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POSITIVE-DEFINITE QUADRATIC FORMS REPRESENTING FINITE SETS OF INTEGERS

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Abstract

For a given subset S of positive integers, a positive-definite integral quadratic form is called S-*universal* if it represents every integer in the set S. We say that an S-universal form has *minimal dimension* if there are no S-universal forms of a lower dimension. The goal of this paper is to study the size of the bound of the discriminant of positive-definite integral S-universal quadratic forms of minimal dimension in the case when S is a finite subset of positive integers.

1. Introduction

Let $f = f(\mathbf{x})$ be a positive-definite integral quadratic form of the form

$$f(\mathbf{x}) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j,$$

where the coefficients a_{ij} are positive integers. A central problem of number theory is determining which positive integers $m \in \mathbf{Z}$ are represented by f, meaning that we can find some integral vector $\mathbf{x} \in \mathbf{Z}^n$ such that $f(\mathbf{x}) = m$. To this date, the determination of those positive integers represented by a given positive-definite integral quadratic form is still far from being fully solved.

In recent years the remarkable Conway-Schneeberger 15-Theorem [1] and Bhargava-Hanke 290-Theorem [2] shed new light in this topic. In later work, M. Bhargava showed that these finiteness theorems are quite common, and announced the following beautiful result:

Theorem 1 (Bhargava). *For any given infinite set S of positive integers, there exists a unique minimal finite subset S_0 of S such that every positive-definite integral quadratic form represents every integer in the set S provided it represents every integer in the set S_0.*

[1]any footnote here

The original proof of this theorem has never been published. This result was generalized by B.M. Kim, M.-H. Kim and B.-K. Oh in [3], they considered representations by a form of positive-definite quadratic forms instead of positive integers.

Let S be a given subset of positive integers, a positive-definite integral quadratic form f is called S-*universal* if it represents every integer in the set S. If S is the set of all positive integers, then any positive definite integral S-universal quadratic form is simply called a *universal* quadratic form. We say that an S-universal form has *minimal dimension* if there are no S-universal forms of lower dimension.

By Theorem 1, an immediate and stunning conclusion is that the set of integers represented by a positive-definite integral quadratic form is determined by a unique finite subset of this set. Then every positive-definite integral quadratic form is S-universal for a certain finite subset S of positive integers.

The purpose of the present paper is to study the size of the bound of the discriminant of positive-definite integral S-universal quadratic forms of minimal dimension in the case when S is a finite subset of positive integers. The historical motivation for this work comes from a paper of Ross [5], in which the author obtained a bound of 861 for the maximum discriminant possible for any universal quaternary forms. In terms of Conway-Schneeberger 15-Theorem perspective, these universal quaternary forms correspond to the $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$-universal quadratic forms of minimal dimension.

In this paper, we shall adopt the standard geometric language of quadratic spaces and lattices, and any unexplained notations and terminologies can be found in [4].

Unless otherwise specified, the term lattice will always refer to an integral $\mathbf{Z}$-lattice L on a (not necessarily fixed) non-degenerate positive-definite quadratic space (V, q) over the field of rational numbers $\mathbf{Q}$. The symmetric bilinear form associated with a quadratic space (V, q) is denoted by B.

If L is a lattice on (V, q) there is a basis $\{x_1, \cdots, x_n\}$ for V such that

$$L = \mathbf{Z}x_1 + \mathbf{Z}x_2 + \cdots + \mathbf{Z}x_n.$$

By the matrix of L in the basis $\{x_1, x_2, \cdots, x_n\}$ we mean the matrix $N = (B(x_i, x_j))$, we write

$$L \simeq N \text{ in } x_1, \ \cdots, \ x_n.$$

The matrix N is called a *matrix presentation* of L, and the determinant of the matrix N is called the discriminant of L, denoted by $d(L)$. For any $v \in L, v \neq 0$. We call $q(v)$ the norm of v.

2. The Escalation Method

In this section we describe the escalation method. (For a detailed account of the escalation method, see [1] and [2].)

Let $S = \{\ell_1 < \ell_2 < \cdots < \ell_t < \cdots\}$ be a subset of positive integers. A lattice L is said to be S-*universal* if it represents every positive integer in S. If a lattice L happens not to be S-universal, define the S-*truant* of L to be the smallest positive integer in S not represented by L. We denote by ℓ_L the S-truant of a lattice L, and we write $\ell_L = 0$ if L is S-universal.

