A SUFFICIENT CONDITION FOR $(\theta^N)_N$ TO HAVE A DISTRIBUTION MODULO ONE, WHEN θ IS IN $\mathbb{F}_2(X)$

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Abstract

Let θ be a given element in $\mathbb{F}_2(X)$. In this article, we give a sufficient condition for the sequence $(\theta^n)_{n\geq 0}$ to have a distribution modulo 1.

1. Introduction

Many number theoretic problems have natural counterparts in the domain of function fields. We are concerned here with the question of the distribution modulo 1 of the powers of an element $\theta \in \mathbb{F}_q(X)$, the counterpart of the question of the distribution modulo 1 of $(3/2)^n$. The reader will notice that the method and result of this note can easily be extended to the case of an algebraic element over $\mathbb{F}_q(X)$; since our result is only partial, we see no interest in stating it in a more general form, as long as generalisation does not bring a better understanding.

Let us start by giving some definition. We denote $\mathbb{F}_q((X))$ by the set of all the Laurent expansions

$$\eta = \sum_{k\geq -k_0} \varepsilon_k(\eta) X^k, \quad k_0 \in \mathbb{N} \text{ and } \varepsilon_k(\eta) \in \mathbb{F}_q.$$

It is a field which contains $\mathbb{F}_q(X)$.

Definition 1 (Densities). Let $\theta \in \mathbb{F}_q((X))$. We say that the sequence $(\theta^n)_{n\geq 0}$ has a distribution modulo 1 if for any $L \geq 1$ and for any $b_L \in \mathbb{F}_q^L$, the sequence

$$\mathcal{N}(\theta, b_L) = \{n \in \mathbb{N} \colon (\varepsilon_1(\theta^n), \ldots, \varepsilon_L(\theta^n)) = b_L\} \tag{1}$$

has an asymptotic density, *i.e.*, if the following limit

$$\lim_{x\to\infty} \frac{1}{x} \operatorname{Card}\{n \leq x \colon n \in \mathcal{N}(\theta, b_L)\} \tag{2}$$

exists.

Similarly, we say that the sequence $(\theta^n)_{n\geq 0}$ has a logarithmic distribution modulo 1 if for any $L \geq 1$ and for any $b_L \in \mathbb{F}_q^L$, the sequence $\mathcal{N}(\theta, b_L)$ has a logarithmic density, i.e., if the following limit

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\in\mathcal{N}(\theta,b_L), n\leq x} \frac{1}{n} \tag{3}$$

exists.

Houndonougbo proved in [5] the existence of the distribution modulo 1 of the sequence $(\theta^n)_{n\geq 0}$, where $\theta = P(X)^\mu + 1/P(X)^\nu$ for positive integers μ and ν and P a non-constant polynomial in $\mathbb{F}_q[X]$: he indeed showed more, namely that the sequence $\mathcal{N}(\theta, (0, 0, \ldots, 0))$ has density 1. Deshouillers proved in [4] that the sequence $(\theta^n)_{n\geq 0}$ also has a distribution modulo 1 when $\theta = P(X)/X^\nu$, i.e., when the Laurent expansion of θ is finite: he showed that for any b_L the sequence $\mathcal{N}(\theta, b_L)$ is q-automatic and that it has a density. Allouche and Deshouillers proved in [1] that for any θ algebraic over $\mathbb{F}_q(X)$, the sequence $\mathcal{N}(\theta, b_L)$ is q-automatic; by a general result of Cobham [3], this implies that the sequence $(\theta^n)_{n\geq 0}$ has a logarithmic distribution modulo 1, but the existence of a distribution modulo 1 is still an open question.

Our aim is to provide a criterion which is sufficient to prove the existence of the distribution modulo 1 of $(\theta^n)_{n\geq 0}$. We made some ten hand numerical experiments on θ with an infinite Laurent expansion; in the cases we considered, this criterion turned out to be satisfied and indeed led to a limit distribution which is the Dirac measure at 0.

