ON CONGRUENCES INVOLVING SOME FIBONOMIAL COEFFICIENTS

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Abstract

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. For $1 \leq k \leq m$, the Fibonomial coefficient is defined as

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_{m-k+1}\cdots F_{m-1}F_m}{F_1\cdots F_k}.$$

In 2013, Marques, Sellers and Trojovský proved that if p is a prime number such that $p \equiv \pm 1 \pmod 5$, then $p \nmid \begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$ for all integers $a \geq 1$. In this paper, we study congruences involving $\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$. In particular, we prove that if p is a prime number with $p \equiv \pm 1 \pmod 5$ and $\min\{k \geq 1 : p \mid F_k\} = p-1$ then $\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv 1 + p + p^2 \pmod{p^3}$, for all $a \geq 3$.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult book [5] to find additional references and history).

In 1915, Fontené published a one-page note [1] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence (A_n) of real or complex numbers.

Since at least the 1960s, there has been much interest in the *Fibonomial coefficients* $\begin{bmatrix} m \\ k \end{bmatrix}_F$, which correspond to the choice $A_n = F_n$. Hence Fibonomial coefficients are defined, for $1 \leq k \leq m$, by

$$\begin{bmatrix} m \\ k \end{bmatrix}_F := \frac{F_{m-k+1}\cdots F_{m-1}F_m}{F_1\cdots F_k}, \tag{1.1}$$

and for $k > m$, $\left[{m \atop k}\right]_F = 0$ (for example, see Gould [2] as well as numerous papers referenced therein). It is surprising that this quantity will always take integer values.

Some authors have been interested in searching for divisibility properties of Fibonomial coefficients. For instance, in 1974, Gould [3] proved several such properties where one of them is analogous to Hermite's identity for binomial coefficients.

In two papers, Marques and Trojovský [13, 14] proved that $p \mid \left[{p^{a+1} \atop p^a}\right]_F$ for all integers $a \geq 1$ and $p \in \{2, 3\}$. Subsequently, Marques, Sellers and Trojovský [16] proved that the number $\left[{p^{a+1} \atop p^a}\right]_F$ is divisible by p for all primes p such that $p \equiv \pm 2 \pmod 5$ and for all integers $a \geq 1$. In 2015, Marques and Trojovský [15] proved that $p^{\lceil (a+\delta_{p,2})/2 \rceil} \parallel \left[{p^{a+1} \atop p^a}\right]_F$, for all integers $a \geq 1$ (here, $\delta_{i,j}$ denotes the Kronecker delta and $a^k \parallel b$ means that $a^k \mid b$, but $a^{k+1} \nmid b$). In this same paper, they proved that if $p \equiv \pm 1 \pmod 5$, then $p \nmid \left[{p^{a+1} \atop p^a}\right]_F$ for all integers $a \geq 1$. A question arises: in this case, what is the residue class of $\left[{p^{a+1} \atop p^a}\right]_F$ modulo p?

Very recently, Trojovský [19] proved that this residue is 1 when $p \equiv \pm 1 \pmod 5$. Here, we study the residue of $\left[{p^{a+1} \atop p^a}\right]_F$ modulo higher powers of p. Apparently, there is no pattern in general. However, we were able to find this residue in the extremal case when $z(p) = \min\{k \geq 1 : p \mid F_k\} = p - 1$ (where $z(n)$ is the order of appearance of n in the Fibonacci sequence. We shall provide more information about this function soon). More precisely, we prove the following theorem

Theorem 1.1. *Let p be a prime such that $z(p) = p - 1$. Then $\left[{p^{a+1} \atop p^a}\right]_F \equiv 1 + p + p^2 \pmod{p^3}$ for all $a \geq 3$.*

In particular, when $z(p) = p - 1$, we have that $\left[{p^{a+1} \atop p^a}\right]_F \equiv 1 + p \pmod{p^2}$.

We organize this paper as follows. In Section 2, we will recall some useful properties of the Fibonacci numbers such as a result concerning the p-adic order of F_n. Section 3 is devoted to the proof of this theorem.

2. Auxiliary Results

Before proceeding further, we recall some facts for the convenience of the reader.

