GIBONACCI EXTENSIONS OF A LUCAS DELIGHT

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Abstract

We extend the well-known Lucas identity $F_{n+1}^2 + F_n^2 = F_{2n+1}$ to gibonacci polynomials; find the corresponding Pell, Jacobsthal, Vieta, and Chebyshev counterparts; and then extract the numeric Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas implications.

1. Introduction

The *generalized gibonacci polynomials* $g_n(x)$ are defined by the recurrence $g_{n+2}(x) = a(x)g_{n+1}(x) + b(x)g_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), g_0(x)$, and $g_1(x)$ are arbitrary complex polynomials; and $n \geq 0$ [8, 10].

Suppose $a(x) = x$ and $b(x) = 1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. Then $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 6, 9].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [5, 6].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $g_0(x) = 2$ and $g_1(x) = 1$, $g_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [2, 3, 7]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Suppose $a(x) = x$ and $b(x) = -1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 7].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $g_0(x) = 1$ and $g_1(x) = x$, $g_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $g_0(x) = 1$ and $g_1(x) = 2x$, $g_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [4, 6, 7].

1.1. Links Among the Subfamilies

By virtue of the relationships among the extended gibonacci subfamilies in Table 1, every ginonacci result has a Jacobsthal, Vieta, and Chebyshev counterpart, where $i = \sqrt{-1}$ [4, 7, 8].

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(x) = U_{n-1}(x/2)$	$v_n(x) = 2T_n(x/2).$

Table 1: Links Among the Gibonacci Subfamilies

In the interest of brevity and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so g_n will mean $g_n(x)$. Again, for brevity, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; $c_n = J_n(x)$ or $j_n(x)$; $d_n = V_n$ or v_n; and $e_n = T_n$ or U_n; and correspondingly, we let $G_n = F_n$ or L_n; $B_n = P_n$ or Q_n; and $C_n = J_n$ or j_n. We also omit a lot of basic, but messy algebra.

1.2. Binet-like Formulas

We can also define explicitly both Fibonacci and Lucas polynomials by the *Binet-like formulas*

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad l_n = \alpha^n + \beta^n,$$

where $\alpha = \dfrac{x + \sqrt{x^2+4}}{2}$ and $\beta = \dfrac{x - \sqrt{x^2+4}}{2}$ are the solutions of the quadratic equation $t^2 - xt - 1 = 0$. It then follows that gibonacci polynomials can be extended to negative subscripts:

$$g_{-n} = \begin{cases} (-1)^{n-1} g_n & \text{if } g_n = f_n \\ (-1)^n g_n & \text{if } g_n = l_n. \end{cases}$$

1.3. Lucas' Identity

In 1876, Lucas discovered the charming identity $F_{n+1}^2 + F_n^2 = F_{2n+1}$ [9]; and [9] gives a beautiful graph-theoretic proof of this identity. It has an equally charming Lucas counterpart: $L_{n+1}^2 + L_n^2 = 5F_{2n+1}$ [9]. In 1999, Melham generalized both results [11]:

$$F_{n+k+1}^2 + F_{n-k}^2 = F_{2n+1} F_{2k+1}; \tag{1}$$
$$L_{n+k+1}^2 + L_{n-k}^2 = 5F_{2n+1} F_{2k+1}. \tag{2}$$

In this article, we extend these two identities to the gibonacci family and then to the other subfamilies. To this end, we first lay some ground work.

1.4. Gibonacci Addition Formulas

Consider the Q-matrix

$$Q = Q(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}.$$

(In 1960, C.H. King studied this matrix with $x = 1$ [9].) It then follows by induction that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}.$$

Equating the corresponding elements of $Q^{m+n} = Q^m \cdot Q^n$, we get the *Fibonacci addition formula*

$$f_{m+n} = f_{m+1}f_n + f_m f_{n-1}.$$

(This can also be established using the Binet-like formula for Fibonacci polynomials.) Consequently,

$$\begin{aligned} f_{2n+1} &= f_{n+1}^2 + f_n^2; \\ f_{m-n} &= (-1)^{n+1}(f_{m+1}f_n - f_m f_{n+1}). \end{aligned}$$

We can show likewise that

$$\begin{aligned} l_{m+n} &= f_{m+1}l_n + f_m l_{n-1}; \\ l_{m-n} &= (-1)^n(f_{m+1}l_n - f_m l_{n+1}). \end{aligned}$$

Thus

$$\begin{aligned} g_{m+n} &= f_{m+1}g_n + f_m g_{n-1}; \\ g_{m-n} &= \begin{cases} -(-1)^n(f_{m+1}g_n - f_m g_{n+1}) & \text{if } g_n = f_n \\ (-1)^n(f_{m+1}g_n - f_m g_{n+1}) & \text{if } g_n = l_n. \end{cases} \end{aligned}$$

2. Gibonacci Extensions

With these tools, we are now ready for the gibonacci extensions of identities (1) and (2).

