n+1,m}(a) \Delta^n (v+1)^{-s}. \tag{45}$$

At point  $v = 1$ , this equality becomes (44). Continuing the process, we may also obtain similar formulæ for  $\zeta(s)$  containing finite differences  $\Delta^n 3^{-s}$ ,  $\Delta^n 4^{-s}$ , ...  $\square$

**Remark 1, related to the polynomials  $N_{n,m}(a)$ .** Polynomials  $N_{n,m}(a)$  generalize many special numbers and have a variety of interesting properties. First of all, we remark that  $N_{n,m}(a)$  are polynomials of degree  $n$  in  $a$  with rational coefficients. This may be seen from the fact that

$$\begin{aligned} N_{n,m}(a) &\equiv \frac{1}{n!} \int_a^{a+m} (x-n+1)_n dx = \frac{1}{n!} \sum_{l=1}^n \frac{S_1(n,l)}{l+1} \{(a+m)^{l+1} - a^{l+1}\} \\ &= \frac{1}{n!} \sum_{l=1}^n \frac{S_1(n,l)}{l+1} \sum_{k=0}^l a^k m^{l+1-k} \binom{l+1}{k}, \end{aligned} \tag{46}$$

where  $n$  and  $m$  are positive integers. This formula is quite simple and very handy for the calculation of  $N_{n,m}(a)$  with the help of CAS. It is therefore clear that for any  $a \in \mathbb{Q}$ , polynomials  $N_{n,m}(a)$  are simply rational numbers. Some of such examples may be of special interest

$$N_{n,1}(-1) = (-1)^n C_n, \quad N_{n,1}(0) = G_n, \quad N_{2n,1}(n-1) = M_{2n},$$

---

<sup>21</sup>We have Ser’s formula when putting  $a = 0$ ,  $m = 1$  in (43), and our Theorem 2 if setting  $a = -1$ ,  $m = 1$  in (44).

where  $M_n$  are central difference coefficients  $M_2 = -1/12$ ,  $M_4 = +11/720$ ,  $M_6 = -191/60480$ ,  $M_8 = +2497/3628800$ , ... , see, e.g., [64], [40, § 9.084], [53, p. 186], OEIS A002195 and A002196. The derivative of  $N_{n,m}(a)$  is

$$\frac{\partial N_{n,m}(a)}{\partial a} = \frac{(a+m-n+1)_n - (a-n+1)_n}{n!} = \binom{a+m}{n} - \binom{a}{n}. \tag{47}$$

Polynomials  $N_{n,m}(a)$  are related to many other special polynomials. This can be readily seen from the generating equation for  $N_{n,m}(a)$ , which we gave in (30) without the proof. Let us now prove it. On the one hand, on integrating  $(1+z)^x$  between  $x = a$  and  $x = a+m$  we undoubtedly have

$$\int_a^{a+m} (1+z)^x dx = \frac{(1+z)^{a+m} - (1+z)^a}{\ln(1+z)}. \tag{48}$$

On the other hand, the same integral may be calculated by expanding  $(1+z)^x$  into the binomial series

$$\int_a^{a+m} (1+z)^x dx = \sum_{n=0}^{\infty} z^n \int_a^{a+m} \binom{x}{n} dx = \sum_{n=0}^{\infty} z^n N_{n,m}(a) \tag{49}$$

by virtue of the uniform convergence. Equating both expressions yields (30). There is also a more direct way to prove the same result and, in addition, to explicitly determine  $N_{0,m}(a)$ . Using (46) and accounting for the absolute convergence, we have for the right part of (30)

$$\sum_{n=1}^{\infty} N_{n,m}(a) z^n = \sum_{l=1}^{\infty} \frac{(a+m)^{l+1} - a^{l+1}}{l+1} \underbrace{\sum_{n=1}^{\infty} \frac{S_1(n,l)}{n!} z^n}_{\ln^l(1+z)/l!} = \frac{1}{\ln(1+z)} \cdot \left\{ \sum_{l=2}^{\infty} \frac{[(a+m)\ln(1+z)]^l}{l!} - \sum_{l=2}^{\infty} \frac{[a\ln(1+z)]^l}{l!} \right\} = \frac{(1+z)^{a+m} - (1+z)^a}{\ln(1+z)} - m, \tag{50}$$

where we used the generating equation for the Stirling numbers of the first kind, see, e.g., [10, p. 408], [9, p. 369]. Hence,  $N_{0,m}(a) = m$ . Polynomials  $N_{n,m}(a)$  are, therefore, close to the Stirling polynomials, to Van Veen's polynomials  $K_n^{(z)}$  [72], to various generalizations of the Bernoulli numbers/polynomials, including the so-called *Nørlund polynomials* [55, p. 602], which are also known as the generalized Bernoulli polynomials of the second kind [19, p. 324, Eq. (2.1)], and to many other special polynomials, see, e.g., [5, Vol. III, § 19], [53, Ch. VI], [49, Vol. I, § 2.8], [18], [57], [58], [34], [35], [78] [15], [14], [62], [45], [63]. The most close connection seems to exist with the *Bernoulli polynomials of the second kind*, also known as

the *Fontana–Bessel polynomials*, which are denoted by  $\psi_n(x)$  by Jordan [41], [42, Ch. 5], [19, p. 324],<sup>22</sup> and have the following generation function

$$\frac{z(z+1)^x}{\ln(z+1)} = \sum_{n=0}^{\infty} \psi_n(x) z^n, \quad |z| < 1; \tag{51}$$

see, e.g., [42, Ch. 5, p. 279, Eq. (8)], [19, p. 324, Eq. (1.11)], and with the Bernoulli polynomials of higher order, usually denoted by  $B_n^{(s)}(x)$ , and defined via

$$\frac{z^s e^{xz}}{(e^z - 1)^s} = \sum_{n=0}^{\infty} \frac{B_n^{(s)}(x)}{n!} z^n, \quad |z| < 2\pi; \tag{52}$$

see, e.g., [53, pp. 127–135], [19, p. 323, Eq. (1.4)], [56], [57, p. 145], [58], [5, Vol. III, § 19.7]. Indeed, using formula (51), we have for the left part of (30)

$$\begin{aligned} \frac{(1+z)^{a+m} - (1+z)^a}{\ln(1+z)} &= \sum_{n=0}^{\infty} \{\psi_n(a+m) - \psi_n(a)\} z^{n-1} \\ &= \sum_{n=0}^{\infty} \{\psi_{n+1}(a+m) - \psi_{n+1}(a)\} z^n \end{aligned}$$

since  $\psi_0(x) = 1$ .<sup>25</sup> Comparing the latter expression to the right part of (30) immediately yields

$$N_{n,m}(a) = \psi_{n+1}(a+m) - \psi_{n+1}(a) = \sum_{k=0}^{m-1} \psi_n(a+k), \quad n \in \mathbb{N}_0, \tag{53}$$

since  $\Delta\psi_{n+1}(x) = \psi_n(x)$ , see, e.g., [42, pp. 265, 268]. In particular, for  $m = 1$  we simply have

$$N_{n,1}(a) = \psi_n(a). \tag{54}$$

Another way to prove (53) is to recall that  $\binom{x}{n} dx = d\psi_{n+1}(x)$ , see, e.g., [41, p. 130], [42, p. 265]. Hence, the antiderivative of the falling factorial is, up to a function of  $n$ , precisely the function  $\psi_{n+1}(x)$ . By virtue of this important property, formula

<sup>22</sup>These polynomials and/or those equivalent or closely related to them, were rediscovered in numerous works and by numerous authors (compare, for example, works [51, p. 1916], [79, p. 3998], [62, § 5.3.2], or compare the Fontana–Bessel polynomials from [4]<sup>23</sup>with  $\psi_n(x)$  introduced by Jordan in the above-cited works and also with polynomials  $P_{n+1}(y)$  employed by Coffey in [23, p. 450]), so we give here only the most frequent notations and definitions for them.

<sup>23</sup>Apparently in [4] Appell refers to Bessel’s work [6] of 1811, which is also indirectly mentioned by Vacca in [73]. In [6], we find some elements related to the polynomials  $\psi_n(x)$ , as well as to  $N_{n,m}(a)$ , but Bessel did not perform their systematical study. In contrast, he studied quite a lot a particular case of them leading to Gregory’s coefficients  $G_n = \psi_n(0)$ ,  $A^0, A', A'', \dots$  in Bessel’s notations, see [6, pp. 10–11], [73, p. 207]. In this context, it may be interesting to remark that Ser [65], [3], investigated polynomials  $\psi_n(x)$  more in details [he denoted them  $P_{n+1}(y)$ ], and this before Jordan, and also calculated several series with  $G_n$ .

(53) follows immediately from our definition of  $N_{n,m}(a)$  given in the statement of the theorem. Furthermore, from (52) it follows that  $B_n^{(n)}(x+1) = n! \psi_n(x)$ , see, e.g., [53, p. 135], [19, Eq. (2.1) & (2.11)], [57, p. 147], whence

$$N_{n,m}(a) = \frac{1}{(n+1)!} \left\{ B_{n+1}^{(n+1)}(a+m+1) - B_{n+1}^{(n+1)}(a+1) \right\}, \quad n \in \mathbb{N}_0. \quad (55)$$