Suppose a lattice L is not S-universal. An S-*escalation* (we abbreviate S-escalation as escalation) of L is defined to be any lattice which is generated by L and a vector whose norm is equal to the S-truant ℓ_L of L. An S-*escalator lattice* (we abbreviate S-escalator lattice as escalator lattice) is a lattice which can be obtained as the result of a sequence of successive escalations of the zero-dimensional lattice.

The unique escalation of the zero-dimensional lattice is the lattice generated by a single vector of norm ℓ_1. This lattice corresponds to the forms $\ell_1 x^2$ (or, in matrix form, $[\ell_1]$).

If $[\ell_1]$ is S-universal, then every $\{\ell_1\}$-universal lattice is S-universal. Otherwise we assume that the S-truant of $[\ell_1]$ is equal to $\ell_k (k \geq 2)$. Hence an escalation of $[\ell_1]$ has inner product matrix of the form

$$\begin{pmatrix} \ell_1 & a \\ a & \ell_k \end{pmatrix}.$$

This means that (up to isometry) there is a basis $\{e_1, e_2\}$ for a binary lattice M such that $q(e_1) = \ell_1$, $q(e_2) = \ell_k$ and $B(e_1, e_2) = a$. By the Cauchy-Schwartz inequality $a^2 < \ell_1 \ell_k$, and since a must be an integer or half-integer, we obtain at most finitely many nonisometric 2-dimensional escalator lattices.

For every positive integer $i = 2, 3, \cdots$, if there exist some i-dimensional escalator lattices which is not S-universal, then we escalate each of these i-dimensional escalator lattices in the same manner. Similarly, we obtain $(i+1)$-dimensional escalator lattices.

We define T_i to be the set of all nonisomorphic i-dimensional escalator lattices for every positive integer $i = 1, 2, 3, \cdots$. We begin by proving several preliminary lemmas.

Lemma 1. *The set T_i is finite for every positive integer $i = 1, 2, 3, \cdots$.*

Proof. The proof is by induction on i. For $i = 1$ the result is trivial. Suppose we have an i-dimensional escalator lattice L_i with S-truant $\ell_i \neq 0$. To obtain L_{i+1} we adjoin a vector ν_i such that $q(\nu_i) = \ell_i$. Let $\{e_1, \cdots, e_i\}$ be a basis of L_i. The quadratic form on L_{i+1} is determined by the value of $B(e_j, \nu_i)$ for $1 \leq j \leq i$. By the Cauchy-Schwartz inequality, there can be only finitely many such symmetric bilinear products. So there are only finitely many possible L_{i+1} for each L_i, and so there are only finitely many possible L_{i+1} if there are finitely many L_i. Hence the result follows. □

On the other hand, the following lemma is the key to our computations.

Lemma 2. *The dimension of the S-universal lattices of minimal dimension is equal to the dimension of the S-universal escalator lattices of minimal dimension.*

Proof. To prove this, notice that if the dimension of any S-universal lattice L of minimal dimension is strictly less than the dimension of the S-universal escalator lattices of minimal dimension, then we may construct with L an escalator lattices sequence

$$\{0\} \subseteq L_1 \subseteq L_2 \subseteq \cdots .$$

In finite steps, we shall obtain a S-universal escalator lattice within L. This implies a contradiction. □

3. S-universal Lattices of Minimal Dimension

In this section we prove our main results.

Let $S = \{\ell_1 < \ell_2 < \cdots < \ell_t\}$ be a finite subset of positive integers. Without loss of generality, we may assume that the square classes $l_i\mathbf{Z}^2 (1 \leq i \leq t)$ are not equal and $t \geq 2$.

Proposition 1. *Let Q_S be the set of S-universal lattices of minimal dimension. Then the number of isometric classes in Q_S is finite.*

Proof. By 103:4 of [4], it suffices to show the discriminants of the S-universal lattices of minimal dimension are bound. To see this, by the above claim notice that any S-universal lattice L of minimal dimension must contain a full rank S-universal escalator sublattice and hence our proposition follows from Lemma 2.2. □

The famous Lagrange's theorem says that the four squares form $x^2 + y^2 + z^2 + \omega^2$ represent all integers. Hence there is no S-universal quinary form of minimal dimension for any finite subset S of positive integers.