Theorem 1. *Let $g_n = f_n$ or l_n, and $\Delta^2 = x^2 + 4$. Then*

$$g_{n+k+1}^2 + g_{n-k}^2 = \begin{cases} f_{2n+1}f_{2k+1} & \text{if } g_n = f_n \\ \Delta^2 f_{2n+1}f_{2k+1} & \text{if } g_n = l_n. \end{cases} \tag{3}$$

Proof. Let $g_n = l_n$. Since $l_{n+1}^2 + l_n^2 = \Delta^2 f_{2n+1}$, we have

$$\begin{aligned}
l_{n+k+1} &= l_{n+(k+1)} = f_{n+1}l_{k+1} + f_n l_k;\\
l_{n-k} &= l_{(n+1)-(k+1)} = (-1)^k(f_{n+1}l_k - f_n l_{k+1});\\
l_{n+k+1}^2 + l_{n-k}^2 &= (f_{n+1}l_{k+1} + f_n l_k)^2 + (f_{n+1}l_k - f_n l_{k+1})^2\\
&= f_{n+1}^2(l_{k+1}^2 + l_k^2) + f_n^2(l_{k+1}^2 + l_k^2)\\
&= (f_{n+1}^2 + f_n^2)(l_{k+1}^2 + l_k^2)\\
&= f_{2n+1} \cdot \Delta^2 f_{2k+1}\\
&= \Delta^2 f_{2n+1} f_{2k+1},
\end{aligned}$$

as desired.

The case, $g_n = f_n$, follows similarly. □

Clearly, Melham's identities follow from Theorem 1. It also implies that

$$\begin{aligned}
p_{n+k+1}^2 + p_{n-k}^2 &= p_{2n+1}p_{2k+1};\\
P_{n+k+1}^2 + P_{n-k}^2 &= P_{2n+1}P_{2k+1};\\
q_{n+k+1}^2 + q_{n-k}^2 &= 4(x^2+1)p_{2n+1}p_{2k+1};\\
Q_{n+k+1}^2 + Q_{n-k}^2 &= 2P_{2n+1}P_{2k+1}.
\end{aligned}$$

3. Gibonacci Implications

By virtue of the relationships in Table 1, Theorem 1 has implications to the Jacobsthal, Vieta, and Chebyshev subfamilies.

3.1. Jacobsthal Consequences

Let $u = 1/\sqrt{x}$ and $g_n = f_n$. Since $J_n(x) = x^{(n-1)/2} f_n(u)$, replace x in identity (3) with u; and multiply the resulting equation with x^{n+k}. This yields

$$J_{n+k+1}^2(x) + x^{2k+1} J_{n-k}^2(x) = J_{2n+1}(x) J_{2k+1}(x).$$

On the other hand, let $g_n = l_n$. Replacing x in (3) with u and multiplying the resulting equation with x^{n+k+1}, we get

$$j_{n+k+1}^2(x) + x^{2k+1} j_{n-k}^2(x) = (4x+1) J_{2n+1}(x) J_{2k+1}(x).$$

Combining the two cases, we have

$$c_{n+k+1}^2 + x^{2k+1} c_{n-k}^2 = \begin{cases} J_{2n+1}(x)J_{2k+1}(x) & \text{if } c_n = J_n(x)\\ (4x+1)J_{2n+1}(x)J_{2k+1}(x) & \text{if } c_n = j_n(x). \end{cases}$$

In particular,

$$C_{n+k+1}^2 + 2^{2k+1}C_{n-k}^2 = \begin{cases} J_{2n+1}J_{2k+1} & \text{if } C_n = J_n \\ 9J_{2n+1}J_{2k+1} & \text{if } C_n = j_n. \end{cases} \tag{4}$$

Next we explore the implications to the Vieta family.