This clearly displays a close connection between  $N_{n,m}(a)$  and the Bernoulli polynomials of both varieties. The latter have been the object of much research by Nörlund<sup>24</sup> [56], [57], [58], [53, Ch. VI], Jordan [41], [42, Ch. 5], Carlitz [19] and some other authors. The polynomials  $N_{n,m}(a)$  may also be given by the following integrals

$$\begin{aligned} N_{n,m}(a) &= \frac{(-1)^n}{\pi} \int_0^\infty \frac{\pi \cos \pi a - \sin \pi a \ln x}{(1+x)^{n+1}} \cdot \frac{[(-x)^m - 1] x^a}{\ln^2 x + \pi^2} dx \\ &= \frac{(-1)^n}{\pi} \int_0^\infty \frac{\pi \cos \pi a + \sin \pi a \ln x}{(1+x)^{n+1}} \cdot \frac{[(-1)^m - x^m] x^{n-m-a-1}}{\ln^2 x + \pi^2} dx, \end{aligned} \quad (56)$$

provided that  $m$  and  $n$  are positive integers, and  $-1 \leq a \leq n-1$  and  $-1-a \leq m \leq n-a-1$ . These representations follow from (53) and this formula

$$\psi_n(x) = \frac{(-1)^{n+1}}{\pi} \int_0^\infty \frac{\pi \cos \pi x - \sin \pi x \ln z}{(1+z)^n} \cdot \frac{z^x dz}{\ln^2 z + \pi^2}, \quad -1 \leq x \leq n-1, \quad (57)$$

whose proof we put in the Appendix. At this stage, it may be useful to provide explicit expressions for the first few polynomials  $N_{n,m}(a)$

$$\begin{aligned} N_{0,m}(a) &= m \\ N_{1,m}(a) &= ma + \frac{1}{2}m^2 \\ N_{2,m}(a) &= \frac{1}{2}ma^2 + \frac{1}{2}am^2 - \frac{1}{2}am + \frac{1}{6}m^3 - \frac{1}{4}m^2 \\ N_{3,m}(a) &= \frac{1}{3}am + \frac{1}{6}m^2 - \frac{1}{2}a^2m - \frac{1}{2}am^2 - \frac{1}{6}m^3 + \frac{1}{6}a^3m + \frac{1}{4}a^2m^2 + \frac{1}{6}am^3 + \frac{1}{24}m^4 \end{aligned} \quad (58)$$

and so on. They may be obtained either directly from (46), or from the similar formulæ for the polynomials  $\psi_n(x)$

$$\psi_n(x) = \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \frac{S_1(n-1, l)}{l+1} x^{l+1} + G_n, \quad n \in \mathbb{N}, \quad {}^{25} \quad (59)$$

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<sup>24</sup>Also written Nörlund.

where  $S_1(0, 0) = 1$ , and from (53). Finally, we remark that the complete asymptotics of the polynomials  $N_{n,m}(a)$  at large  $n$  are given by

$$N_{n,m}(a) \sim \frac{(-1)^{n+1}}{\pi n^{a+1}} \sum_{l=0}^{\infty} \frac{1}{\ln^{l+1} n} \cdot \left[ \sin \pi a \cdot \Gamma(a+1) \right]^{(l)}, \quad \begin{matrix} n \rightarrow \infty \\ \operatorname{Re} a \geq -1 \end{matrix}, \quad (60)$$

where  $^{(l)}$  stands for the  $l$ th derivative, and  $m$  is a finite natural number. In particular, retaining first two terms, we have

$$N_{n,m}(a) \sim \frac{(-1)^{n+1}}{\pi n^{a+1} \ln n} \cdot \left\{ \sin \pi a \cdot \Gamma(a+1) + \frac{\pi \cos \pi a \cdot \Gamma(a+1) + \sin \pi a \cdot \Gamma(a+1) \cdot \Psi(a+1)}{\ln n} \right\} \quad (61)$$

at  $n \rightarrow \infty$ . Both results can be obtained without difficulty from the complete asymptotics of  $B_n^{(n)}(x)$  given by Nørlund [58, p. 38]. Note that if  $a \in \mathbb{N}_0$ , the first term of asymptotics (60)–(61) vanishes, and thus  $N_{n,m}(a)$  decreases slightly faster. Remark also that making  $a \rightarrow 0$  and  $a \rightarrow -1$  in (60)–(61), we find the asymptotics of numbers  $G_n$  and  $C_n$  respectively.<sup>26</sup>

**Theorem 4.** *For any positive integer  $k > 1$ , the  $\zeta$ -functions  $\zeta(s, v)$  and  $\zeta(s)$  obey the following functional relationships*

$$\begin{aligned} \zeta(s, v) = & - \sum_{l=1}^{k-1} \frac{(k-l+1)_l}{(s-l)_l} \cdot \zeta(s-l, v) + \sum_{l=1}^k \frac{(k-l+1)_l}{(s-l)_l} \cdot v^{l-s} + \quad (62) \\ & + k \sum_{n=0}^{\infty} (-1)^n G_{n+1}^{(k)} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{-s}, \end{aligned}$$

and

$$\zeta(s) = - \sum_{l=1}^{k-1} \frac{(k-l+1)_l}{(s-l)_l} \zeta(s-l) + \frac{k}{s-k} + k \sum_{n=0}^{\infty} (-1)^n G_{n+1}^{(k)} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s}$$

<sup>25</sup>In particular,  $\psi_1(x) = x + \frac{1}{2}$ ,  $\psi_2(x) = \frac{1}{2}x^2 - \frac{1}{12}$ ,  $\psi_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{24}$ ,  $\psi_4(x) = \frac{1}{24}x^4 - \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{19}{720}$ , etc. see, e.g., [42, p. 272]. The value  $\psi_0(x) = 1$  cannot be computed with the help of this formula and directly follows from the generating equation (51) (the limit of the left part when  $z \rightarrow 0$  tends to 1 independently of  $x$ ). Note also that  $\psi_n(0) = G_n$ , except for  $n = 0$ . Formula (59) may also be found in [42, p. 267] with a little correction to take into account: the upper bound of the sum in (6) should read  $n$  instead of  $n+1$ .

<sup>26</sup>In [10, p. 414, Eq. (51)], we obtained the complete asymptotics for  $C_n = C_{2,n}/n! = |B_n^{(n)}|/n!$  at large  $n$ . However, it seems appropriate to notice that the equivalent result may be straightforwardly derived from Nørlund’s asymptotics of  $B_n^{(n)}(x)$ , since  $B_n^{(n)}(0) = B_n^{(n)}$ , see [58, pp. 27, 38, 40].

respectively, where  $G_n^{(k)}$  are Gregory's coefficients of higher order defined as

$$G_n^{(k)} \equiv \frac{1}{n!} \sum_{l=1}^n \frac{S_1(n, l)}{l+k}. \tag{63}$$

In fact, we come to prove a more general result, of which the above formulæ are two particular cases.

*Proof.* Let  $\rho(x)$  be the normalized weight such that

$$\int_a^{a+m} \rho(x) dx = 1, \quad \text{and let} \quad N_{n,m}^{(\rho)}(a) \equiv \frac{1}{n!} \int_a^{a+m} (x-n+1)_n \rho(x) dx.$$

Performing the same procedure as in the case of (28) and assuming the uniform convergence, we obtain

$$\zeta(s, v) = \underbrace{\sum_{n=0}^{\infty} \int_a^{a+m} \frac{\rho(x)}{(v+x+n)^s} dx}_{F_{n,m,a}[\rho(x)]} + \sum_{n=1}^{\infty} N_{n,m}^{(\rho)}(a) \Delta^{n-1} v^{-s}. \tag{64}$$

Albeit this generalization appears rather theoretical, it, however, may be useful if the functional  $F_{n,m,a}[\rho(x)]$  admits a suitable closed-form and if the series  $\sum F_{n,m,a}[\rho(x)]$  converges. Thus, if we simply put  $\rho(x) = 1/m$ , then we retrieve our formula (28). If we put  $\rho(x) = kx^{k-1}$ , where  $k \in \mathbb{N}$ , and set  $a = 0, m = 1$ , then it is not difficult to see that

$$N_{n,1}^{(kx^{k-1})}(0) = \frac{k}{n!} \sum_{l=1}^n \frac{S_1(n, l)}{l+k} \equiv kG_n^{(k)}, \tag{65}$$

Now, remarking that the repeated integration by parts yields

$$\begin{aligned} \int x^{k-1} (v+x+n)^{-s} dx &= \frac{1}{k} \sum_{l=1}^{k-1} \frac{(-1)^{l+1} (v+x+n)^{l-s} \cdot x^{k-l} \cdot (k-l+1)_l}{(1-s)_l} + \\ &+ \frac{(-1)^{k+1} (v+x+n)^{k-s} \cdot (k-1)!}{(1-s)_k}, \end{aligned} \tag{66}$$

and evaluating the infinite series  $\sum F_{n,1,0}[kx^{k-1}]$ , formula (64) reduces to the desired result stated in (62), since  $(1-s)_l = (-1)^l (s-l)_l$ . Note that we have (26) as a particular case of (62) at  $k = 1$ . Moreover, if we put  $v = 1$  and simplify the second sum in the first line, then we immediately arrive at this curious formula for the  $\zeta$ -function

$$\zeta(s) = - \sum_{l=1}^{k-1} \frac{(k-l+1)_l}{(s-l)_l} \cdot \zeta(s-l) + \frac{k}{s-k} + k \sum_{n=0}^{\infty} G_{n+1}^{(k)} \Delta^n 1^{-s}, \tag{67}$$

which is also announced in Theorem 4. □

It is interesting to remark that formula (48) from [10, p. 414] also contains similar shifted values of the  $\zeta$ -functions.<sup>27</sup> Another functional relationship of the same kind was discovered by Allouche et al. in [2, Sect. 3, p. 362].

**Nota Bene.** The numbers  $G_n^{(k)}$  are yet another generalization of Gregory’s coefficients  $G_n = G_n^{(1)}$ , and we already encountered them in [10, pp. 413–414]. In the latter, we, *inter alia*, showed that  $|G_n^{(k)}| \sim n^{-1} \ln^{-k-1} n$  at  $n \rightarrow \infty$  and also evaluated the series  $\sum (-1)^n G_n^{(k)}/n$  for  $k = 2, 3, 4, \dots$  [10, Eqs. 38, 50]. Besides, although it is much easier to define  $G_n^{(k)}$  via the finite sum with the Stirling numbers of the first kind (63), they equally may be defined via the generating function

$$z \frac{(-1)^{k+1}(k-1)!}{\ln^k(1+z)} + (1+z) \sum_{l=1}^{k-1} \frac{(-1)^{l+1}(k-l+1)_{l-1}}{\ln^l(1+z)} = \frac{1}{k} + \sum_{n=1}^{\infty} G_n^{(k)} z^n, \quad (68)$$

where  $|z| < 1$  and  $k \in \mathbb{N}$ .<sup>28</sup> This formula may be obtained by proceeding in the manner analogous to (50) and coincides with (6) at  $k = 1$ . Note that the left part of (68) contains the powers of  $\ln^{-1}(1+z)$  up to and including  $k$ , the fact which prompted us to call  $G_n^{(k)}$  *Gregory’s coefficients of higher order*.