From now on, we assume that $q = 2$ and that $\theta \in \mathbb{F}_2(X)$. In order to describe the 2-automata which generate the sequences $\mathcal{N}(\theta, b_L)$ we follow [1] and first introduce some definition.

For $n \geq 0$, we consider the Laurent expansions

$$\theta^n = \sum_{k\geq -k_0(n)} \varepsilon_k(\theta^n) X^k.$$

Since θ is rational, its expansion is ultimately periodic and the following definition makes sense.

Definition 2 (Parameter). The *parameter* of an element θ in $\mathbb{F}_2(X)$ is the smallest even positive integer T satisfying

$$\varepsilon_{-h}(\theta) = 0 \text{ if } h \geq T$$

and

$$\varepsilon_{h+T}(\theta) = \varepsilon_h(\theta) \text{ if } h \geq T.$$

From now on we denote T as the parameter of θ. For $n \geq 0$ and $K, L \geq 0$, we define

$$\mathcal{B}(n,K,L) = (\varepsilon_{-K}(\theta^n), \varepsilon_{-K+1}(\theta^n), \ldots, \varepsilon_{L-1}(\theta^n), \varepsilon_L(\theta^n)) \in \mathbb{F}_2^{L+1+T}, \tag{4}$$

$$\mathcal{B}(n,L) = \mathcal{B}(n,T,L) \tag{5}$$

and

$$\mathcal{M}(n) = (m(n,0), \ldots, m(n,T-1)) \in \mathbb{F}_2^T, \tag{6}$$

$$\text{where, for } t \in \mathbb{Z}\colon m(n,t) = \sum_{h=0}^{\infty} \varepsilon_{-t-hT}(\theta^n) \in \mathbb{F}_2,$$

which is well-defined since this sum contains only a finite number of non-zero elements.

The key ingredient in [1] is the fact that, for $L \geq T$, the two $(2T+L+1)$-tuples $(\mathcal{M}(2n), \mathcal{B}(2n,L))$ and $(\mathcal{M}(2n+1), \mathcal{B}(2n+1,L))$ only depend on $(\mathcal{M}(n), \mathcal{B}(n,L))$. Since [1] is not easily available, we give here a proof of this fact.

Proposition 1. *Let $L \geq T$; there exist two maps ρ and τ from $\mathbb{F}_2^{2T+L+1}$ into itself such that for every $n \geq 0$ one has,*

$$(\mathcal{M}(2n), \mathcal{B}(2n,L)) = \rho\left((\mathcal{M}(n), \mathcal{B}(n,L))\right), \tag{7}$$

$$(\mathcal{M}(2n+1), \mathcal{B}(2n+1,L)) = \tau\left((\mathcal{M}(n), \mathcal{B}(n,L))\right). \tag{8}$$

Proof. We first observe that

$$\forall k \in \mathbb{Z}\colon \varepsilon_{2k}(\theta^{2n}) = \varepsilon_k(\theta^n), \tag{9}$$

$$\forall k \in \mathbb{Z}\colon \varepsilon_{2k+1}(\theta^{2n}) = 0, \tag{10}$$

$$\text{For } t \text{ even in } [0,T)\colon m(2n,t) = m(n,t/2) + m(n,t/2+T/2), \tag{11}$$

$$\text{For } t \text{ odd in } [0,T)\colon m(2n,t) = 0. \tag{12}$$

This implies that as soon as one knows $\mathcal{B}(n,L)$, all the coefficients of θ^{2n} with indices beween $-2T-1$ and $2L+1$ are known: so are $\mathcal{B}(2n, 2T+1, 2L+1)$ and *a fortiori* $\mathcal{B}(2n,L)$. Similarly, the knowledge of $\mathcal{M}(n)$ implies that of $\mathcal{M}(2n)$. This implies (7).