3.2. Vieta Consequences

Let $u = -ix$ and $g_n = f_n$. Since $V_n(x) = i^{n-1}f_n(u)$, replace x with u in identity (3); and multiply the resulting equation with i^{2n+2k}. We then get

$$V_{n+k+1}^2 - V_{n-k}^2 = V_{2n+1}V_{2k+1}.$$

Suppose $g_n = l_n$. Using the link $v_n(x) = i^n l_n(u)$, identity (3) similarly yields

$$v_{n+k+1}^2 - v_{n-k}^2 = (x^2 - 4)V_{2n+1}V_{2k+1}.$$

These two cases together imply that

$$d_{n+k+1}^2 - d_{n-k}^2 = \begin{cases} V_{2n+1}V_{2k+1} & \text{if } d_n = V_n \\ (x^2-4)V_{2n+1}V_{2k+1} & \text{if } d_n = v_n. \end{cases} \tag{5}$$

3.2.1. Fibonacci Byproducts

Since $xV_n(t) = f_{2n}(x)$ and $xv_n(t) = l_{2n}(x)$, identity (5) has Fibonacci and Lucas consequences, where $t = x^2 + 2$. To this end, replace x with t in (5), and multiply the resulting equation with x^2. We then get

$$g_{2n+2k+2}^2 - g_{2n-2k}^2 = \begin{cases} f_{4n+2}f_{4k+2} & \text{if } g_n = f_n \\ x^2(x^2+4)f_{4n+2}f_{4k+2} & \text{if } g_n = l_n. \end{cases} \tag{6}$$

It follows from (6) that

$$\begin{aligned} G_{2n+2k+2}^2 - G_{2n-2k}^2 &= \begin{cases} F_{4n+2}F_{4k+2} & \text{if } G_n = F_n \\ 5F_{4n+2}F_{4k+2} & \text{if } G_n = L_n; \end{cases} \\ b_{2n+2k+2}^2 - b_{2n-2k}^2 &= \begin{cases} p_{4n+2}p_{4k+2} & \text{if } b_n = p_n \\ 16x^2(x^2+1)p_{4n+2}p_{4k+2} & \text{if } b_n = q_n; \end{cases} \\ B_{2n+2k+2}^2 - B_{2n-2k}^2 &= \begin{cases} P_{4n+2}P_{4k+2} & \text{if } B_n = P_n \\ 8P_{4n+2}P_{4k+2} & \text{if } B_n = Q_n. \end{cases} \end{aligned}$$

3.2.2. Jacobsthal Byproducts

The relationships $x^{n-1}V_n(z) = J_{2n}(x)$ and $x^n v_n(z) = j_{2n}(x)$ imply that identity (5) has Jacobsthal consequences as well, where $z = (2x+1)/x$:

$$c_{2n+2k+2}^2 - x^{4k+2}c_{2n-2k}^2 = \begin{cases} J_{4n+2}(x)J_{4k+2}(x) & \text{if } c_n = J_n(x) \\ (4x+1)J_{4n+2}(x)J_{4k+2}(x) & \text{if } c_n = j_n(x). \end{cases}$$

(We omitted the details for the sake of brevity.)

Consequently,

$$C_{2n+2k+2}^2 - 4^{2k+1}C_{2n-2k}^2 = \begin{cases} J_{4n+2}J_{4k+2} & \text{if } C_n = J_n \\ 9J_{4n+2}J_{4k+2} & \text{if } C_n = j_n. \end{cases}$$

3.3. Chebyshev Consequences

Finally, we pursue the Chebyshev implications of Theorem 1. Letting $w = 2x$, it follows from identity (5) that

$$e_{n+k+1}^2 - e_{n-k}^2 = \begin{cases} U_{2n+2}U_{2k} & \text{if } e_n = U_n \\ 4(x^2-1)U_{2n}U_{2k} & \text{if } e_n = T_n. \end{cases}$$

Next we invoke Theorem 1 to develop additional Pell, Jacobsthal, Vieta, and Chebyshev dividends. The next two theorems form the cornerstone of our gibonacci investigations.

Theorem 2. *Let $g_n = f_n$ or l_n. Then*

$$f_{2k+1}g_{n+k+2}^2 = f_{2k+3}g_{n+k+1}^2 + f_{2k+3}g_{n-k}^2 - f_{2k+1}g_{n-k+1}^2. \tag{7}$$

Proof. Let $g_n = l_n$. It then follows from identity (3) that

$$\begin{aligned} l_{n+k+1}^2 + l_{n-k}^2 &= \Delta^2 f_{2n+1}f_{2k+1}; \\ l_{n+k+2}^2 + l_{n-k-1}^2 &= \Delta^2 f_{2n+1}f_{2k+3}. \end{aligned}$$

These two equations imply that

$$f_{2k+3}\left(l_{n+k+1}^2 + l_{n-k}^2\right) = f_{2k+1}\left(l_{n+k+2}^2 + l_{n-k-1}^2\right).$$

This yields identity (7) when $g_n = l_n$.