Lemma 2.1. *We have*

(a) For all primes p, one has that $F_{p-\left(\frac{5}{p}\right)} \equiv 0 \pmod p$, where $\left(\frac{a}{q}\right)$ denotes the Legendre symbol of a with respect to a prime $q > 2$.

(b) (Addition formula) $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$.

(c) (Multiple angle formula) $F_{kn+c} = \sum_{i=0}^{k} \binom{k}{i} F_{c-i} F_n^i F_{n+1}^{k-i}$.

(d) *(Definition of Pisano period) If $\pi(m)$ is the Pisano period of m (i.e., the smallest period of the Fibonacci sequence modulo m), then $a \equiv b \pmod{\pi(m)}$ implies that $F_a \equiv F_b \pmod{m}$.*

(e) *If p is a prime number with $p \equiv \pm 1 \pmod{10}$ and let $k \geq 1$ be an integer, then $\pi(p^k)$ divides $p^{k-1}(p-1)$.*

The previous facts are quite classical and their proofs can be found, for example, in [22].

Before stating the next lemma, we recall that for a positive integer n, the *order* (*or rank*) *of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (some authors also call it the *order of apparition*, as it was called by Lucas, or the *Fibonacci entry point*). There are several results on $z(n)$ in the literature. For example, recently, the third author [7, 8, 9, 10, 11, 12] found closed formulas for this function at some integers related to the Fibonacci and Lucas numbers.

Lemma 2.2. (*Cf. [8, Lemma 2.2 (c)]*) *If $n \mid F_m$, then $z(n) \mid m$.*

Note that Lemma 2.1 (a) together with Lemma 2.2 implies that $z(p) \mid p - (\frac{5}{p})$ for all primes $p \neq 5$. Also, it is well-known that $(\frac{5}{p}) = -1$ or 1 according to the residue of p modulo 5. More precisely, we have that if $p \neq 5$ is a prime, then $z(p) \mid p+1$ if $p \equiv \pm 2 \pmod 5$ and $z(p) \mid p-1$ otherwise.

Lemma 2.3. (*Cf. [11, Lemma 2.3]*) *For all primes $p \neq 5$, we have $\gcd(z(p), p) = 1$.*

We recall that the p-adic order (or valuation) of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r. The p-adic order of Fibonacci numbers has been completely characterized, see [4, 6, 20, 21]. For instance, from the main results of Lengyel [6], we extract the following result.

Lemma 2.4. *If $n \geq 1$ and $p \neq 2$ and 5, then*

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + e(p), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

Here $e(p) := \nu_p(F_{z(p)})$.

A proof of a more general result can be found in [6].

3. Proof of Theorem 1.1

We have that

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F = \prod_{j=1}^{p^a} \frac{F_{p^a(p-1)+j}}{F_j}.$$

Since $\pi(p^3) \mid p^2(p-1) \mid p^a(p-1)$ (by Lemma 2.1 (e)), we have that $p^a(p-1)+j \equiv j \pmod{\pi(p^3)}$ for all $j \in \mathcal{M}$ (where $\mathcal{M}$ is the set of all $j \in [1, p^a]$ such that $p-1 \nmid j$). Thus $\prod_{j\in\mathcal{M}} F_{p^a(p-1)+j} \equiv \prod_{j\in\mathcal{M}} F_j \pmod{p^3}$. Since $\gcd(\prod_{j\in\mathcal{M}} F_j, p^3) = 1$, then

$$\frac{\prod_{j\in M} F_{p^a(p-1)+j}}{\prod_{j\in\mathcal{M}} F_j} \equiv 1 \pmod{p^3}.$$

Thus

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv \prod_{j=1}^{p^{a-1}+p^{a-2}+\cdots+1} \frac{F_{p^a(p-1)+j(p-1)}}{F_{j(p-1)}} \pmod{p^3}.$$

Let us split our proof into some small claims in order to make our exposition more clear.

Claim 1. *If $p \nmid j$, then* $\frac{F_{p^a(p-1)+j(p-1)}}{F_{j(p-1)}} \equiv 1 \pmod{p^3}$.