We noticed that the knowledge of $\mathcal{B}(n,L)$ gives us that of $\mathcal{B}(2n,2T+1,2L+1)$. Let us show that the knowledge of $\mathcal{B}(n,L)$ and of $\mathcal{M}(n)$ gives us the knowledge of

$$m(2n,t)=\sum_{h=0}^{\infty}\varepsilon_{-t-hT}(\theta^{2n}) \text{ for } t\in[-2L-T-1,3T+1]. \tag{13}$$

Indeed, if $t\in[0,T-1]$, then $m(2n,t)$ is an element of $\mathcal{M}(2n)$; Otherwise $m(2n,t)$ is an element of $\mathcal{M}(2n)$ which is modified by a few terms which belong to $\mathcal{B}(2n,2T+1,2L+T+1)$, *e.g.* $m(2n,-2)=\varepsilon_2(\theta^{2n})+m(2n,T-2)$, $m(2n,T)=-\varepsilon_0(\theta^{2n})+m(2n,0)$.

For any k we have

$$\begin{aligned}
\varepsilon_k(\theta^{2n+1}) &= \sum_{r=-\infty}^{+\infty}\varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n})=\sum_{r=0}^{+\infty}\varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n})\\
&= \sum_{r=0}^{2T-1}\varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n})+\sum_{r=2T}^{\infty}\varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n})\\
&= \sum_{r=0}^{2T-1}\varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n})+\sum_{\nu=0}^{T-1}\varepsilon_{T+\nu}(\theta)\sum_{\substack{r\geq 2T\\ r\equiv\nu \bmod T}}\varepsilon_{k+T-r}(\theta^{2n})\\
&= \sum_{r=0}^{2T-1}\varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n})+\sum_{\nu=0}^{T-1}\varepsilon_{T+\nu}(\theta)\,m(2n,T+\nu-k).
\end{aligned}$$

The last relation shows that as soon as one knows $\mathcal{B}(n,L)$ and $\mathcal{M}(n,L)$ (and the digits of θ with indices between $-T$ and $2T$ which are our inititial data), we have enough information to determine $\mathcal{B}(2n+1,L)$ (cf. (13) and the fact that for $k\in[-T,L]$ we have $T+\nu-k\in[T-L,3T-1]\subset[-2L-T-1,3T+1]$).

We finally study $\mathcal{M}(2n+1)$. Let $t\in[0,T-1)$. Reasoning as above, we have.

$$\begin{aligned}
m(2n+1,t) &= \sum_{h=0}^{\infty}\sum_{r=0}^{2T-1}\varepsilon_{-T+r}(\theta)\varepsilon_{-t-r-(h-1)T}(\theta^{2n})\\
&+ \sum_{h=0}^{\infty}\sum_{r=2T}^{\infty}\varepsilon_{-T+r}(\theta)\varepsilon_{-t-r-(h-1)T}(\theta^{2n})\\
&= S_1+S_2, \quad \text{say.}
\end{aligned}$$

By interchanging the sums in the first term on the right-hand side, we see that it is equal to

$$S_1=\sum_{r=0}^{2T-1}\varepsilon_{-T+r}(\theta)\,m(2n,t+r-T). \tag{14}$$

Since $r\in[0,2T-1]$ and $t\in[0,T-1]$, we have $-T\leq t+r-T\leq 2T-2$ and thus the term in (14) is known as soon as $\mathcal{M}(n)$ is known. Let us look at the second term. We have

$$\begin{aligned}
S_2 &= \sum_{h=0}^{\infty}\sum_{r=2T}^{\infty}\varepsilon_{-T+r}(\theta)\varepsilon_{-t-r-(h-1)T}(\theta^{2n})\\
&= \sum_{h=0}^{\infty}\sum_{s=T}^{\infty}\varepsilon_{s}(\theta)\varepsilon_{-t-s-hT}(\theta^{2n})\\
&= \sum_{h=0}^{\infty}\sum_{\nu=0}^{T-1}\sum_{\substack{r\geq T\\ s\equiv\nu \bmod T}}\varepsilon_{s}(\theta)\varepsilon_{-t-s-hT}(\theta^{2n}).
\end{aligned}$$