The derivation of the identity follows similarly when $g_n = f_n$. □

The next result follows from Theorem 2 when $k = 1$.

Corollary 1. *Let $g_n = f_n$ or l_n. Then*

$$(x^2+1)g_{n+3}^2 = (x^4+3x^2+1)g_{n+2}^2 + (x^4+3x^2+1)g_{n-1}^2 - (x^2+1)g_{n-2}^2. \tag{8}$$

It follows from equation (8) that the squares of Fibonacci and Lucas polynomials satisfy the fifth order recurrence

$$(x^2+1)a_{n+3} = (x^4+3x^2+1)a_{n+2} + (x^4+3x^2+1)a_{n-1} - (x^2+1)a_{n-2},$$

where $a_n = a_n(x)$ and $n \geq 2$. When $a_n = f_n^2, a_0 = 0, a_1 = 1, a_2 = x^2, a_3 = (x^2+1)^2, a_4 = (x^3+2x)^2$, and $a_5 = (x^4+3x^2+1)^2$; and when $a_n = l_n^2, a_0 = 4, a_1 = x^2, a_2 = (x^2+2)^2, a_3 = (x^3+3x)^2$, and $a_4 = (x^4+4x^2+1)^2$ and $a_5 = (x^5+5x^3+5x)^2$.

It also follows that

$$\begin{aligned} 2G_{n+3}^2 &= 5G_{n+2}^2 + 5G_{n-1}^2 - 2G_{n-2}^2; \\ (4x^2+1)b_{n+3}^2 &= (16x^4+12x^2+1)b_{n+2}^2 + (16x^4+12x^2+1)b_{n-1}^2 - (4x^2+1)b_{n-2}^2; \\ 5B_{n+3}^2 &= 29B_{n+2}^2 + 29B_{n-1}^2 - 5B_{n-2}^2. \end{aligned}$$

The next theorem shows a different way of employing identity (3) to develop a gibonacci relationship. As can be predicted, it also has interesting byproducts.

Theorem 3. *Let $g_n = f_n$ or l_n. Then*

$$f_{2k+1}\left(g_{n+k+2}^2 + g_{n-k-1}^2\right) + f_{2k+3}\left(g_{n+k+1}^2 + g_{n-k}^2\right) = \begin{cases} 2f_{2n+1}f_{2k+1}f_{2k+3} & \text{if } g_n = f_n \\ 2\Delta^2 f_{2n+1}f_{2k+1}f_{2k+3} & \text{if } g_n = l_n. \end{cases}$$

Proof. Suppose $g_n = l_n$. By identity (3), we have

$$l_{n+k+1}^2 + l_{n-k}^2 = \Delta^2 f_{2n+1}f_{2k+1}; \tag{9}$$
$$l_{n+k+2}^2 + l_{n-k-1}^2 = \Delta^2 f_{2n+1}f_{2k+3}. \tag{10}$$

Multiply (9) with f_{2k+3}, and (10) with f_{2k+1}. Adding the resulting equations yields the desired result when $g_n = l_n$.

The case $g_n = f_n$ follows similarly. □

The following result is a direct consequence of this theorem.

Corollary 2. *Let $g_n = f_n$ or l_n. Then*

$$\begin{aligned} (x^2+1)g_{n+3}^2 + Ag_{n+2}^2 + Ag_{n-1}^2 & \\ + (x^2+1)g_{n-2}^2 &= \begin{cases} 2A(x^2+1)f_{2n+1} & \text{if } g_n = f_n \\ 2A(x^2+1)\Delta^2 f_{2n+1} & \text{if } g_n = l_n, \end{cases} \end{aligned} \tag{11}$$

where $A = A(x) = x^4 + 3x^2 + 1$.

This corollary implies that

$$2G_{n+3}^2 + 5G_{n+2}^2 + 5G_{n-1}^2 + 2G_{n-2}^2 = \begin{cases} 20F_{2n+1} & \text{if } G_n = F_n \\ 100F_{2n+1} & \text{if } G_n = L_n; \end{cases} \tag{12}$$

$$(4x^2+1)b_{n+3}^2 + Bg_{n+2}^2 + Bg_{n-1}^2 + (4x^2+1)g_{n-2}^2 = \begin{cases} 2B(4x^2+1)p_{2n+1} & \text{if } b_n = p_n \\ 8B(x^2+1)(4x^2+1)p_{2n+1} & \text{if } b_n = q_n; \end{cases}$$

$$5B_{n+3}^2 + 29B_{n+2}^2 + 29B_{n-1}^2 + 5B_{n-2}^2 = \begin{cases} 290P_{2n+1} & \text{if } B_n = P_n \\ 580P_{2n+1} & \text{if } B_n = Q_n, \end{cases}$$

where $B = B(x) = A(2x)$.