**Theorem 5.** Let  $\mathbf{u} \equiv \{u_n\}_{n \geq 0}$  be the sequence of bounded complex numbers and let denote  $\Delta u_n = u_{n+1} - u_n$ ,  $\Delta^2 u_n = \Delta(\Delta u_n) = u_{n+2} - 2u_{n+1} + u_n$ , and so on, exactly as in (32). Consider now the following functional

$$\zeta(s, v, \mathbf{u}) = \sum_{n=0}^{\infty} \frac{u_n}{(v+n)^s}, \quad \begin{array}{l} \operatorname{Re} s > 1 \\ v \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \end{array}$$

generalizing the Euler–Riemann  $\zeta$ -function  $\zeta(s) = \zeta(s, 1, \mathbf{1})$ , the Hurwitz  $\zeta$ -function  $\zeta(s, v) = \zeta(s, v, \mathbf{1})$ , the functional  $F(s)$  from [2, Sect. 3, p. 362]  $F(s) = \zeta(s, 1, \mathbf{u})$ . The Dirichlet series  $\zeta(s, v, \mathbf{u})$  admits two following series representations

$$\begin{aligned} \zeta(s, v, \mathbf{u}) &= \frac{1}{m(s-1)} \sum_{n=0}^{m-1} \zeta(s-1, v+a+n+1, \Delta \mathbf{u}) + \\ &+ \frac{u_0}{m(s-1)} \sum_{n=0}^{m-1} (v+a+n)^{1-s} - \frac{1}{m} \sum_{n=1}^{\infty} (-1)^n N_{n,m}(a) \zeta(s, v+n, \Delta^n \mathbf{u}) + \\ &+ \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+1,m}(a) \sum_{l=0}^n \Delta^l u_0 \sum_{k=0}^{n-l} (-1)^k \binom{n-l}{k} (k+l+v)^{-s} \end{aligned} \quad (69)$$

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<sup>27</sup>Formula (48) from [10] and its proof were first released on 5 January 2015 in the 6th arXiv version of the paper. 28 September 2015, a particular case of the same formula for nonnegative integer  $s$  was also presented by Xu, Yan and Shi [77, p. 94, Theorem 2.9], who, apparently, were not aware of the arXiv preprint of our work [10] (we have not found the preprint of [77]).

<sup>28</sup>For  $k = 1$  the sum over  $l$  should be taken as 0.

and

$$\begin{aligned} \zeta(s, v, \mathbf{u}) &= \frac{1}{m(s-1)} \sum_{n=0}^{m-1} \left\{ \zeta(s-1, v+a+n+1, \Delta \mathbf{u}) + u_0(v+a+n)^{1-s} \right\} + \\ &\quad + \frac{1}{m} \sum_{n=1}^{\infty} N_{n,m}(a) \Delta^{n-1} \zeta(s, v+1, \Delta \mathbf{u}) + \\ &\quad + \frac{u_0}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+1,m}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{-s}, \end{aligned} \tag{70}$$

where  $\Delta^n \mathbf{u} \equiv \{\Delta^n u_l\}_{l \geq 0}$  and the operator  $\Delta^0$  does nothing by convention.

*Proof.* From the definition of  $\zeta(s, v, \mathbf{u})$  it follows that this Dirichlet series possesses the following recurrence relation in  $v$

$$\begin{aligned} \Delta \zeta(s, v, \mathbf{u}) &\equiv \zeta(s, v+1, \mathbf{u}) - \zeta(s, v, \mathbf{u}) = \sum_{n=0}^{\infty} \frac{-\Delta u_n}{(v+n+1)^s} - u_0 v^{-s} \\ &= -\zeta(s, v+1, \Delta \mathbf{u}) - u_0 v^{-s}. \end{aligned} \tag{71}$$

The second-order recurrence relation reads

$$\Delta^2 \zeta(s, v, \mathbf{u}) \equiv \Delta(\Delta \zeta(s, v, \mathbf{u})) = \zeta(s, v+2, \Delta^2 \mathbf{u}) + \Delta u_0(v+1)^{-s} - u_0 \Delta v^{-s},$$

and more generally we have

$$\Delta^k \zeta(s, v, \mathbf{u}) = (-1)^k \zeta(s, v+k, \Delta^k \mathbf{u}) + \sum_{l=0}^{k-1} (-1)^{l+1} \cdot \Delta^l u_0 \cdot \Delta^{k-1-l}(v+l)^{-s}.$$

Writing the Gregory–Newton interpolation formula for  $\zeta(s, x+v, \mathbf{u})$  we, therefore, obtain

$$\begin{aligned} \zeta(s, x+v, \mathbf{u}) &= \zeta(s, v, \mathbf{u}) + \sum_{n=1}^{\infty} \frac{(x-n+1)_n}{n!} \Delta^n \zeta(s, v, \mathbf{u}) = \\ &= \zeta(s, v, \mathbf{u}) + \sum_{n=1}^{\infty} \frac{(-1)^n (x-n+1)_n}{n!} \zeta(s, v+n, \Delta^n \mathbf{u}) + \\ &\quad + \sum_{n=1}^{\infty} \frac{(x-n+1)_n}{n!} \sum_{l=0}^{n-1} (-1)^{l+1} \cdot \Delta^l u_0 \cdot \Delta^{n-1-l}(v+l)^{-s}. \end{aligned}$$

Effecting the term-by-term integration over the interval  $x \in [a, a+m]$  yields

$$\begin{aligned} \zeta(s, v, \mathbf{u}) &= \frac{1}{m(s-1)} \left\{ \zeta(s-1, v+a, \mathbf{u}) - \zeta(s-1, v+a+m, \mathbf{u}) \right\} - \\ &\quad - \frac{1}{m} \sum_{n=1}^{\infty} (-1)^n N_{n,m}(a) \zeta(s, v+n, \Delta^n \mathbf{u}) - \end{aligned}$$

$$-\frac{1}{m} \sum_{n=1}^{\infty} N_{n,m}(a) \sum_{l=0}^{n-1} (-1)^{l+1} \cdot \Delta^l u_0 \cdot \Delta^{n-1-l}(v+l)^{-s}.$$

Finally, simplifying the expression in curly brackets

$$\begin{aligned} \zeta(s-1, v+a, \mathbf{u}) - \zeta(s-1, v+a+m, \mathbf{u}) &= \\ &= \sum_{n=0}^{m-1} \underbrace{\left[ \zeta(s-1, v+a+n, \mathbf{u}) - \zeta(s-1, v+a+n+1, \mathbf{u}) \right]}_{-\Delta \zeta(s-1, v+a+n, \mathbf{u})} = \\ &= \sum_{n=0}^{m-1} \zeta(s-1, v+a+n+1, \Delta \mathbf{u}) + u_0 \sum_{n=0}^{m-1} (v+a+n)^{1-s}, \end{aligned} \tag{72}$$

we arrive at

$$\begin{aligned} \zeta(s, v, \mathbf{u}) &= \frac{1}{m(s-1)} \sum_{n=0}^{m-1} \left\{ \zeta(s-1, v+a+n+1, \Delta \mathbf{u}) + u_0 (v+a+n)^{1-s} \right\} - \\ &\quad - \frac{1}{m} \sum_{n=1}^{\infty} (-1)^n N_{n,m}(a) \zeta(s, v+n, \Delta^n \mathbf{u}) - \\ &\quad - \frac{1}{m} \sum_{n=1}^{\infty} N_{n,m}(a) \sum_{l=0}^{n-1} (-1)^{l+1} \cdot \Delta^l u_0 \cdot \Delta^{n-1-l}(v+l)^{-s}, \end{aligned} \tag{73}$$

which is a generalization of (42) to the Dirichlet series  $\zeta(s, v, \mathbf{u})$ . □

Now, from Eq. (72) we may also proceed in a slightly different manner. Applying operator  $\Delta^{n-1}$  to (71) gives

$$\Delta^n \zeta(s, v, \mathbf{u}) = -\Delta^{n-1} \zeta(s, v+1, \Delta \mathbf{u}) - u_0 \Delta^{n-1} v^{-s}.$$

Therefore, (72) becomes

$$\begin{aligned} \zeta(s, x+v, \mathbf{u}) &= \zeta(s, v, \mathbf{u}) - \sum_{n=1}^{\infty} \frac{(x-n+1)_n}{n!} \Delta^{n-1} \zeta(s, v+1, \Delta \mathbf{u}) - \\ &\quad - u_0 \sum_{n=1}^{\infty} \frac{(x-n+1)_n}{n!} \Delta^{n-1} v^{-s}. \end{aligned} \tag{74}$$

Integrating termwise over  $x \in [a, a+m]$  and accounting for (72) yields

$$\begin{aligned} \zeta(s, v, \mathbf{u}) &= \frac{1}{m(s-1)} \sum_{n=0}^{m-1} \left\{ \zeta(s-1, v+a+n+1, \Delta \mathbf{u}) + u_0 (v+a+n)^{1-s} \right\} + \\ &\quad + \frac{1}{m} \sum_{n=1}^{\infty} N_{n,m}(a) \Delta^{n-1} \zeta(s, v+1, \Delta \mathbf{u}) + \frac{u_0}{m} \sum_{n=1}^{\infty} N_{n,m}(a) \Delta^{n-1} v^{-s}, \end{aligned} \tag{75}$$

which, unlike (73), directly relates  $\zeta(\dots, \dots, \mathbf{u})$  to  $\zeta(\dots, \dots, \Delta \mathbf{u})$ .  $\square$

Although formulæ (73), (75) may seem quite theoretical, they both have a multitude of interesting applications and consequences for the concrete  $L$ -functions. For example, if  $u_n$  is a polynomial of degree  $k - 1$  in  $n$ , the  $k$ -th finite difference  $\Delta^k \mathbf{u}$  vanishes and so do higher differences. Hence  $\zeta(s, v + n, \Delta^n \mathbf{u}) = 0$  for  $n \geq k$  and the right part of the previous equation becomes much simpler. On the other hand, if  $u_n$  is a polynomial in  $n$ , the considered  $L$ -function may always be written as a linear combination of the  $\zeta$ -functions, so that we have a relation between the  $\zeta$ -functions of arguments  $s, s - 1, s - 2, \dots$ . We come to discuss some of such examples in the next Corollary.