We use the periodicity of the digits of θ and write $s=\nu+T+kT$. We have

$$\begin{aligned}
S_2 &= \sum_{\nu=0}^{T-1}\varepsilon_{T+\nu}(\theta)\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\varepsilon_{-t-\nu-T-(h+k)T}(\theta^{2n})\\
&= \sum_{\nu=0}^{T-1}\varepsilon_{T+\nu}(\theta)\sum_{\ell=0}^{\infty}\left(\sum_{\substack{h\geq 0,k\geq 0\\ h+k=\ell}}1\right)\varepsilon_{-t-\nu-T-\ell T}(\theta^{2n}).
\end{aligned}$$

It is enough to consider each inside sum over ℓ. We notice that if $t+\nu$ is odd, then all the terms $\varepsilon_{-t-\nu-T-\ell T}(\theta^{2n})$ are zero and so is the sum of those terms over ℓ. We also notice that the sum $\sum_{\substack{h\geq 0,k\geq 0\\ h+k=\ell}}1$ is equal to 1 when ℓ is even and to 0 when ℓ is odd. Combining those two remarks and writing $\ell=2\lambda$, we have, when $t+\nu$ is even

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\left(\sum_{\substack{h\geq 0,k\geq 0\\ h+k=\ell}}1\right)\varepsilon_{-t-\nu-T-\ell T}(\theta^{2n}) = \sum_{\lambda=0}^{\infty}\varepsilon_{-(\nu+t+T)/2-\lambda T}(\theta^{n})\\
=\ & m(n,(\nu+t+T)/2);
\end{aligned}$$

when $\nu+t+T\leq 2T$, then $m(n,(\nu+t+T)/2)$ is an element in $\mathcal{M}(n)$; otherwise, we write $m(n,(\nu+t+T)/2)=m(n,(\nu+t-T)/2)-\varepsilon_{(T-\nu-t)/2}(\theta^{n})$, which the difference of an element of $\mathcal{M}(n)$ and an element of $\mathcal{B}(n,L)$.

Thus S_2 is also known as soon as $\mathcal{B}(n,L)$ and $\mathcal{M}(n)$ are known. This ends the proof of Proposition 1. $\square$

This permits us to build a directed graph Γ_L with edges indexed by 0 or 1 as follows. We first consider the set of vertices

$$\mathcal{R}_L=\{(\mathcal{M}(n),\mathcal{B}(n,L)) : n\geq 0\}. \tag{15}$$

We then build two edges starting from $r_L\in\mathcal{R}_L$ depending on $\varepsilon=0$ or $\varepsilon=1$ in the following way: since $r_L=(\mathcal{M}(n),\mathcal{B}(n,L))$ for some integer $n\geq 0$, for each

$\varepsilon \in \{0, 1\}$, we define the state

$$\delta_L(r_L, \varepsilon) := (\mathcal{M}(2n + \varepsilon), \mathcal{B}(2n + \varepsilon, L)) \tag{16}$$

and the edge $(r_L, \delta_L(r_L, \varepsilon))$, which are well-defined. The above-mentioned observation of [1] implies that our definition is indeed independent of the choice of n such that $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$. The reader who needs to refresh her/his knowledge on *Automatic Sequences* is strongly recommended to visit [2][1], especially subsections 4.1 and 5.1. The refreshment being performed, it is not difficult to see that the sequence $\mathcal{N}(\theta, b_L)$ is 2-automatic: it is recognized by the automaton $\mathcal{A}(b_L) = \{\mathcal{R}_L, \{0, 1\}, \delta_L, r_{0,L}, F(b_L)\}$, where $\mathcal{R}_L$ and δ_L have already been defined,

$$r_{0,L} = (\mathcal{M}(0), \mathcal{B}(0, L))$$

and

$F(b_L)$ is the set of those $r \in \mathcal{R}_L$, the last L components of which are b_L.

It will be convenient to extend the function δ_L to a new function still called δ_L, defined over the words w on $\{0, 1\}$, satisfying

$$\forall r \in \mathcal{R}_L, \forall \varepsilon \in \{0, 1\}, \forall w \in \{0, 1\}^* \colon \delta_L(r, \emptyset) = r, \delta_L(r, \varepsilon w) = \delta_L(\delta_L(r, w), \varepsilon).$$

Let us recall a criterion on the graph Γ_L which insures that the sequence $\mathcal{N}(\theta, b_L)$ has a density.