It follows from (12) that

$$2L_{n+3}^2 + 5L_{n+2}^2 + 5L_{n-1}^2 + 2L_{n-2}^2 = 5(2F_{n+3}^2 + 5F_{n+2}^2 + 5F_{n-1}^2 + 2F_{n-2}^2);$$

this is obviously true since $L_n^2 - 5F_n^2 = 4(-1)^n$ [9]. Likewise,

$$5Q_{n+3}^2 + 29Q_{n+2}^2 + 29Q_{n-1}^2 + 5Q_{n-2}^2 = 2(5P_{n+3}^2 + 29P_{n+2}^2 + 29P_{n-1}^2 + 5P_{n-2}^2)$$

is also true since $Q_n^2 - 2P_n^2 = (-1)^n$ [6].

Corollary 2, coupled with Corollary 1, yields the following result.

Corollary 3. *Let $g_n = f_n$ or l_n. Then*

$$g_{n+2}^2 + g_{n-1}^2 = \begin{cases} (x^2+1)f_{2n+1} & \text{if } g_n = f_n \\ \Delta^2(x^2+1)f_{2n+1} & \text{if } g_n = l_n. \end{cases} \tag{13}$$

Consequently,

$$G_{n+2}^2 + G_{n-1}^2 = \begin{cases} 2F_{2n+1} & \text{if } G_n = F_n \\ 10F_{2n+1} & \text{if } G_n = L_n; \end{cases}$$

$$b_{n+2}^2 + b_{n-1}^2 = \begin{cases} (4x^2+1)p_{2n+1} & \text{if } b_n = p_n \\ 4(x^2+1)p_{2n+1} & \text{if } b_n = q_n; \end{cases}$$

$$B_{n+2}^2 + B_{n-1}^2 = \begin{cases} 5P_{2n+1} & \text{if } B_n = P_n \\ 10P_{2n+1} & \text{if } B_n = Q_n. \end{cases}$$

4. Additional Consequences

Next we employ the relationships in Table 1 to investigate the implications of Corollaries 1–3 to the Jacobsthal, Vieta, and Chebyshev subfamilies.

4.1. Jacobsthal Implications

4.1.1. Corollary 1 Revisited

Letting $u = 1/\sqrt{x}$ in identity (8) yields

$$x(x+1)g_{n+3}^2 = (x^2+3x+1)g_{n+2}^2 + (x^2+3x+1)g_{n-1}^2 - x(x+1)g_{n-2}^2, \qquad (14)$$

where $g_n = g_n(u)$.

Suppose $g_n = f_n$. Multiplying the corresponding equation with x^{n+2}, we get

$$(x+1)J_{n+3}^2(x) = (x^2+3x+1)J_{n+2}^2(x) + (x^2+3x+1)x^3J_{n-1}^2(x) - (x+1)x^5J_{n-2}^2(x).$$

When $g_n = l_n$, equation (14) similarly yields

$$(x+1)j_{n+3}^2(x) = (x^2+3x+1)j_{n+2}^2(x) + (x^2+3x+1)x^3j_{n-1}^2(x) - (x+1)x^5j_{n-2}^2(x).$$

Combining the two cases, we have

$$(x+1)c_{n+3}^2 = (x^2+3x+1)c_{n+2}^2 + (x^2+3x+1)x^3c_{n-1}^2 - (x+1)x^5c_{n-2}^2. \qquad (15)$$

Consequently,

$$3C_{n+3}^2 = 11C_{n+2}^2 + 88C_{n-1}^2 - 96C_{n-2}^2.$$

4.1.2. Corollary 2 Revisited

With $u = 1/\sqrt{x}$ in identity (11) and some basic algebra, we get

$$\begin{aligned}(x+1)x^3g_{n+3}^2 + Bx^2g_{n+2}^2 + Bx^2g_{n-1}^2 & \\ + (x+1)x^3g_{n_2}^2 &= \begin{cases} 2Bx(x+1)f_{2n+1} & \text{if } g_n = f_n \\ 2B(x+1)(4x+1)f_{2n+1} & \text{if } g_n = l_n, \end{cases}\end{aligned} \qquad (16)$$

where $B = B(x) = x^2+3x+1$.