**Corollary 6.** *The following formulæ relating the  $\zeta$ -functions of different arguments and types hold*

$$\begin{aligned} (v + a + \frac{1}{2}m - 1) \cdot \zeta(s, v) &= -\frac{\zeta(s - 1, v + a)}{s - 1} + \zeta(s - 1, v) + \\ &+ \frac{1}{m(s - 1)} \sum_{n=0}^{m-1} \frac{m - n - 1}{(v + a + n)^{s-1}} + \\ &+ \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+2, m}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k + v)^{-s}, \end{aligned} \tag{76}$$

$$\begin{aligned} (v + a - \frac{1}{2}) \cdot \zeta(s, v) &= -\frac{\zeta(s - 1, v + a)}{s - 1} + \zeta(s - 1, v) + \\ &+ \sum_{n=0}^{\infty} (-1)^n \psi_{n+2}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k + v)^{-s}, \end{aligned} \tag{77}$$

$$\begin{aligned} (v + a) \cdot \zeta(s, v) &= \frac{(v + a)^{1-s}}{2(s - 1)} - \frac{\zeta(s - 1, v + a)}{s - 1} + \zeta(s - 1, v) + \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n N_{n+2, 2}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k + v)^{-s}, \end{aligned} \tag{78}$$

$$(v - \frac{1}{2}) \cdot \zeta(s, v) = \frac{s - 2}{s - 1} \cdot \zeta(s - 1, v) + \sum_{n=0}^{\infty} (-1)^n G_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k + v)^{-s}, \tag{79}$$

$$\frac{m}{2} \cdot \zeta(s) = \frac{s - 2}{s - 1} \cdot \zeta(s - 1) + \frac{mH_m^{(s-1)} - H_m^{(s-2)}}{m(s - 1)} + \tag{80}$$

$$\begin{aligned}
 & + \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+2,m}(0) \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s}, \\
 \zeta(s) & = \frac{2(s-2)}{s-1} \cdot \zeta(s-1) + 2 \sum_{n=0}^{\infty} (-1)^n G_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s}, \quad (81)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\zeta(s-1, a)}{s-1} & = \frac{a^{1-s}}{s-1} - (a + \frac{1}{2}m) \cdot \zeta(s) + \zeta(s-1) + \\
 & + \frac{1}{m(s-1)} \sum_{n=1}^m \frac{m-n}{(a+n)^{s-1}} + \quad (82) \\
 & + \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n N_{n+2,m}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} (k+1)^{-s},
 \end{aligned}$$

where  $m$  is a natural number and  $H_m^{(s)}$  is the generalized harmonic number (see p. 36).

*Proof.* Let  $u_n = P_1(n) = \alpha + \beta n$ , where  $\alpha$  and  $\beta$  are some coefficients. Then,

$$u_0 = \alpha, \quad \Delta u_n = \beta, \quad \Delta^2 u_n = 0, \dots, \quad \Delta^k u_n = 0,$$

and hence  $\zeta(s, v, \Delta \mathbf{u}) = \beta \zeta(s, v)$ . On the other hand, from a simple arithmetic argument it also follows that

$$\zeta(s, v, \mathbf{u}) = (\alpha - \beta v) \zeta(s, v) + \beta \zeta(s-1, v).$$

Using the recurrence relation of the Hurwitz  $\zeta$ -function and the fact that

$$\Delta^{n-1} \zeta(s, v+1) = -\Delta^{n-2} v^{-s} - \Delta^{n-1} v^{-s}, \quad (83)$$

as well as recalling that  $N_{1,m}(a) = ma + \frac{1}{2}m^2$  (see (58)) and setting, for simplicity,  $\alpha = 1$  and  $\beta = 1$ ,<sup>29</sup> formula (75) becomes

$$\begin{aligned}
 (1 - v - a - \frac{1}{2}m) \zeta(s, v) & = \frac{1}{m(s-1)} \sum_{n=0}^{m-1} \left\{ \zeta(s-1, v+a+n+1) + \right. \\
 & \left. + (v+a+n)^{1-s} \right\} - \zeta(s-1, v) - \frac{1}{m} \sum_{n=0}^{\infty} N_{n+2,m}(a) \Delta^n v^{-s}. \quad (84)
 \end{aligned}$$

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<sup>29</sup>It is possible to perform these calculations with arbitrary  $\alpha$  and  $\beta$ , but the resulting expressions become very cumbersome.

By recursively decreasing the second argument in  $\zeta(s-1, v+a+n+1)$ , we obtain the following functional relationship

$$(1 - v - a - \frac{1}{2}m)\zeta(s, v) = \frac{\zeta(s-1, v+a)}{s-1} - \zeta(s-1, v) - \frac{1}{m(s-1)} \sum_{n=0}^{m-1} \frac{m-n-1}{(v+a+n)^{s-1}} - \frac{1}{m} \sum_{n=0}^{\infty} N_{n+2,m}(a) \Delta^n v^{-s}, \tag{85}$$

where  $m \in \mathbb{N}$  and the first sum in the second line should be taken as zero for  $m = 1$ . This is our formula (76) and it has many interesting particular cases. For instance, for  $m = 1$  and  $m = 2$  we obtain (77) and (78) respectively, since  $N_{n+2,1}(a) = \psi_{n+2}(a)$ . Furthermore, making  $a = 0$  we get (79), because  $\psi_n(0) = G_n$ . Setting  $v = 1$  and  $a \in \mathbb{N}_0$  gives us corresponding expressions for  $\zeta(s)$ . For example, at  $v = 1$  and  $a = 0$  relationship (85) becomes

$$\frac{m}{2}\zeta(s) = \frac{s-2}{s-1}\zeta(s-1) + \frac{1}{m(s-1)} \sum_{n=1}^m \frac{m-n}{n^{s-1}} + \frac{1}{m} \sum_{n=0}^{\infty} N_{n+2,m}(0) \Delta^n 1^{-s}, \tag{86}$$

where the finite sum in the middle may be also written in terms of the generalized harmonic numbers; we, thus, arrive at (80). Furthermore, putting  $m = 1$  we obtain a strikingly simple functional relationship with Gregory's coefficients, equation (81). From (85) we may also obtain a relationship between  $\zeta(s)$  and  $\zeta(s, a)$ . Putting  $v = 1$  gives us relationship (82). Finally, proceeding similarly with  $u_n = P_2(n)$  and using

$$\Delta^n \zeta(s, v, \mathbf{u}) = \Delta^{n-2} \zeta(s, v+2, \Delta^2 \mathbf{u}) + \Delta u_0 \Delta^{n-2} (v+1)^{-s} - u_0 \Delta^{n-1} v^{-s},$$

we may obtain formulæ relating  $\zeta$ -functions of arguments  $s, s-1$  and  $s-2$ . The same procedure may be applied to  $u_n = P_3(n)$  and so on. □

**Corollary 7.** *The generalized Stieltjes constants  $\gamma_m(v)$  may be given by the following series representations*

$$\gamma_m(v) = -\frac{\ln^{m+1} v}{m+1} + \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^m(k+v)}{k+v}, \tag{87}$$

where  $\operatorname{Re} v > 0$ ,

$$\gamma_m(v) = -\frac{\ln^{m+1}(v-1)}{m+1} - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^m(k+v)}{k+v}, \tag{88}$$

where  $\operatorname{Re} v > 1$ ,

$$\begin{aligned} \gamma_m(v) = & -\frac{1}{r(m+1)} \sum_{l=0}^{r-1} \ln^{m+1}(v+a+l) + \\ & + \frac{1}{r} \sum_{n=0}^{\infty} (-1)^n N_{n+1,r}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^m(k+v)}{k+v}, \end{aligned} \tag{89}$$

where  $r \in \mathbb{N}$ ,  $\operatorname{Re} a > -1$  and  $\operatorname{Re} v > -\operatorname{Re} a$ , and

$$\begin{aligned} \gamma_m(v) = & \frac{1}{1-v-a-\frac{1}{2}r} \left\{ \frac{(-1)^m}{m+1} \zeta^{(m+1)}(0, v+a) - (-1)^m \zeta^{(m)}(0, v) + \right. \\ & + \frac{1}{r(m+1)} \sum_{n=0}^{r-2} (r-n-1) \ln^{m+1}(v+a+n) - \\ & \left. - \frac{1}{r} \sum_{n=0}^{\infty} (-1)^n N_{n+2,r}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^m(k+v)}{k+v} \right\} \end{aligned} \tag{90}$$

under the same conditions.

*Proof.* The generalized Stieltjes constants  $\gamma_m(v)$ ,  $m \in \mathbb{N}_0$ ,  $v \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ , are introduced analogously to the ordinary Stieltjes constants

$$\zeta(s, v) = \frac{1}{s-1} - \Psi(v) + \sum_{m=1}^{\infty} \frac{(-1)^m \gamma_m(v)}{m!} (s-1)^m, \quad s \in \mathbb{C} \setminus \{1\}, \tag{91}$$

with  $\gamma_0(v) = -\Psi(v)$ , see, e.g., [8, p. 541, Eq. (14)].<sup>30</sup> Thus, from expansions (26), (27), (28) and (85), by proceeding in the same manner as in Corollary 1, we deduce the announced series representations.<sup>31</sup> Notice also that the above formulæ may be rewritten in a slightly different way by means of the recurrence relation for the generalized Stieltjes constants  $\gamma_m(v+1) = \gamma_m(v) - v^{-1} \ln^m v$ .  $\square$

**Remark 2, related to the digamma function (the  $\Psi$ -function) and to Euler’s constant  $\gamma$ .** Since

$$(-1)^n \Delta^n v^{-1} = \frac{n!}{(v)_{n+1}} \tag{92}$$

we have for the zeroth Stieltjes constant, and hence for the digamma function  $\Psi(v)$ , the following expansions

$$\Psi(v) = \ln v - \sum_{n=1}^{\infty} \frac{|G_n| (n-1)!}{(v)_n}, \quad \operatorname{Re} v > 0, \tag{93}$$

$$\Psi(v) = \ln(v-1) + \sum_{n=1}^{\infty} \frac{C_n (n-1)!}{(v)_n}, \quad \operatorname{Re} v > 1, \tag{94}$$

$$\Psi(v) = \ln(v+a) + \sum_{n=1}^{\infty} \frac{(-1)^n \psi_n(a) (n-1)!}{(v)_n}, \tag{95}$$

<sup>30</sup>For more information on  $\gamma_m(v)$ , see [7], [8], and the literature given in the last reference. Note that since  $\zeta(s, 1) = \zeta(s)$ , the generalized Stieltjes constants  $\gamma_m(1) = \gamma_m$ .