In fact, by the addition formula,

$$\frac{F_{p^a(p-1)+j(p-1)}}{F_{j(p-1)}} = F_{p^a(p-1)-1} + \frac{F_{p^a(p-1)}}{F_{j(p-1)}} F_{j(p-1)+1} \equiv 1 + \frac{F_{p^a(p-1)}}{F_{j(p-1)}} F_{j(p-1)+1} \pmod{p^3}.$$

Note that

$$\nu_p\left(\frac{F_{p^a(p-1)}}{F_{j(p-1)}} \cdot F_{j(p-1)+1}\right) = a - \nu_p(j) = a \geq 3$$

and the result follows.

Thus, we obtain that

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv \prod_{j=1}^{r} \frac{F_{p^a(p-1)+jp(p-1)}}{F_{jp(p-1)}} \pmod{p^3},$$

where $r := p^{a-2} + \cdots + 1$. Let us write the following:

$$\prod_{j=1}^{r} \frac{F_{p^a(p-1)+jp(p-1)}}{F_{jp(p-1)}} = \prod_{j=1}^{r} \left(F_{p^a(p-1)-1} + \frac{F_{p^a(p-1)}}{F_{jp(p-1)}} \cdot F_{jp(p-1)+1}\right) = \prod_{j=1}^{r} (\lambda - y(j)),$$

where $\lambda := F_{p^a(p-1)-1}$ and $y(j) := -F_{p^a(p-1)} F_{jp(p-1)+1}/F_{jp(p-1)}$. However, it is a well-known fact that $\prod_{j=1}^{r}(\lambda - y(j)) = \lambda^r - \sigma_1(\overline{y})\lambda^{r-1} + \cdots + (-1)^r \sigma_r(\overline{y})$, where σ_j is the j-th elementary symmetric function and $\overline{y} = (y(1), \ldots, y(r))$. Thus,

$$\prod_{j=1}^{r} \frac{F_{p^a(p-1)+jp(p-1)}}{F_{jp(p-1)}} \equiv 1 + \sigma_1(\overline{x}) + \cdots + \sigma_r(\overline{x}) \pmod{p^3},$$

where $\overline{x} := (x(1), \ldots, x(r))$ with $x(j) = -y(j)$. Also we used the fact that $\lambda \equiv 1 \pmod{p^3}$.

Claim 2. *If $k \in [2, r]$, then $\sigma_k(\overline{x}) \equiv 0 \pmod{p^3}$.*

In fact, we have that

$$\sigma_k(\overline{x}) = \sum_{1 \le i_1 < \cdots < i_k \le r} x(i_1) \cdots x(i_k).$$

Observe that each term in the summatory is

$$x(i_1) \cdots x(i_k) = \frac{(F_{p^a(p-1)})^k}{\prod_{j=1}^{k} F_{i_j p(p-1)}} \cdot \prod_{j=1}^{k} F_{i_j p(p-1)+1}.$$

Thus $\nu_p(x(i_1) \cdots x(i_k)) = ak - k - \sum_{j=1}^{k} \nu_p(i_j)$. However $i_k \le r < 2p^{a-2}$ yields that $\nu_p(i_k) \le a-2$ and $\nu_p(i_j) \le a-3$, for all $j \in [1, k-1]$ (without loss of generality). Hence $\nu_p(x(i_1) \cdots x(i_k)) \ge ak - k - (k-1)(a-3) - (a-2) = 2k - 1 \ge 3$ for $k \ge 2$. So $x(i_1) \cdots x(i_k) \equiv 0 \pmod{p^3}$ and then $\sigma_k(\overline{x}) \equiv 0 \pmod{p^3}$ as desired.

Hence,

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv 1 + \sigma_1(\overline{x}) \equiv 1 + x(1) + \cdots + x(r) \pmod{p^3}.$$

Claim 3. *If $j \in [1, r] \backslash \{ip^{a-3} : i \in [1, p+1]\}$, then $x(j) \equiv 0 \pmod{p^3}$.*

This follows directly, because $\nu_p(x(j)) = a - 1 - \nu_p(j) \ge a - 1 - (a-4) = 3$. Here we used that $\nu_p(j) \le a-4$, for the hypothesis on j and the fact that $(p+2)p^{a-3} > r$.