Let $g_n = f_n$. Multiplying (16) with x^{n+2}, this yields

$$(x+1)J_{n+3}^2(x) + BJ_{n+2}^2(x) + Bx^3J_{n-1}^2(x) + (x+1)x^5J_{n-2}^2(x) = 2B(x+1)J_{2n+1}(x).$$

When $g_n = l_n$, equation (16) similarly yields

$$(x+1)j_{n+3}^2(x) + Bj_{n+2}^2(x) + Bx^3j_{n-1}^2(x) + (x+1)x^5j_{n-2}^2(x) = 2B(x+1)(4x+1)J_{2n+1}(x).$$

Combining the two cases, we have

$$\begin{aligned}(x+1)c_{n+3}^2 + Bc_{n+2}^2 + Bx^3c_{n-1}^2 & \\ + (x+1)x^5c_{n-2}^2 &= \begin{cases} 2B(x+1)J_{2n+1}(x) & \text{if } c_n = J_n(x) \\ 2B(x+1)(4x+1)J_{2n+1}(x) & \text{if } c_n = j_n(x). \end{cases}\end{aligned} \qquad (17)$$

Consequently, we have

$$3C_{n+3}^2 + 11C_{n+2}^2 + 88C_{n-1}^2 + 96C_{n-2}^2 = \begin{cases} 66J_{2n+1} & \text{if } C_n = J_n \\ 594J_{2n+1} & \text{if } C_n = j_n. \end{cases} \qquad (18)$$

Identity (18) has an interesting byproduct:

$$3j_{n+3}^2+11j_{n+2}^2+88j_{n-1}^2+96j_{n-2}^2=9(3J_{n+3}^2+11J_{n+2}^2+88J_{n-1}^2+96J_{n-2}^2);$$

this is clearly true since $j_n^2-9J_n^2=(-1)^n 2^{n+2}$.

It follows from equations (15) and (17) that

$$c_{n+2}^2+x^3c_{n-1}^2=\begin{cases}(x+1)J_{2n+1}(x) & \text{if } c_n=J_n(x)\\ (x+1)(4x+1)J_{2n+1}(x) & \text{if } c_n=j_n(x).\end{cases}$$

This implies

$$C_{n+2}^2+8C_{n-1}^2=\begin{cases}3J_{2n+1} & \text{if } C_n=J_n\\ 27J_{2n+1} & \text{if } C_n=j_n.\end{cases}$$

4.2. Vieta Implications

4.2.1. Corollary 1 Revisited

Let $t=-ix$. It then follows from identity (8) that

$$(1-x^2)g_{n+3}^2=(x^4-3x^2+1)g_{n+2}^2+(x^4-3x^2+1)g_{n-1}^2-(1-x^2)g_{n-2}^2, \quad (19)$$

where $g_n=g_n(t)$.

Suppose $g_n=f_n$. Multiplying the resulting equation with i^{2n+4} yields

$$(x^2-1)V_{n+3}^2=(x^4-3x^2+1)V_{n+2}^2-(x^4-3x^2+1)V_{n-1}^2+(x^2-1)V_{n-2}^2.$$

When $g_n=l_n$, we similarly get

$$(x^2-1)v_{n+3}^2=(x^4-3x^2+1)v_{n+2}^2-(x^4-3x^2+1)v_{n-1}^2+(x^2-1)v_{n-2}^2.$$

Thus

$$(x^2-1)d_{n+3}^2=(x^4-3x^2+1)d_{n+2}^2-(x^4-3x^2+1)d_{n-1}^2+(x^2-1)d_{n-2}^2. \quad (20)$$

4.2.2. Fibonacci and Lucas Byproducts

Let $z=x^2+2$. Since $xV_n(z)=f_{2n}(x)$ and $xv_n(z)=l_{2n}(x)$, identity (20) has Fibonacci and Lucas implications. Replacing x with z in equation (20) and multiplying the resulting equation with x^2 yield

$$Ag_{2n+6}^2=Bg_{2n+4}^2-Bg_{2n-2}^2+Ag_{2n-4}^2, \quad (21)$$

where $A=A(x)=x^4+4x^2+3$ and $B=B(x)=x^8+8x^6+21x^4+20x^2+5$.

This implies

$$\begin{aligned}8G_{2n+6}^2 &= 55G_{2n+4}^2-55G_{2n-2}^2+8G_{2n-4}^2;\\ A(2x)b_{2n+6}^2 &= B(2x)b_{2n+4}^2-B(2x)b_{2n-2}^2+A(2x)b_{2n-4}^2;\\ 35B_{2n+6}^2 &= 1189B_{2n+4}^2-1189B_{2n-2}^2+35B_{2n-4}^2.\end{aligned}$$

(For the curious-minded, we add that 1189 is a fascinating number [9].)