<sup>31</sup>It may also be noted that the particular case  $m = 1$  of (87) was earlier given by Coffey [22, p. 2052, Eq. (1.18)].

$$\Psi(v) = \frac{1}{r} \sum_{l=0}^{r-1} \ln(v+a+l) + \frac{1}{r} \sum_{n=1}^{\infty} \frac{(-1)^n N_{n,r}(a)(n-1)!}{(v)_n}, \tag{96}$$

$$\Psi(v) = \frac{1}{\frac{1}{2}r + v + a - 1} \left\{ \ln \Gamma(v+a) + v - \frac{1}{2} \ln 2\pi - \frac{1}{2} + \right. \\ \left. + \frac{1}{r} \sum_{n=0}^{r-2} (r-n-1) \ln(v+a+n) + \frac{1}{r} \sum_{n=1}^{\infty} \frac{(-1)^n N_{n+1,r}(a)(n-1)!}{(v)_n} \right\}, \tag{97}$$

with  $r \in \mathbb{N}$ ,  $\operatorname{Re} a > -1$  and  $\operatorname{Re} v > -a$ , because of (54).<sup>32</sup> First two representations coincide with the not well-known Binet–Nørlund expansions for the digamma function [10, pp. 428–429, Eqs. (91)–(94)],<sup>33</sup> while (95)–(97) seem to be new. From (93) and (94), it also immediately follows that  $\ln(v-1) < \Psi(v) < \ln v$  for  $v > 1$ , since the sums with  $G_n$  and  $C_n$  keep their sign.<sup>34</sup> We may also obtain series expansions for the digamma function from (62), but the resulting expressions strongly depend on  $k$ . For instance, putting  $k = 2$  and expanding both sides into the Laurent series (91), we obtain the following formula

$$\Psi(v) = 2 \ln \Gamma(v) - 2v \ln v + 2v + 2 \ln v - \ln 2\pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^n G_n^{(2)}(n-1)!}{(v)_n}, \tag{98}$$

which relates the  $\Gamma$ -function to its logarithmic derivative.<sup>35</sup> For higher  $k$  these expressions become quite cumbersome and also imply the derivatives of the  $\zeta$ -function at negative integers. In particular, for  $k = 3$ , we deduce

$$\Psi(v) = 3 \ln \Gamma(v) - 6\zeta'(-1, v) + 3v^2 \ln v - \frac{3}{2}v^2 - 6v \ln(v) + \tag{99} \\ + 3v + 3 \ln v - \frac{3}{2} \ln 2\pi + \frac{1}{2} + 3 \sum_{n=1}^{\infty} \frac{(-1)^n G_n^{(3)}(n-1)!}{(v)_n},$$

provided the convergence of the last series. Formula (98) is also interesting in that it gives series with rational terms for  $\ln \Gamma(v)$  if  $v \in \mathbb{Q}$  (we only need to use Gauss’ digamma theorem for this [8, p. 584, Eq. (B.4)]). Note also that all series (93)–(99) converge very rapidly for large  $v$ . For instance, putting in (96) or in (97)  $v = 2\pi$ ,

<sup>32</sup>In the last formula the sum in the middle should be taken as zero for  $r = 1$ . Note also that  $\zeta(0, x) = \frac{1}{2} - x$  and  $\zeta'(0, x) = \ln \Gamma(x) - \frac{1}{2} \ln 2\pi$ . Similarly to (95) formulæ (89)–(90) and (97) may also be written in terms of  $\psi_n(a)$  instead of  $N_{n,r}(a)$  when  $r = 1$ , see (54).

<sup>33</sup>Formula (94) reduces to [10, p. 429, Eq. (94), first formula] by putting  $v$  instead of  $v - 1$  and by making use of the recurrence relationship for the digamma function  $\Psi(v+1) = \Psi(v) + v^{-1}$ .

<sup>34</sup>This simple and important result is not new, but its derivation from (93) and (94) seems to be novel, and in addition, is elementary.

<sup>35</sup>There were many attempts aiming to find possible relationships between these two functions. For instance, in 1842 Carl Malmsten, by trying to find such a relationship, obtained a variant of Gauss’ theorem for  $\Psi(v)$  at  $v \in \mathbb{Q}$ , see [7, p. 37, Eq. (23)], [8, p. 584, Eq. (B.4)].

$a = 3$ ,  $m = 2$ , and taking only 10 terms in the last sum, we get the value of  $\Psi(2\pi)$  with 9 correct digits.

Putting in the previous formulæ for the digamma function argument  $v \in \mathbb{Q}$ , we may also obtain series with rational terms for Euler’s constant. The most simple is to put  $v = 1$ . In this case, formula (93) reduces to the famous Fontana–Mascheroni series

$$\Psi(1) = -\gamma = -\sum_{n=1}^{\infty} \frac{|G_n|}{n}, \tag{100}$$

see, e.g., [73, p. 207], [8, p. 539], [10, pp. 406, 413,429–430], [9, p. 379], while (96) gives us

$$\gamma = -\frac{1}{m} \sum_{l=1}^m \ln(a+l) - \frac{1}{m} \sum_{n=1}^{\infty} \frac{(-1)^n N_{n,m}(a)}{n}, \quad \begin{matrix} m \in \mathbb{N} \\ a > -1 \end{matrix}. \tag{101}$$

This series generalizes (100) to a large family of series (we have the aforementioned series at  $a = 0$  since  $N_{n,1}(0) = \psi_n(0) = G_n$ ). For example, setting  $a = -\frac{1}{2}$  and  $m = 1$  (the mean value between  $a = -1$  corresponding to the coefficients  $C_n$  and  $a = 0$  corresponding to  $G_n$ ), we have, by virtue of (54), the following series

$$\begin{aligned} \gamma &= \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n N_{n,1}(-\frac{1}{2})}{n} = \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n \psi_n(-\frac{1}{2})}{n} = \\ &= \ln 2 - 0 - \frac{1}{48} - \frac{1}{72} - \frac{223}{23\,040} - \frac{103}{14\,400} - \frac{32\,119}{5\,806\,080} - \frac{1111}{250\,880} - \dots \end{aligned} \tag{102}$$

relating two fundamental constants  $\gamma$  and  $\ln 2$ . This series, however, converges quite slowly, as  $\sum n^{-3/2} \ln^{-1} n$  by virtue of (61). A more rapidly convergent series may be obtained by setting large integer  $a$ . At the same time, it should be noted that precisely for large  $a$ , first terms of the series may unexpectedly grow, but after some term they decrease and the series converges. For instance, taking  $a = 7$ , we have

$$\begin{aligned} \gamma &= -3 \ln 2 - \sum_{n=1}^{\infty} \frac{(-1)^n \psi_n(7)}{n} = -3 \ln 2 + \frac{15}{2} - \frac{293}{24} + \\ &\quad + \frac{1079}{72} - \dots - \frac{8183}{9\,331\,200} - \frac{530\,113}{4\,790\,016\,000} - \dots \end{aligned} \tag{103}$$

Also, the pattern of the sign is not obvious, but for the large index  $n$  all the terms should be negative. By the way, adding the series with  $a = -\frac{1}{2}$  to that with  $a = 1$ , we eliminate  $\ln 2$  and thus get a series with rational terms only for Euler’s constant

$$\begin{aligned} \gamma &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \left\{ \psi_n(-\frac{1}{2}) + \psi_n(1) \right\} = \frac{3}{4} - \frac{11}{96} - \frac{1}{72} - \frac{311}{46\,080} - \\ &\quad - \frac{5}{1152} - \frac{7291}{2\,322\,432} - \frac{243}{100\,352} - \frac{14\,462\,317}{7\,431\,782\,400} - \dots \end{aligned} \tag{104}$$

converging at the same rate as  $\sum n^{-3/2} \ln^{-1} n$ . Other choices of  $a$  are also possible in order to get series with rational terms only for  $\gamma$ .<sup>36</sup> In fact, it is not difficult to show that we can eliminate the logarithm by properly choosing  $a$ , namely

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \left\{ \psi_n(a) + \psi_n\left(-\frac{a}{1+a}\right) \right\}, \quad a > -1, \quad (105)$$

which, at  $a \in \mathbb{Q}$ , represents a huge family of series with rational terms only for Euler’s constant. This series converges at the same rate as  $\sum n^{-a-2} \ln^{-1} n$  for  $-1 < a \leq 0$  and  $\sum n^{-\frac{a+2}{a+1}} \ln^{-1} n$  for  $a \geq 0$ , except for the integer values of  $a$  for which  $\ln^{-1} n$  should be replaced by  $\ln^{-2} n$ . In other words, the rate of convergence of (105) cannot be worse than  $\sum n^{-1-\varepsilon} \ln^{-1} n$ , where  $\varepsilon$  is a positive however small parameter, and better than  $\sum n^{-2} \ln^{-2} n$ . More generally, from (101) it follows that if  $a_1, \dots, a_k$  and  $m$  are chosen so that  $(1 + a_1)_m \cdots (1 + a_k)_m = 1$ , then

$$\gamma = \frac{1}{m^k} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{l=1}^k N_{n,m}(a_l), \quad a_1, \dots, a_k > -1. \quad (106)$$

Furthermore, if for some  $q_1, \dots, q_k$  and  $m$ , the quantities  $a_1, \dots, a_k$  are chosen so that  $(1 + a_1)_{\frac{q_1}{m}} \cdots (1 + a_k)_{\frac{q_k}{m}} = 1$ , then we have a more general formula

$$\gamma = \frac{1}{m(q_1 + \dots + q_k)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{l=1}^k q_l N_{n,m}(a_l), \quad a_1, \dots, a_k > -1, \quad (107)$$

which is the most complete generalization of the Fontana–Mascheroni series (100).