So,

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv 1 + \sum_{i=1}^{p+1} x(ip^{a-3}) \pmod{p^3}.$$

Claim 4. *If $i, j \in [1, p-1]$ satisfy $i + j = p$, then $x(ip^{a-3}) + x(jp^{a-3}) \equiv 0 \pmod{p^3}$.*

In fact, we have

$$x(ip^{a-3}) + x(jp^{a-3}) = \frac{F_{p^a(p-1)}}{F_{ip^{a-2}(p-1)} F_{jp^{a-2}(p-1)}} \left(F_{ip^{a-2}(p-1)+1} F_{jp^{a-2}(p-1)} \right.$$
$$\left. + F_{ip^{a-2}(p-1)} F_{jp^{a-2}(p-1)+1} \right).$$

Now, we use the Fibonacci recurrence together with the addition formula to arrive at

$$x(ip^{a-3}) + x(jp^{a-3}) = \frac{F_{p^a(p-1)}}{F_{ip^{a-2}(p-1)} F_{jp^{a-2}(p-1)}} \left(F_{ip^{a-2}(p-1)} F_{jp^{a-2}(p-1)} + F_{p^{a-1}(p-1)} \right).$$

Here we used that $i + j = p$. Note that $\nu_p(F_{ip^{a-2}(p-1)} F_{jp^{a-2}(p-1)}) = 2a - 4 + 2e(p)$ and $\nu_p(F_{p^{a-1}(p-1)}) = a - 1 + e(p)$. Since $2a - 4 + 2e(p) > a - 1 + e(p)$ (for $a \ge 3$), we have that

$$\nu_p(F_{ip^{a-2}(p-1)} F_{jp^{a-2}(p-1)} + F_{p^{a-1}(p-1)}) = a - 1 + e(p),$$

where we used the fact that $\nu_p(m+n) = \min\{\nu_p(m), \nu_p(n)\}$ if $\nu_p(m) \neq \nu_p(n)$. This yields that

$$\nu_p(x(ip^{a-3}) + x(jp^{a-3})) = a + e(p) + a - 1 + e(p) - (2(a-2) + e(p)) = 3$$

as desired.

By applying the previous fact for the pairs $(i, j) = (k+1, p-1-k)$ for $k \in [0, (p-3)/2]$, we conclude that

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv 1 + x(p^{a-2}) + x((p+1)p^{a-3}) \pmod{p^3}.$$

Claim 5. *We have that* $x(p^{a-2}) \equiv p \pmod{p^3}$.

To prove that, we use the multiple angle formula (Lemma 2.1 (c)) for $k = p, n = p^{a-1}(p-1)$ and $c = 0$. Thus,

$$x(p^{a-2}) = \frac{F_{p^a(p-1)}}{F_{p^{a-1}(p-1)}} F_{p^{a-1}(p-1)+1} = \sum_{i=1}^{p} \binom{p}{i} F_{-i} F_{p^{a-1}(p-1)}^{i-1} F_{p^a(p-1)+1}^{p-i+1}.$$

Since

$$\nu_p\left(\binom{p}{i} F_{p^{a-1}(p-1)}^{i-1}\right) = \begin{cases} 1 + (i-1)(a-1+e(p)), & i \in [2, p-1] \\ (p-1)(a-1+e(p)), & i = p, \end{cases}$$

it follows that $\nu_p(\binom{p}{i} F_{p^{a-1}(p-1)}^{i-1}) \geq 3$ for all $i \in [2, p]$. Therefore

$$x(p^{a-2}) = \frac{F_{p^a(p-1)}}{F_{p^{a-1}(p-1)}} F_{p^{a-1}(p-1)+1} \equiv p F_{p^{a-1}(p-1)+1}^{p} \equiv p \pmod{p^3},$$

where we used that $F_{p^{a-1}(p-1)+1}^{p} \equiv 1 \pmod{p^3}$. The result is proved.

So,

$$\begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F \equiv 1 + p + x((p+1)p^{a-3}) \pmod{p^3}. \tag{3.1}$$

Thus, in order to complete the proof of the theorem, it remains to prove the following fact:

Claim 6. *We have that* $x((p+1)p^{a-3}) \equiv p^2 \pmod{p^3}$.