In the directed graph Γ_L, we say that two vertices r and s are *equivalent* if there is a directed path leading from r to s and a directed path leading from s to r; this permits to consider *equivalent classes*, which form a tree, which leads to the notion of *final class*; we finally say that an equivalent class is *regular* if there exists an integer ℓ such that for any pair (r, s), there is a directed path of length ℓ leading from r to s. We have the following criterion.

Proposition 2. *Let $L \geq T$ be a given integer and $b_L \in \mathbb{F}_2^L$ be a given vector. If the graph Γ_L of the automaton $\mathcal{A}(b_L)$ has a single final class and if this class is regular, then, the sequence $\mathcal{N}(\theta, b_L)$ has an asymptotic density.*

Theorem 8.4.7, p. 272 of [2] deals with the special case where all the states constitute a single final regular equivalence class. Proposition 2 is a mere extension of this result where the key tool is Perron-Frobenius theorem.

Back to our question, we notice that if $u \leq v$, then $\mathcal{N}(\theta, b_u)$ is a finite union of sequences $\mathcal{N}(\theta, b_v)$, and it is thus enough for our purpose to consider the sequences $\mathcal{N}(\theta, b_L)$ for all sufficiently large values of L.

We wish to prove here that if the criterion applies to an automaton $\mathcal{A}(b_L)$, it also applies to the automaton $\mathcal{A}(b_{L+1})$, which leads to the following theorem.

[1] Thanks, Jeff, for this invaluable monography... and for the rest!

Theorem 1. *Let θ be an element in $\mathbb{F}_2(X)$ and let T be its parameter. If the graph Γ_T has a single final class and if this final class is regular, then the sequence $(\theta^n)_{n\geq 0}$ has a distribution modulo 1.*

We remark that in the statement of Theorem 1, the automaton $\mathcal{A}(b_T)$ only occurs through its graph Γ_T, which itself depends on $\{\mathcal{R}_T, \delta_T, r_{0,T}\}$ but not on $F(b_T)$.

Corollary 1. *Let θ be an element in $\mathbb{F}_2(X)$ and let T be its parameter. If the graph Γ_T has a single equivalence class, then the sequence $(\theta^n)_{n\geq 0}$ has a distribution modulo 1.*

2. Connection Between the Automata $\mathcal{A}(b_L)$ and $\mathcal{A}(b_{L+1})$

Our key tool to understand the connection between the automata $\mathcal{A}(b_{L+1})$ and $\mathcal{A}(b_L)$ is a natural map from $\mathcal{R}_{L+1}$ onto $\mathcal{R}_L$; we define it and give its main properties in the following proposition.

Proposition 3. *Let θ and T be as in Theorem 1 and let $L \geq T$. The map σ_L from $\mathcal{R}_{L+1}$ to $\mathbb{F}_2^{2T+L+1}$, defined by suppressing the last component of an element, has the following properties*

$$\sigma_L\left(\mathcal{R}_{L+1}\right) = \mathcal{R}_L, \tag{17}$$

$$\forall r \in \mathcal{R}_L, \text{ at most two elements } s \in \mathcal{R}_{L+1} \text{ such that } \sigma_L(s) = r, \tag{18}$$

$$\sigma_L(r_{0,L+1}) = r_{0,L}, \tag{19}$$

$$\forall r \in \mathcal{R}_{L+1}, \text{ and } \varepsilon \in \{0,1\}, \text{ we have} : \sigma_L\left(\delta_{L+1}(r,\varepsilon)\right) = \delta_L(\sigma_L(r),\varepsilon). \tag{20}$$