Analogously, one can obtain the series expansions for  $\gamma$  from (97). Indeed, putting for simplicity  $v = 1$  in the latter expression yields

$$\gamma = -\frac{2}{m+2a} \left\{ \ln \Gamma(a+1) - \frac{1}{2} \ln 2\pi + \frac{1}{2} + \frac{1}{m} \sum_{n=1}^{m-1} (m-n) \ln(a+n) + \frac{1}{m} \sum_{n=1}^{\infty} \frac{(-1)^n N_{n+1,m}(a)}{n} \right\}, \quad (108)$$

where  $m \in \mathbb{N}$  and  $\operatorname{Re} a > -1$ . For  $m = 1$  the sum in the middle should be taken as zero, so that

$$\gamma = -\frac{2}{1+2a} \left\{ \ln \Gamma(a+1) - \frac{1}{2} \ln 2\pi + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n \psi_{n+1}(a)}{n} \right\}, \quad \operatorname{Re} a > -1. \quad (109)$$

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<sup>36</sup>For a list of the most known series with the rational terms only for Euler’s constant, see, e.g., [9, p. 379]. From the historical viewpoint it may also be interesting to note that series (36) from [9, p. 380], in its second form, was also given by F. Franklin already in 1883 in a paper read at a meeting of the University Mathematical Society [32]. Moreover, this series, of course, may be even much older since it is obtained by a quite elementary technique.

At integer and demi-integer values of  $a$ , we have a quite simple expression for Euler's constant  $\gamma$ , which does not contain the  $\Gamma$ -function; conversely, it may also be regarded as yet another series for  $\ln \Gamma(z)$ . Another interesting consequence of this formula is that it readily permits to obtain an interesting series for the digamma function. By multiplying both sides of this formula by  $m + 2a$  and by calculating the derivative of the resulting expression with respect to  $a$ , see (47), we obtain

$$\Psi(a + 1) = -\gamma - \frac{1}{m} \sum_{n=1}^{m-1} \frac{m-n}{a+n} - \frac{1}{m} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left\{ \binom{a+m}{n+1} - \binom{a}{n+1} \right\}, \tag{110}$$

where  $m = 2, 3, 4, \dots$  and  $\operatorname{Re} a > -1$ . At  $m = 1$ , we directly get the result from (109)

$$\Psi(a + 1) = -\gamma - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{d\psi_{n+1}(a)}{da} = -\gamma - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \binom{a}{n}, \quad \operatorname{Re} a > -1, \tag{111}$$

the series which is sometimes attributed to Stern [57, p. 251].

It is also possible to deduce series expansions with rational terms for Euler's constant from series (98). Putting  $v = 1$ , we obtain the following series

$$\begin{aligned} \gamma = \ln 2\pi - 2 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n G_n^{(2)}}{n} &= \ln 2\pi - 2 + \frac{2}{3} + \\ &+ \frac{1}{24} + \frac{7}{540} + \frac{17}{2880} + \frac{41}{12\,600} + \frac{731}{362\,880} + \dots \end{aligned} \tag{112}$$

converging at the same rate as  $\sum n^{-2} \ln^{-3} n$  (see Nota Bene on p. 20). It is interesting that the numerators of this series, except in its first term, coincide with those of the third row of the inverse Akiyama-Tanigawa algorithm from  $1/n$  (see OEIS A193546) while its denominator does not seem to be known to the OEIS. In fact, the reader may easily verify that all these series are new, and at the moment of writing of this paper were not known to the OEIS [except (112)].

**Corollary 8.** *The generalized Maclaurin coefficients  $\delta_m(v)$  of the regular function  $\zeta(s, v) - (s - 1)^{-1}$  admit the following series representations*

$$\delta_m(v) = f_m(v) + \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^m(k + v), \tag{113}$$

$$\delta_m(v) = f_m(v - 1) - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^m(k + v), \tag{114}$$

$$\delta_m(v) = \frac{1}{r} \sum_{l=0}^{r-1} f_m(v + a + l) + \frac{1}{r} \sum_{n=0}^{\infty} (-1)^n N_{n+1,r}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^m(k + v), \tag{115}$$

where we let

$$f_m(v) = (-1)^m m! \left\{ 1 - v - v \sum_{k=1}^m (-1)^k \frac{\ln^k v}{k!} \right\}$$

for brevity.

*Proof.* Generalizing expansion (17) to the Hurwitz  $\zeta$ -function, we may introduce  $\delta_m(v)$ ,  $m \in \mathbb{N}$ ,  $v \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ , as the coefficients in the expansion

$$\zeta(s, v) = \frac{1}{s-1} + \frac{3}{2} - v + \sum_{m=1}^{\infty} \frac{(-1)^m \delta_m(v)}{m!} s^m, \quad s \in \mathbb{C} \setminus \{1\}. \quad (116)$$

It is, therefore, not difficult to see that  $\delta_m(v) = (-1)^m \{ \zeta^{(m)}(0, v) + m! \}$ . The desired formulæ are obtained by a direct differentiation of (26)–(28) respectively. Similarly to the generalized Stieltjes constants, functions  $\delta_m(v)$  enjoy a recurrence relation  $\delta_m(v+1) = \delta_m(v) - \ln^m v$ , which may be used to rewrite (113)–(114) in a slightly different form if necessary.  $\square$

**Remark 3, related to the logarithm of the  $\Gamma$ -function.** Recalling that  $\delta_1(v) = -\ln \Gamma(v) + \frac{1}{2} \ln 2\pi - 1$  and noticing that  $f_1(v) = v - 1 - v \ln v$ , gives us these three series expansions

$$\ln \Gamma(v) = v \ln v - v + \frac{1}{2} \ln 2\pi - \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v), \quad (117)$$

where  $\operatorname{Re} v > 0$ ,

$$\ln \Gamma(v) = (v-1) \ln(v-1) - v + 1 + \frac{1}{2} \ln 2\pi + \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v), \quad (118)$$

where  $\operatorname{Re} v > 1$ , and

$$\begin{aligned} \ln \Gamma(v) = & \frac{1}{r} \sum_{l=0}^{r-1} (v+a+l) \ln(v+a+l) - v - a - \frac{r}{2} + \frac{1}{2} \ln 2\pi + \\ & + \frac{1}{2} - \frac{1}{r} \sum_{n=0}^{\infty} (-1)^n N_{n+1,r}(a) \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v), \end{aligned} \quad (119)$$

$\operatorname{Re} v > -\operatorname{Re} a$ ,  $\operatorname{Re} a > -1$ ,  $r \in \mathbb{N}$  for the logarithm of the  $\Gamma$ -function. First of these representations is equivalent to a little-known formula for the logarithm of the  $\Gamma$ -function, which appears in epistolary exchanges between Charles Hermite and Salvatore Pincherle dating back to 1900 [38, p. 63, two last formulæ], [59, vol. IV, p. 535, third and fourth formulæ], while the second and the third representations seem to be novel.

Note also that the parameter  $a$ , in all the expansions in which it appears, plays the role of the “rate of convergence”: the greater this parameter, the faster the convergence, especially if  $a$  is an integer.

**Remark 4, related to some similar expansions containing the same finite differences.** It seems appropriate to note that there exist other expansions of the same nature, which merit being mentioned here. For instance

$$\ln \Gamma(v+x) = \ln \Gamma(v) + \sum_{n=0}^{\infty} (-1)^n \binom{x}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v), \tag{120}$$

$$\ln \Gamma(v) = -v + \frac{1}{2} + \frac{1}{2} \ln 2\pi + \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v) \ln(k+v), \tag{121}$$

$$\Psi(v) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v), \tag{122}$$

$$\gamma_m(v) = -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^{m+1}(k+v), \quad m \in \mathbb{N}; \tag{123}$$

see, e.g., [38, p. 59], [59, vol. IV, p. 531], [57, p. 251], [26], [25]. The three latter formulæ are usually deduced from Hasse’s series (2), but they equally may be obtained from a more general formula

$$\frac{dF(v)}{dv} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^{n+1} F(v) = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} F(v+k), \tag{124}$$

which is a well-known result in the theory of finite differences, see, e.g., [57, pp. 240–242]. Moreover, Hasse’s series itself (2) is a simple consequence of this formula. Putting  $F(v) = \zeta(s, v)$ , we actually have

$$\Delta \zeta(s, v) = \zeta(s, v+1) - \zeta(s, v) = -v^{-s} \quad \text{and} \quad \frac{\partial \zeta(s, v)}{\partial v} = -s \zeta(s+1, v). \tag{125}$$

Hence, (124) reads

$$s \zeta(s+1, v) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^n v^{-s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (v+k)^{-s}. \tag{126}$$

Rewriting this expression for  $s-1$  instead of  $s$  and then dividing both sides by  $s-1$  immediately yield Hasse’s series (2).