Note that

$$x((p+1)p^{a-3}) = \frac{F_{p^a(p-1)}}{F_{p^{a-2}(p^2-1)}} \cdot F_{p^{a-2}(p^2-1)+1}.$$

Now, we use the multiple angle formula for $k = p^2$, $n = p^{a-2}(p-1)$ and $c = 0$. We obtain

$$\frac{F_{p^a(p-1)}}{F_{p^{a-2}(p^2-1)}}F_{p^{a-2}(p^2-1)+1} = \sum_{i=1}^{p^2}\binom{p^2}{i}F_{-i}\frac{F^i_{p^{a-2}(p-1)}}{F_{p^{a-2}(p^2-1)}}F^{p^2-i+1}_{p^{a-2}(p-1)+1}.$$

Observe that

$$\nu_p\left(\binom{p^2}{i}\frac{F^i_{p^{a-2}(p-1)}}{F_{p^{a-2}(p^2-1)}}\right) = \begin{cases} 1+(i-1)(a-2+e(p))), & i\in[2,p^2-1] \\ (p^2-1)(a-2+e(p))), & i=p^2. \end{cases}$$

Therefore

$$\nu_p\left(\binom{p^2}{i}\frac{F^i_{p^{a-2}(p-1)}}{F_{p^{a-2}(p^2-1)}}\right)\geq 3$$

and so

$$\frac{F_{p^a(p-1))}}{F_{p^{a-2}(p^2-1)}}\cdot F_{p^{a-2}(p^2-1)+1} \equiv p^2\frac{F_{p^{a-2}(p-1)}}{F_{p^{a-2}(p^2-1)}}F^{p^2}_{p^{a-2}(p-1)+1} \equiv p^2\frac{F_{p^{a-2}(p-1)}}{F_{p^{a-2}(p^2-1)}} \pmod{p^3}. \tag{3.2}$$

We claim that

$$\frac{F_{p^{a-2}(p^2-1)}}{F_{p^{a-2}(p-1)}} \equiv p+1 \pmod{p^3}.$$

In fact, by using the multiple angle formula for $k=p+1$, $n=p^{a-2}(p-1)$ and $c=0$, we get

$$\frac{F_{p^{a-2}(p^2-1)}}{F_{p^{a-2}(p-1)}} = \sum_{i=1}^{p+1}\binom{p+1}{i}F_{-i}F^{i-1}_{p^{a-2}(p-1)}F^{p+1-i}_{p^{a-2}(p-1)+1}.$$

By analyzing the p-adic valuation, we have

$$\nu_p\left(\binom{p+1}{i}F^{i-1}_{p^{a-2}(p-1)}\right)\geq \begin{cases} 1+(i-1)(a-2+e(p)), & i\in[2,p-1] \\ (p-1)(a-2)+e(p), & i\in\{p,p+1\}. \end{cases}$$

Thus $\nu_p(\binom{p+1}{i}F^{i-1}_{p^{a-2}(p-1)})\geq 3$, for $i\in[2,p+1]$ (here we used that $p\geq 11$, since $z(p)=p-1$). Then

$$\frac{F_{p^{a-2}(p^2-1)}}{F_{p^{a-2}(p-1)}} \equiv p+1 \pmod{p^3},$$

since $F^p_{p^{a-2}(p-1)+1}\equiv 1 \pmod{p^3}$.

Now, we use the congruence $1\equiv p^3+1\equiv (p+1)(p^2-p+1) \pmod{p^3}$ together with the previous fact to derive

$$\frac{F_{p^{a-2}(p-1)}}{F_{p^{a-2}(p^2-1)}} \equiv p^2-p+1 \pmod{p^3}, \tag{3.3}$$

where we used that $\nu_p(F_{p^{a-2}(p-1)}/F_{p^{a-2}(p^2-1)}) = 0$. Therefore, by combining (3.2) and (3.3), we obtain

$$\frac{F_{p^a(p-1)}}{F_{p^{a-2}(p^2-1)}} F_{p^{a-2}(p^2-1)+1} \equiv p^2(p^2-p+1) \equiv p^2 \pmod{p^3}.$$

In conclusion, by substituting this congruence in (3.1), we obtain the desired result. □

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