Proof. By definition, cf. (15), for a state r_{L+1} in $\mathcal{R}_{L+1}$, there exists an integer n such that r_{L+1} is the $(2T+L+2)$-tuple $(\mathcal{M}(n), \mathcal{B}(n, L+1))$. If we suppress its last component we get the $(2T+L+1)$-tuple $(\mathcal{M}(n), \mathcal{B}(n, L))$, which is an element of $\mathcal{R}_L$. In the other direction, if we start with an element r_L in $\mathcal{R}_L$, there exits an n such that $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$, and for this n we have $r_L = \sigma_L((\mathcal{M}(n), \mathcal{B}(n, L+1))$. Thus, the suppression of the last component defines the map σ_L which satisfies the Property (17). Property (18) comes from the fact that the last component of an element of $\mathcal{R}_{L+1}$ belongs to $\{0,1\}$. We have $\sigma_L(r_{0,L+1}) = \sigma_L\left((\mathcal{M}(0), \mathcal{B}(0, L+1)\right) = (\mathcal{M}(0), \mathcal{B}(0, L)) = r_{0,L}$, which proves Property (19). Finally, let $r \in \mathcal{R}_{L+1}$, there exists an integer n such that $r = (\mathcal{M}(n), \mathcal{B}(n, L+1))$; for $\varepsilon \in \{0,1\}$, we have, cf. (16), $\delta_{L+1}(r,\varepsilon) := (\mathcal{M}(2n+\varepsilon), \mathcal{B}(2n+\varepsilon, L+1))$ and so $\sigma_L\left(\delta_{L+1}(r,\varepsilon)\right) = \sigma_L\left((\mathcal{M}(2n+\varepsilon), \mathcal{B}(2n+\varepsilon, L+1))\right) = (\mathcal{M}(2n+\varepsilon), \mathcal{B}(2n+\varepsilon, L)) = \delta_L(\sigma_L(r),\varepsilon)$, which proves (20). □

We say that $r_{L+1} \in \mathcal{R}_{L+1}$ is *sitting above* r_L for some $r_L \in \mathcal{R}_L$, if $\sigma(r_{L+1}) = r_L$.

Claim 1. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L and let $\mathcal{C}_{L+1}$ be one of the final classes of Γ_{L+1}. Then any element of $\mathcal{C}_{L+1}$ is sitting above some element of $\mathcal{C}_L$.*

Proof. Let $r_{L+1} \in \mathcal{C}_{L+1}$ be a given element. Then we have, $\sigma_L(r_{L+1}) \in \mathcal{R}_L$. If $\sigma_L(r_{L+1}) \in \mathcal{C}_L$, then we are done. If not, then there exists a word $w \in \{0,1\}^*$ such that $\delta_L(\sigma_L(r_{L+1}), w) \in \mathcal{C}_L$. Since r_{L+1} belongs to $\mathcal{C}_{L+1}$ which is a final class, the state $\delta_{L+1}(r_{L+1}, w)$ belongs to $\mathcal{C}_{L+1}$. Therefore, there exists a word $w' \in \{0,1\}^*$ such that $\delta_{L+1}(r_{L+1}, ww') = r_{L+1}$. Thus, by the definition of $\mathcal{C}_L$, we see that $\delta_L(\sigma_L(r_{L+1}), ww') \in \mathcal{C}_L$. Since r_{L+1} is sitting above $\sigma_L(r_{L+1})$, by (20) we conclude that $r_{L+1} = \delta_{L+1}(r_{L+1}, ww')$ is sitting above $\delta_L(\sigma_L(r_{L+1}), ww') \in \mathcal{C}_L$. □

Now, we look at a converse of Claim 1.