4.2.3. Jacobsthal Byproducts

It follows by the relationships $J_{2n}(x) = x^{n-1}V_n(w)$ and $j_{2n}(x) = x^n v_n(w)$ that identity (20) has Jacobsthal byproducts also, where $w = (2x+1)/x$. Replacing x with w in (20) yields

$$(3x^2+4x+1)x^2 d_{n+3}^2 = Dd_{n+2}^2 - Dd_{n-1}^2 + (3x^2+4x+1)x^2 d_{n-2}^2, \qquad (22)$$

where $d_n = d_n(w)$ and $D = D(x) = 5x^4 + 20x^3 + 21x^2 + 8x + 1$.

Suppose $d_n = V_n$. Multiplying (21) with x^{2n+4} yields

$$(3x^2+4x+1)J_{2n+6}^2(x) = DJ_{2n+4}^2(x) - Dx^6 J_{2n-2}^2(x) + (3x^2+4x+1)x^{10} J_{2n-4}^2(x).$$

Similarly, when $d_n = v_n$, we get

$$(3x^2+4x+1)j_{2n+6}^2(x) = Dj_{2n+4}^2(x) - Dx^6 j_{2n-2}^2(x) + (3x^2+4x+1)x^{10} j_{2n-4}^2(x).$$

Combining the two cases, we have

$$\begin{aligned}(3x^2+4x+1)c_{2n+6}^2 &= Dc_{2n+4}^2 - Dx^6 c_{2n-2}^2 + (3x^2+4x+1)x^{10} c_{2n-4}^2\\ 21C_{2n+6}^2 &= 341C_{2n+4}^2 - 21{,}824C_{2n-2}^2 + 21{,}504C_{2n-4}^2. \qquad (23)\end{aligned}$$

4.2.4. Corollary 2 Revisited

With $t = -ix$, it follows from identity (11) that

$$\begin{aligned}(1-x^2)g_{n+3}^2 + Eg_{n+2}^2 + Eg_{n-1}^2&\\ + (1-x^2)g_{n-2}^2 &= \begin{cases} 2(1-x^2)Ef_{2n+1} & \text{if } g_n = f_n\\ 2(1-x^2)(4-x^2)Ef_{2n+1} & \text{if } g_n = l_n, \end{cases} \qquad (24)\end{aligned}$$

where $g_n = g_n(t)$ and $E = E(x) = x^4 - 3x^2 + 1$.

Multiply equation (24) with i^{2n+4} when $g_n = f_n$, and with i^{2n+6} when $g_n = l_n$. Combining both cases, we get

$$\begin{aligned}(x^2-1)d_{n+3}^2 + Ed_{n+2}^2 - Ed_{n-1}^2&\\ - (x^2-1)d_{n-2}^2 &= \begin{cases} 2(x^2-1)EV_{2n+1} & \text{if } d_n = V_n\\ 2(x^2-1)(x^2-4)EV_{2n+1} & \text{if } d_n = v_n. \end{cases} \qquad (25)\end{aligned}$$

It follows by (20) and (25) that

$$d_{n+2}^2 - d_{n-1}^2 = \begin{cases} (x^2-1)V_{2n+1} & \text{if } d_n = V_n\\ (x^2-1)(x^2-4)V_{2n+1} & \text{if } d_n = v_n. \end{cases}$$

By virtue of the relationships $xV_n(z) = f_{2n}(x)$ and $xv_n(z) = l_{2n}(x)$, (25) has Fibonacci, Lucas, and Pell consequences (in the interest of brevity, we omit all the details):

$$\begin{aligned}
Ag^2_{2n+6} + Bg^2_{2n+4} - Bg^2_{2n-2} - Ag^2_{2n-4} &= \begin{cases} 2xABf_{4n+2} & \text{if } g_n = f_n \\ 2x(x^2+4)ABf_{4n+2} & \text{if } g_n = l_n; \end{cases} \qquad (26)\\
8G^2_{2n+6} + 55G^2_{2n+4} - 55G^2_{2n-2} - 8G^2_{2n-4} &= \begin{cases} 880F_{4n+2} & \text{if } G_n = F_n \\ 4400F_{4n+2} & \text{if } G_n = L_n; \end{cases}\\
Hb^2_{2n+6} + Kb^2_{2n+4} - Kb^2_{2n-2} - Hb^2_{2n-4} &= \begin{cases} 4xHKp_{4n+2} & \text{if } b_n = p_n \\ 16x(x^2+1)HKp_{4n+2} & \text{if } b_n = q_n; \end{cases}\\
35B^2_{2n+6}+1189B^2_{2n+4}-1189B^2_{2n-2}-35B^2_{2n-4} &= \begin{cases} 166,460P_{4n+2} & \text{if } B_n = P_n \\ 332,920P_{4n+2} & \text{if } B_n = Q_n, \end{cases}
\end{aligned}$$

where $A = A(x) = x^4+4x^2+3; B = B(x) = x^8+8x^6+21x^4+20x^2+5; H = A(2x)$; and $K = B(2x)$.