Let T_i be the set of all nonisomorphic i-dimensional escalators for every $i = 1, 2, 3, 4$.

Theorem 2. *Let L be a binary S-universal $\mathbf{Z}$-lattice of minimal dimension. Then we have*

$$d(L) \leq l_1 l_2.$$

Proof. This follows immediately from the assumption that the square classes $l_1\mathbf{Z}^2$ and $l_2\mathbf{Z}^2$ are not equal. □

Example 1. Let $S = \{1, 3\}$. We see that there are two nonisometric S-universal escalator lattices having integer matrix of minimal dimension. The unique escalation of the zero-dimensional lattice is the lattice generated by a single vector of norm 1. This lattice corresponds to the forms x^2 which fails to represent the number 3. Hence an escalation of [1] has inner product matrix of the form

$$\begin{pmatrix} 1 & a \\ a & 3 \end{pmatrix}.$$

By the Cauchy-Schwartz inequality, $a^2 < 3$, so $a = 0, \pm 1$. So we obtain two nonisometric two-dimensional escalator lattices, namely those lattices having Gauss-reduced Gram matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \text{and} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Theorem 3. *Let L be a ternary S-universal $\mathbf{Z}$-lattice of minimal dimension. Then we have*

$$d(L) \leq l_1 l_2 \cdot \max\{\ell_M | M \in T_2\}.$$

Proof. By the condition of minimal dimension we know that there are certainly escalators M of T_2 that

$$\ell_M \neq 0.$$

Since any ternary S-universal lattice L of minimal dimension must contain a full rank S-universal escalator sublattice $K \in T_3$, hence

$$d(L) \leq d(K) \leq l_1 l_2 \cdot \max\{\ell_M | M \in T_2\}.$$

So the theorem is proved. □

Example 2. Let $S = \{1, 3, 5\}$. As shown in Example 3.3, two $\{1, 3\}$-universal escalator lattices having integer matrix of minimal dimension are $x^2 + 2y^2$ and x^2+3y^2 with truant 5. If we escalate each of these two-dimensional escalator lattices, we find that we obtain 15 nonisometric ternary S-universal escalator lattices having integer matrix, namely those lattices having Gauss-reduced Gram matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix}, \text{and} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

Theorem 4. *Let L be a quaternary S-universal $\mathbf{Z}$-lattice of minimal dimension. Then we have*

$$d(L) \leq l_1 l_2 \cdot \max\{\ell_K | K \in T_2\} \cdot \max\{\ell_M | M \in T_3\}.$$

Proof. By the condition of minimal dimension we know that

$$T_i \neq \emptyset$$

for every $i = 1, 2, 3, 4$. If L is a quaternary S-universal $\mathbf{Z}$-lattice of minimal dimension, we can construct an escalator sequence

$$\{0\} \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq M_4 \subseteq L$$

where $M_i \in T_i$ for every $i = 1, 2, 3, 4$.

A direct computation gives

$$\max\{d(M) | M \in T_3\} \leq l_1 l_2 \cdot \max\{\ell_K | K \in T_2\}.$$

Then we see that

$$d(L) \leq d(M_4) \leq l_1 l_2 \cdot \max\{\ell_K | K \in T_2\} \cdot \max\{\ell_M | M \in T_3\}.$$

This proves our theorem. □

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References

[1] M. Bhargava, On the Conway-Schneeberge fifteen theorem, *Contemp. Math* **272** (2000), 27-37.

[2] M. Bhargava and J. Hanke, Universal quadratic forms and the 290-Theorem, preprint.

[3] B.M. Kim, M.-H. Kim and B.-K. Oh, A finiteness theorem for representability of quadratic forms by forms, J. Reine Angew. Math **581** (2005), 23-30.

[4] O. T. O'Meara, *Introduction to Quadratic Forms*, Springer, New York, 1971.

[5] A. E. Ross, On a problem of Ramanujan, *Amer. J. Math* **68** (1946), 29-46.