Claim 2. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L and let $\mathcal{C}_{L+1}$ be one of the final classes of Γ_{L+1}. For any element $r \in \mathcal{C}_L$, there is an element $s \in \mathcal{C}_{L+1}$ which is sitting above r.*

Proof. Let r be an element in $\mathcal{C}_L$ and t an element in $\mathcal{C}_{L+1}$. There exists a word $w \in \{0,1\}^*$ such that $\delta_L(\sigma_L(t), w) \in \mathcal{C}_L$. Since $r \in \mathcal{C}_L$, there exists a word w' such that $\delta_L(\sigma_L(t), ww') = r$. Now, we look at $\delta_{L+1}(t, ww')$. Since $t \in \mathcal{C}_{L+1}$, we conclude that $\delta_{L+1}(t, ww') \in \mathcal{C}_{L+1}$. Since t is sitting above $\sigma_L(t)$, by (20), we have $\delta_{L+1}(t, ww') \in \mathcal{C}_{L+1}$ is sitting above r. □

Claim 3. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L. If $\mathcal{C}_{L+1}$ is a final class in Γ_{L+1}, then $|\mathcal{C}_{L+1}| \geq |\mathcal{C}_L|$.*

Proof. If $r \neq s$ are two different elements in $\mathcal{C}_L$, then, by Claim 2, there exist two elements r', s' of $\mathcal{C}_{L+1}$ such that r' is sitting above r and s' is sitting above s. Since $r \neq s$, by the definition, we see that $r' \neq s'$. Hence the claim. □

Claim 4. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L. Then there can be at most two distinct final classes in Γ_{L+1}.*

Proof. We first remark that two classes are distinct if and only if they are disjoint; now, the claim simply follows from Claim 3 and (18): each final class has at least $|\mathcal{C}_L|$ elements and for any element r in $\mathcal{C}_L$ we can have at most two elements of $\mathcal{R}_{L+1}$ sitting above r. □

Claim 5. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L. Suppose that $\mathcal{C}_{L+1}^{(1)}$ and $\mathcal{C}_{L+1}^{(2)}$ are two distinct final classes in Γ_{L+1} sitting above $\mathcal{C}_L$. They are disjoint and for every element $r \in \mathcal{C}_L$, there is exactly one element $r_1 \in \mathcal{C}_{L+1}^{(1)}$ and one element*

$r_2 \in \mathcal{C}^{(2)}_{L+1}$ which are sitting above r. Moreover, for any $(r_1, r_2) \in \mathcal{C}^{(1)}_{L+1} \times \mathcal{C}^{(2)}_{L+1}$ and for any word $w \in \{0,1\}^*$, we have

$$\delta_{L+1}(r_1, w) \neq \delta_{L+1}(r_2, w). \tag{21}$$

Proof. Since the two classes $\mathcal{C}^{(1)}_{L+1}$ and $\mathcal{C}^{(2)}_{L+1}$ are distinct, they are disjoint. By Claim 2, above each element $r \in \mathcal{C}_L$, there exist $r_1 \in \mathcal{C}^{(1)}_{L+1}$ and $r_2 \in \mathcal{C}^{(2)}_{L+1}$ sitting above r ; since the two classes are disjoint, we have $r_1 \neq r_2$. By (18), this implies that above r, there can be only one element from each of the $\mathcal{C}^{(i)}_{L+1}$. The last assertion follows the fact that for any word w, the element $\delta_{L+1}(r_i, w)$ is in $\mathcal{C}^{(i)}_{L+1}$. □

3. Proof of Theorem 1

By Proposition 2, it is enough to prove that for any $L \geq T$ and any b_L, the graph Γ_L of the automaton $\mathcal{A}(b_L)$ has a single final class, and this class is regular. Let us recall that Γ_L is independent of b_L. We shall prove our assertion by induction on L.

The assumption of Theorem 1 is simply the case $L = T$: the graph Γ_T has a single final class and this class is regular.

Let us assume that for some $L \geq T$, the graph Γ_L has a single final class and this class is regular; let $\mathcal{C}_L$ be this class.