It follows from (22) and (26) that

$$g^2_{2n+4} - g^2_{2n-2} = \begin{cases} x(x^4+4x^2+3)f_{4n+2} & \text{if } g_n = f_n \\ x(x^2+4)(x^4+4x^2+3)f_{4n+2} & \text{if } g_n = l_n; \end{cases}$$

This implies

$$G^2_{2n+4} - G^2_{2n-2} = \begin{cases} 8F_{4n+2} & \text{if } G_n = F_n \\ 40F_{4n+2} & \text{if } G_n = L_n. \end{cases}$$

4.2.5. Jacobsthal Byproducts

As before, identity (25) also has Jacobsthal implications. Replacing x with $w = (2x+1)/x$, it yields

$$\begin{aligned}
&Fx^6d^2_{n+3} + Dx^4d^2_{n+2} - Dx^4d^2_{n-1}\\
&\qquad - Fx^6d^2_{n-2} = \begin{cases} 2DFx^2V_{2n+1} & \text{if } d_n = V_n \\ 2DF(4x+1)V_{2n+1} & \text{if } d_n = v_n, \end{cases} \qquad (27)
\end{aligned}$$

where $d_n = d_n(w); D = D(x) = 5x^4+20x^3+21x^2+8x+1$; and $F = F(x) = 3x^2+4x+1$.

Considering the cases $d_n = V_n$ or v_n, we then get

$$Fc^2_{2n+6}+Dc^2_{2n+4}-Dx^6c^2_{2n-2}-Fx^{10}c^2_{2n-4} = \begin{cases} 2DFx^2J_{4n+2}(x) & \text{if } c_n = J_n(x) \\ 2DF(4x+1)J_{4n+2}(x) & \text{if } c_n = j_n(x). \end{cases}$$

In particular, we have

$$\begin{aligned}
&21C^2_{2n+6} + 341C^2_{2n+4} - 21,824C^2_{2n-2}\\
&\qquad - 21,504C^2_{2n-4} = \begin{cases} 14,322J_{4n+2} & \text{if } C_n = J_n \\ 128,898J_{4n+2} & \text{if } C_n = j_n. \end{cases} \qquad (28)
\end{aligned}$$

It follows by (23) and (28) that

$$341C_{2n+4}^2 - 21,824C_{2n-2}^2 = \begin{cases} 7,161J_{4n+2} & \text{if } C_n = J_n \\ 64,449J_{4n+2} & \text{if } C_n = j_n. \end{cases}$$

Next we present the Chebyshev consequences of the corollaries.

5. Chebyshev Implications

5.1. Corollary 1 Revisited

Replacing x with $2x$ in identity (20) results in the equation

$$(4x^2-1)d_{n+3}^2 = (16x^4-12x^2+1)d_{n+2}^2 - (16x^4-12x^2+1)d_{n-1}^2 + (4x^2-1)d_{n-2}^2,$$

where $d_n = d_n(2x)$. Considering the cases $d_n = V_n$ or v_n, we get

$$Le_{n+3}^2 = Me_{n+2}^2 - Me_{n-1}^2 + Le_{n-2}^2, \tag{29}$$

where $L = L(x) = 4x^2 - 1$ and $M = M(x) = 16x^4 - 12x^2 + 1$.

5.2. Corollary 2 Revisited

Using similar steps, it follows from identity (25) that

$$Le_{n+3}^2 + Me_{n+2}^2 - Me_{n-1}^2 - Le_{n-2}^2 = \begin{cases} 2LMU_{2n+2} & \text{if } e_n = U_n \\ 2LM(x^2-4)U_{2n} & \text{if } e_n = T_n. \end{cases} \tag{30}$$

5.3. Corollary 3 Revisited

Identity (29), together with (30), implies that

$$e_{n+2}^2 - e_{n-1}^2 = \begin{cases} (4x^2-1)U_{2n+2} & \text{if } e_n = U_n \\ (x^2-4)(4x^2-1)U_{2n} & \text{if } e_n = T_n. \end{cases}$$

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