It is also evident that expressions containing the fractions  $\frac{1}{n+1}$  are more simple than those with the coefficients  $G_n, C_n$  or  $N_{n,m}(a)$ . Applying (124) to the Dirichlet

series  $\zeta(s, v, \mathbf{u})$  introduced in Theorem 5 and proceeding analogously, we may obtain without difficulty the following result

$$\zeta(s, v, \mathbf{u}) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^n \zeta(s-1, v+1, \Delta \mathbf{u}) + \frac{u_0}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^n v^{1-s} \tag{127}$$

which is the analog of (75). Considering again  $\zeta(s, v, \mathbf{u})$  with  $u_n = 1 + n$ , we easily derive the following functional relationship

$$(v-1)\zeta(s, v) = \frac{s-2}{s-1} \zeta(s-1, v) - \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} \Delta^n v^{1-s}, \quad s \in \mathbb{C} \setminus \{1, 2\}, \tag{128}$$

or explicitly

$$\zeta(s, v) = \frac{1}{(v-1)(s-1)} \left\{ (s-2)\zeta(s-1, v) - \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{1-s} \right\}, \tag{129}$$

$s \in \mathbb{C} \setminus \{1, 2\}$ , which is, in some senses, analogous to (79) and which is yet another generalization of Ser’s formula (4).<sup>37</sup> Another functional relationship for  $\zeta(s)$  may be obtained by setting  $v = \frac{1}{2}$ , because  $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$  [this obviously also applies to other formulæ we obtained for  $\zeta(s, v)$ ]. Now, multiply both sides of the latter equality by  $s-1$  and expand it into the Taylor series about  $(s-1)$ .<sup>38</sup> Equating the coefficients of  $(s-1)^1$  yields the following relationship between the gamma and the digamma functions

$$\Psi(v) = \frac{1}{v-1} \left\{ \ln \Gamma(v) + v - \frac{1}{2} \ln 2\pi - \frac{1}{2} - \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v) \right\}. \tag{130}$$

Similarly, by equating the coefficients of  $(s-1)^{m+1}$  we have for the generalized Stieltjes constants

$$\begin{aligned} \gamma_m(v) = \frac{1}{v-1} & \left\{ (-1)^m \zeta^{(m)}(0, v) - \frac{(-1)^m}{m+1} \zeta^{(m+1)}(0, v) + \right. \\ & \left. + \frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^{m+1}(k+v) \right\}, \end{aligned} \tag{131}$$

where  $m \in \mathbb{N}$ ,  $\text{Re } v > 0$  and where the right part should be regarded as a limit when  $v = 1$ .<sup>39</sup>

<sup>37</sup>We precisely obtain the latter as a particular case of (128) when  $v = 1$ .

<sup>38</sup>The  $\zeta$ -functions should be expanded into the Laurent series accordingly to (91).

<sup>39</sup>It is interesting to compare this formula with (90).

Furthermore, proceeding similarly and using formula (67) from [57, p. 241], one may obtain the following expansions

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+2)^{1-s}, \tag{132}$$

$$\zeta(s, v-1) = \frac{1}{s-1} \sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{1-s}, \tag{133}$$

$$\ln \Gamma(v-1) = -v + \frac{3}{2} + \frac{1}{2} \ln 2\pi + \sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v) \ln(k+v), \tag{134}$$

$$\Psi(v-1) = \sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v), \tag{135}$$

$$\gamma = -\sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+2), \tag{136}$$

$$\gamma_m = -\frac{1}{m+1} \sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^{m+1}(k+2), \tag{137}$$

$$\gamma_m(v-1) = -\frac{1}{m+1} \sum_{n=0}^{\infty} H_{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^{m+1}(k+v), \tag{138}$$

where  $\operatorname{Re} v > 1$ ,  $m$  is a natural number and  $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$  is the  $n$ th harmonic number. Using the corresponding recurrence relations, these formulæ may also be written for  $v$  instead of  $v-1$  in the left part. Besides, for the Dirichlet series  $\zeta(s, v, \mathbf{u})$  from Theorem 5, we also have

$$\begin{aligned} \zeta(s, v-1, \mathbf{u}) &= \frac{1}{s-1} \sum_{n=0}^{\infty} (-1)^n H_{n+1} \Delta^n \zeta(s-1, v+1, \Delta \mathbf{u}) + \\ &\quad + \frac{u_0}{s-1} \sum_{n=0}^{\infty} (-1)^n H_{n+1} \Delta^n v^{1-s}. \end{aligned}$$

If, for example,  $u_n = n+1$ , then

$$\begin{aligned} \zeta(s, v-1) &= \frac{1}{(v-2)(s-1)} \left\{ (v-1)^{1-s} + (s-2) \zeta(s-1, v-1) - \right. \\ &\quad \left. - \sum_{n=0}^{\infty} H_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+v)^{1-s} \right\}, \end{aligned} \tag{139}$$

$s \in \mathbb{C} \setminus \{1, 2\}$ ,  $\operatorname{Re} v > 1$ . By expanding both sides of this formula into the Laurent series about  $s = 1$ , we get

$$\Psi(v-1) = \frac{1}{v-2} \left\{ \ln \Gamma(v) + v - \frac{3}{2} - \frac{1}{2} \ln 2\pi - \sum_{n=0}^{\infty} H_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(k+v) \right\}, \tag{140}$$

$$\begin{aligned} \gamma_m(v-1) = \frac{1}{v-2} \left\{ -\frac{\ln^{m+1}(v-1)}{m+1} + (-1)^m \zeta^{(m)}(0, v-1) - \right. \\ \left. - \frac{(-1)^m}{m+1} \zeta^{(m+1)}(0, v-1) + \frac{1}{m+1} \sum_{n=0}^{\infty} H_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \ln^{m+1}(k+v) \right\} \end{aligned} \tag{141}$$

holding for  $\operatorname{Re} v > 1$ . If, in addition, we set  $v = 2$  in (139), we get

$$\zeta(s) = \frac{1}{s-1} \left\{ -1 + \sum_{n=0}^{\infty} H_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} (k+2)^{-s} \right\}, \quad s \in \mathbb{C} \setminus \{1\}, \tag{142}$$

and hence

$$\gamma = -\sum_{n=0}^{\infty} H_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln(k+2)}{k+2}, \tag{143}$$

$$\gamma_m = -\frac{1}{m+1} \sum_{n=0}^{\infty} H_{n+2} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\ln^{m+1}(k+2)}{k+2}, \quad m \in \mathbb{N}. \tag{144}$$

Finally, differentiating (130) with respect to  $v$  and simplifying the sum with the binomial coefficients yields a formula for the trigamma function  $\Psi_1(v)$

$$\Psi_1(v) = \frac{1}{v-1} \left\{ 1 - \sum_{n=0}^{\infty} \frac{n!}{(n+2) \cdot (v)_{n+1}} \right\}, \quad \operatorname{Re} v > 0, \tag{145}$$

where for  $v = 1$  the right part should be calculated via an appropriated limiting procedure. Similarly, from (135) we get

$$\Psi_1(v-1) = \sum_{n=0}^{\infty} \frac{H_{n+1} n!}{(v)_{n+1}}, \quad \operatorname{Re} v > 1. \tag{146}$$

which may also be written as

$$\Psi_1(v) = v \sum_{n=0}^{\infty} \frac{H_{n+1} n!}{(v)_{n+2}}, \quad \operatorname{Re} v > 0, \tag{147}$$

if we put  $v + 1$  instead of  $v$ . These formulæ may be compared with

$$\Psi_1(v) = \sum_{n=0}^{\infty} \frac{n!}{(n+1) \cdot (v)_{n+1}}, \quad \operatorname{Re} v > 0, \tag{148}$$

which may be readily obtained from (122) by following the same line of reasoning.

**Theorem 6.** <sup>40</sup> *The function  $\zeta(s, v)$  may be represented by the following series with the Stirling numbers of the first kind*

$$\begin{aligned} \zeta(s, v) &= \frac{k!}{(s-k)_k} \sum_{n=0}^{\infty} \frac{|S_1(n+k, k)|}{(n+k)!} \sum_{l=0}^{n+k-1} (-1)^l \binom{n+k-1}{l} (l+v)^{k-s} \\ &= \frac{k!}{(s-k)_k} \sum_{n=0}^{\infty} \frac{|S_1(n+k, k)|}{(n+k)!} \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \sum_{l=0}^n (-1)^l \binom{n}{l} (l+r+v)^{k-s} \end{aligned} \tag{149}$$

where  $k \in \mathbb{N}$ . It may also be written in terms of simpler numbers, especially for small  $k$ ; see (157) hereafter. For example, for  $k = 1$ , we obtain Hasse's series (2); for  $k = 2$ , we have

$$\begin{aligned} \zeta(s, v) &= \frac{2}{(s-1)(s-2)} \sum_{n=0}^{\infty} \frac{H_{n+1}}{n+2} \sum_{l=0}^{n+1} (-1)^l \binom{n+1}{l} (l+v)^{2-s} \\ &= \frac{2}{(s-1)(s-2)} \sum_{n=0}^{\infty} \frac{H_{n+1}}{n+2} \sum_{l=0}^n (-1)^l \binom{n}{l} \left\{ (l+v)^{2-s} - (l+v+1)^{2-s} \right\}, \end{aligned} \tag{150}$$

for  $k = 3$

$$\zeta(s, v) = \frac{3}{(s-1)(s-2)(s-3)} \sum_{n=0}^{\infty} \frac{H_{n+2}^2 - H_{n+2}^{(2)}}{n+3} \sum_{l=0}^{n+2} (-1)^l \binom{n+2}{l} (l+v)^{3-s},$$

and so on, where  $H_n^{(s)} = 1^{-s} + 2^{-s} + 3^{-s} + \dots + n^{-s}$  are the generalized harmonic numbers, also known as the incomplete  $\zeta$ -function.

*Proof.* From the theory of finite differences, it is known that

$$\frac{d^k F(v)}{dv^k} = k \sum_{n=0}^{\infty} \frac{B_n^{(n+k)}}{(n+k)n!} \Delta^{n+k} F(v), \quad k \in \mathbb{N}, \tag{151}$$

where  $B_n^{(n+k)}$  are the Bernoulli numbers of higher order, see, e.g., [57, p. 242].<sup>41</sup> These numbers are the particular case of the Bernoulli polynomials of higher order

<sup>40</sup>We place this theorem after the other results because, on the one hand, it logically continues the previous remark, and on the other hand, this result merits to be presented as a separate theorem, rather than a simple formula in the text.