We first prove that the graph Γ_{L+1} has a single final class. By Claim 1, any single final class of Γ_{L+1} is sitting above $\mathcal{C}_L$; thus by Claim 4, there are at most two final classes in Γ_{L+1}. If we have indeed two distinct final classes, then we can apply Claim 5: let r be an element in $\mathcal{C}_L$: there exist two elements r_1 and r_2 which are sitting above r and which belong to the two different classes above $\mathcal{C}_L$. Choose an integer h such that $2^h > L + 1$ and consider a word w consisting of h zeroes. By (20), the elements $\delta_{L+1}(r_i, w)$ are sitting above $\delta_L(r, w)$ for $i = 1$ and 2. This means that they differ at most by their last digit. Let n_1 and n_2 be two integers such that

$$r_i = (\mathcal{M}(n_i), \mathcal{B}(n_i, L+1)) \text{ for } i = 1, 2.$$

Then, we have

$$\begin{aligned}\delta_{L+1}(r_i, w) &= \left(\mathcal{M}\left(n_i 2^h\right), \mathcal{B}\left(n_i 2^h, L+1\right)\right) \\ &= \left(\mathcal{M}\left(n_i 2^h\right), \mathcal{B}\left(n_i 2^h, L\right) \circ \varepsilon_{L+1}(\theta^{n_i 2^h})\right)\end{aligned}$$

for $i = 1, 2$, where the symbol $\circ$ represents the concatenation. Since $2^h > L + 1$, we have

$$\varepsilon_{L+1}(\theta^{n_1 2^h}) = \varepsilon_{L+1}(\theta^{n_2 2^h}) = 0,$$

which implies

$$\delta_{L+1}(r_1, w) = \delta_{L+1}(r_2, w),$$

a contradiction to Claim 5. Hence, there is only one final class in the graph Γ_{T+1}.

Let $\mathcal{C}_{L+1}$ be the unique final class in Γ_{L+1}. It remains to show that this class is regular.

If $|\mathcal{C}_{L+1}| = |\mathcal{C}_L|$, then, by Claim 2, for every $r_L \in \mathcal{C}_L$, there is exactly only one $r_{L+1} \in \mathcal{C}_{L+1}$ such that r_{L+1} is sitting above r_L and conversely. Therefore, by (20), $\delta_{L+1}(r_{L+1}, w)$ is sitting above $\delta_L(r_L, w)$ for every word w. Since $\mathcal{C}_L$ is regular, this implies that $\mathcal{C}_{L+1}$ is also regular.

Suppose that $|\mathcal{C}_{L+1}| > |\mathcal{C}_L|$. Then, there exists r in $\mathcal{C}_L$ such that the two elements $r \circ 0$ and $r \circ 1$ are in $\mathcal{C}_{L+1}$. Choose an integer h such that $2^h > L+1$ and let w be the word consisting of h zeroes. By the above argument, we have

$$\delta_{L+1}(r \circ 0, w) = \delta_{L+1}(r \circ 1, w),$$

and we denote this element by s. Since $\mathcal{C}_L$ is regular, there exists an integer K such that for any $k \geq K$, there is a word w_k of length k satisfying

$$\delta_L(\sigma_L(s), w_k) = r;$$

thus $\delta_{L+1}(s, w_k)$ is either $r \circ 0$ or $r \circ 1$. But in either case, we have $\delta_{L+1}(s, w_k w) = s$, so that for any $\ell \geq K + h$ there is a path of length ℓ which connects s to s. Since $\mathcal{C}_{L+1}$ is an equivalent class, any element u can be connected to any element v by a path of length exactly $2|\mathcal{C}_{L+1}| + K + h$, which implies that $\mathcal{C}_{L+1}$ is regular. Theorem 1 is proved. □

Proof of Corollary 1. Since the graph of $\mathcal{A}(b_T)$ has unique class, say, $\mathcal{C}_T$, it is the final class. Choose an integer h such that $2^h > T$. Let w be a word consisting only h zeroes. Then $\delta_T(r_{0,T}, w) = \big(\mathcal{M}(2^h), \mathcal{B}(2^h, T)\big) = r_h$, say. Therefore there exists a word w' such that $\delta_T(r_h, w') = r_{0,T}$. Thus, we have

$$\delta_T(r_{0,T}, ww') = r_{0,T} \text{ with } |ww'| = K, \text{ say.}$$

Since it is an equivalent class, any element u can be connected to any other element v by a path of length $K + 2|\mathcal{C}_T|$. Hence it is regular. Therefore, by Theorem 1, we get the corollary. □

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