<sup>41</sup>This formula is actually a generalization of (124), which we get when  $k = 1$ .



see, e.g., [24, p. 217], [31], [16], [17]. Now, the repeated differentiation of  $\zeta(s, v)$  yields

$$\frac{\partial^k \zeta(s, v)}{\partial v^k} = (-1)^k (s)_k \zeta(s + k, v), \quad k \in \mathbb{N}, \tag{159}$$

and since  $\Delta\zeta(s, v) = \zeta(s, v + 1) - \zeta(s, v) = -v^{-s}$ , formula (160) becomes

$$\zeta(s + k, v) = \frac{(-1)^{k-1} k!}{(s)_k} \sum_{n=0}^{\infty} \frac{S_1(n + k, k)}{(n + k)!} \Delta^{n+k-1} v^{-s}, \quad k \in \mathbb{N}. \tag{160}$$

Rewriting this expression for  $s$  instead of  $s+k$  and using the fact that  $\text{sgn } S_1(m, n) = (-1)^{m \pm n}$ , we arrive at

$$\zeta(s, v) = \frac{k!}{(s - k)_k} \sum_{n=0}^{\infty} \frac{|S_1(n + k, k)|}{(n + k)!} (-1)^{n+k-1} \Delta^{n+k-1} v^{k-s}, \quad k \in \mathbb{N}, \tag{161}$$

which is identical with (165). The latter formula may also be written in terms of the  $n$ th finite difference instead of the  $(n + k - 1)$ th one. Indeed, since  $\Delta^{n+k-1} v^{k-s} = \Delta^{k-1}(\Delta^n v^{k-s})$ , then by virtue of (32), we have

$$\zeta(s, v) = \frac{k!}{(s - k)_k} \sum_{n=0}^{\infty} \frac{|S_1(n + k, k)|}{(n + k)!} \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} (-1)^n \Delta^n (v+r)^{k-s}, \tag{162}$$

where  $k \in \mathbb{N}$ . Particular cases of the above formulæ may be of interest. For instance, if  $k = 1$ , then by virtue of (157) the latter formula immediately reduces to Hasse's series (2). If  $k = 2$ , then

$$\begin{aligned} \zeta(s, v) &= \frac{2}{(s-1)(s-2)} \sum_{n=0}^{\infty} \frac{H_{n+1}}{n+2} (-1)^{n+1} \Delta^{n+1} v^{2-s} \\ &= \frac{2}{(s-1)(s-2)} \sum_{n=0}^{\infty} \frac{H_{n+1}}{n+2} (-1)^n \left\{ \Delta^n v^{2-s} - \Delta^n (v+1)^{2-s} \right\}. \end{aligned} \tag{163}$$

For  $k = 3$ , we get

$$\begin{aligned} \zeta(s, v) &= \frac{3}{(s-1)(s-2)(s-3)} \sum_{n=0}^{\infty} \frac{H_{n+2}^2 - H_{n+2}^{(2)}}{n+3} (-1)^{n+2} \Delta^{n+2} v^{3-s} \\ &= \frac{3}{(s-1)(s-2)(s-3)} \sum_{n=0}^{\infty} \frac{H_{n+2}^2 - H_{n+2}^{(2)}}{n+3} (-1)^n \left\{ \Delta^n v^{3-s} - \right. \\ &\qquad \qquad \qquad \left. - 2\Delta^n (v+1)^{3-s} + \Delta^n (v+2)^{3-s} \right\} \end{aligned} \tag{164}$$

and so on. □

The formulæ obtained in the above theorem give rise to many interesting expressions. For instance, putting  $v = 1$  we get the following series representation for the classic Euler–Riemann  $\zeta$ -function

$$\zeta(s) = \frac{k!}{(s - k)_k} \sum_{n=0}^{\infty} \frac{|S_1(n + k, k)|}{(n + k)!} \sum_{l=0}^{n+k-1} (-1)^l \binom{n + k - 1}{l} (l + 1)^{k-s}, \quad (165)$$

$k \in \mathbb{N}$ . It is also interesting that the right of this expression, following the fraction  $k!/(s - k)_k$ , contains zeros of the first order at  $s = 2, 3, \dots, k$ , while it has a fixed quantity when  $s \rightarrow 1$ . This, perhaps, may be useful for the study of the Riemann hypothesis, since the right part also contains the zeros in the strip  $0 < \operatorname{Re} s < 1$ . Furthermore, expanding both sides of (165) into the Laurent series about  $s - 1, s - 2, \dots$  one can obtain the series expansions for  $\Psi(v), \gamma_m(v), \Psi_1(v)$ , etc. For example, the Laurent series in a neighborhood of  $s = 1$  of (163) yields

$$\Psi(v) = -1 - 2 \sum_{n=0}^{\infty} \frac{H_{n+1}}{n + 2} \sum_{l=0}^{n+1} (-1)^l \binom{n + 1}{l} (l + v) \ln(l + v), \quad (166)$$

while that in a neighborhood of  $s = 2$  gives a series for the trigamma function

$$\Psi_1(v) = -2 \sum_{n=0}^{\infty} \frac{H_{n+1}}{n + 2} \sum_{l=0}^{n+1} (-1)^l \binom{n + 1}{l} \ln(l + v), \quad (167)$$

which may also be obtained from (166) by a direct differentiation.

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### Appendix. The Integral Formula for the Bernoulli Polynomials of the Second Kind (Fontana–Bessel Polynomials) $\psi_n(x)$

Integral representation (57) may be obtained by various methods. Below we propose a contour integration method, which leads quite rapidly to the desired result.

Rewrite the generating equation for  $\psi_n(x)$ , formula (51), for  $u$  instead of  $z$ . Setting  $z = -(1 + u) = (1 + u)e^{+i\pi}$ , or equivalently  $1 + u = ze^{-i\pi}$ , the latter formula becomes

$$\frac{z + 1}{\ln z - \pi i} (-z)^x = \frac{(z + 1) z^x e^{-i\pi x}}{\ln z - \pi i} = \sum_{n=0}^{\infty} (-1)^{n-1} \psi_n(x) (z + 1)^n, \quad |z + 1| < 1.$$

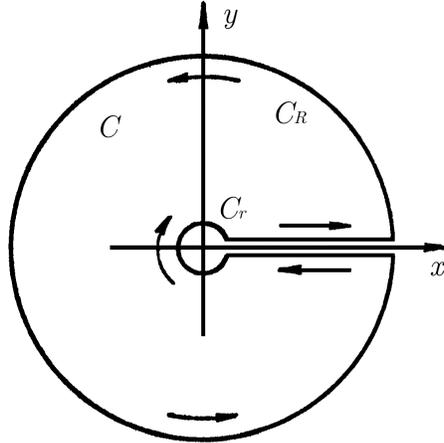


Figure 1: Integration contour  $C$  ( $r$  and  $R$  are radii of the small and big circles respectively, where  $r \ll 1$  and  $R \gg 1$ ).

where in the right part we replaced  $e^{-i\pi(n-1)}$  by  $(-1)^{n-1}$ . Evaluating now the following line integral along a contour  $C$  (see Fig. 1), and then letting  $R \rightarrow \infty$ ,  $r \rightarrow 0$ , we have<sup>42</sup>

$$\oint_C \frac{z^x e^{-i\pi x}}{(1+z)^n (\ln z - \pi i)} dz = \int_r^R \dots + \int_{C_R} \dots + \int_{Re^{2i\pi}}^{re^{2i\pi}} \dots + \int_{C_r} \dots \stackrel{R \rightarrow \infty}{\underset{r \rightarrow 0}{\equiv}} \quad (168)$$

$$= \int_0^\infty \left\{ \frac{e^{-\pi i x}}{\ln z - \pi i} - \frac{e^{+\pi i x}}{\ln z + \pi i} \right\} \cdot \frac{z^x dz}{(1+z)^n} = 2i \int_0^\infty \frac{\pi \cos \pi x - \sin \pi x \ln z}{\ln^2 z + \pi^2} \cdot \frac{z^x dz}{(1+z)^n}$$

since for  $R \rightarrow \infty$

$$\left| \int_{C_R} \frac{z^x e^{-i\pi x}}{(1+z)^n (\ln z - \pi i)} dz \right| = O\left(\frac{1}{R^{n-x-1} \ln R}\right) = o(1), \quad n \geq x + 1,$$

and for  $r \rightarrow 0$

$$\left| \int_{C_r} \frac{z^x e^{-i\pi x}}{(1+z)^n (\ln z - \pi i)} dz \right| = O\left(\frac{r^{x+1}}{\ln r}\right) = o(1), \quad x \geq -1,$$

where  $n \in \mathbb{N}$  and  $x$  is assumed to be real. But the above contour integral may also be evaluated by means of the Cauchy residue theorem. Since the integrand has

<sup>42</sup>Note that for the clarity we should keep  $z^x e^{-i\pi x}$  rather than simply  $(-z)^x$ .

only two singularities, a branch point at  $z = 0$  which we excluded and a pole of the  $(n + 1)$ th order at  $z = e^{i\pi} = -1$ , the Cauchy residue theorem gives us

$$\oint_C \frac{z^x e^{-i\pi x}}{(1+z)^n (\ln z - \pi i)} dz = 2\pi i \operatorname{res}_{z=-1} \frac{z^x e^{-i\pi x}}{(1+z)^n (\ln z - \pi i)} = \tag{169}$$

$$= \frac{2\pi i}{n!} \cdot \left. \frac{\partial^n}{\partial z^n} \frac{z^x e^{-i\pi x} (z+1)}{\ln z - \pi i} \right|_{z=-1} = 2\pi i (-1)^{n-1} \psi_n(x).$$

Equating (168) with (169), therefore, yields

$$\psi_n(x) = \frac{(-1)^{n+1}}{\pi} \int_0^\infty \frac{\pi \cos \pi x - \sin \pi x \ln z}{(1+z)^n} \cdot \frac{z^x dz}{\ln^2 z + \pi^2}, \tag{170}$$

where  $n \in \mathbb{N}$  and  $-1 \leq x \leq n - 1$ . This formula may also be written in a variety of other forms, for example,

$$\psi_n(x) = \frac{(-1)^{n+1}}{\pi} \int_0^\infty \frac{\pi \cos \pi x + \sin \pi x \ln z}{(1+z)^n} \cdot \frac{z^{n-x-2} dz}{\ln^2 z + \pi^2} \tag{171}$$

$$= \frac{(-1)^{n+1}}{\pi} \int_{-\infty}^{+\infty} \frac{\pi \cos \pi x - v \sin \pi x}{(1+e^v)^n} \cdot \frac{e^{v(x+1)}}{v^2 + \pi^2} dv \tag{172}$$

$$= \frac{(-1)^{n+1}}{\pi} \int_{-\infty}^{+\infty} \frac{\pi \cos \pi x + v \sin \pi x}{(1+e^v)^n} \cdot \frac{e^{v(n-x-1)}}{v^2 + \pi^2} dv \tag{173}$$

with the same  $n$  and  $x$ . Note that at  $x = 0$  we retrieve Schröder's formulæ for  $G_n$  [11]. □