A COMBINATORIAL PROOF OF AN ORDERED-PARTITION EXPANSION OF DETERMINANTS GIVEN BY INSKO, JOHNSON AND SULLIVAN

Sajal Kumar Mukherjee [1]
Department of Mathematics, Visva-Bharati, Santiniketan, India
shyamal.sajalmukherjee@gmail.com

A. K. Bhuniya
Department of Mathematics, Visva-Bharati, Santiniketan, India
anjankbhuniya@gmail.com

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Abstract

In this paper we give a purely combinatorial proof of an ordered-partition expansion of determinants inspired by multivariate finite operator calculus. This argument also includes a combinatorial proof of an interesting identity about Stirling numbers of the second kind. Also, we give a topological proof of Ryser's formula for permanents.

1. Introduction

In [1], Erik Insko, Katie Johnson and Shaun Sullivan proved a Ryser-type formula for determinants. They called it "An ordered-partition expansion of determinants." This expansion evolved from a conjecture about a transfer formula in multivariate finite operator calculus. Before stating their theorem, let us define some notation and terminology. We will denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let S be a finite set, then an *ordered set partition* (in short, an *ordered partition*) of S is an ordered tuple $(\beta_1, \beta_2, \ldots, \beta_r)$ of pairwise disjoint subsets, whose union is S. For example, $(\{1, 3\}, \{2\}), (\{2\}, \{1, 3\})$ are two (there are many others) ordered partitions of [3]. Now let us state the theorem.

Theorem 1. *For a square matrix $A = (a_{ij})_{n\times n}$,*

$$|A| = \sum_{B\vdash[n]} (-1)^{n-|B|} \prod_{\beta_k\in B} \prod_{i\in\beta_k} \sum_{j\in\acute{\beta}_k} a_{ij},$$

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where the outer summation runs over all ordered partitions $B = (\beta_1, \beta_2, \ldots, \beta_r)$ of $[n]$, and the inner summation runs over all integers j in the union of $\acute{\beta_k} = \bigcup_{l=1}^{k} \beta_l$ of first k parts of the partition B.

For a function $f : [n] \to [n]$, define $a_f := \prod_{i=1}^{n} a_{if(i)}$. Note that a_f is defined to be $\prod_{i=1}^{n} a_{f(i)i}$ in [1]. But this presents no additional problem in the present situation, as the determinant of a matrix and its transpose are equal, and the same is true for permanents also. Now, the determinantal expression in Theorem 1.1 can be restated as $|A| = \sum_{B \vdash [n]} (-1)^{n-|B|} (\sum_f a_f)$; where the inner summation runs over all functions $f : [n] \to [n]$ satisfying the following property: if $i \in \beta_k$, then $f(i) \in \bigcup_{j=1}^{k} \beta_j$, where $B = (\beta_1, \beta_2, \ldots, \beta_r)$. After expansion, this will take the form $\sum_f c_f a_f$, f running through all the functions $f : [n] \to [n]$. In [1], Erik Insko, Katie Johnson and Shaun Sullivan proved that if f is bijective $c_f = sign(f)$, and for nonbijective f they showed c_f to be zero by an elegant topological argument analyzing the Euler characteristics of subsets of a suitably chosen permutahedron. Since this proof is highly topological and their expansion also resembles Ryser's formula for permanents, stating that for a square matrix $A = (a_{ij})_{n \times n}$, $perm(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} a_{ij}$, the authors asked two questions at the end of their paper [1] :

1. Is there any combinatorial proof of Theorem 1 in [1]?
2. Is it possible to prove Ryser's formula topologically in the spirit of their argument in Theorem 1 in [1]?

In this paper, we answer these two questions, i.e., we give a simple combinatorial argument to prove the "ordered-partition expansion" and also we give a topological proof of Ryser's formula at the end. From now on we will use IJS to abbreviate the names of the authors of [1].

2. Combinatorial Proof of the Expansion

Before proceeding, we recall a definition from [1]. For a function $f : [n] \to [n]$, define $S_f := \{B \vdash [n] : i \in \beta_k \Rightarrow f(i) \in \bigcup_{i=1}^{k} \beta_j \ \forall k\}$, where $B = (\beta_1, \beta_2, \ldots, \beta_r)$.

Let us give a short description of the structure of the poset of ordered partitions. Let $\mathcal{P}_n$ denote the poset of ordered partitions of the set $[n]$. At the top of the poset $\mathcal{P}_n$, we have $n!$ ordered partitions consisting of singletons. IJS call them 'singleton partitions.' Directly below a given ordered partition $B = (\beta_1, \beta_2, \ldots, \beta_r)$ in $\mathcal{P}_n$ are the ordered partitions $B_i = (\beta_1, \beta_2, \ldots, \beta_i \cup \beta_{i+1}, \ldots, \beta_r)$, where $1 \le i \le r-1$, i.e., directly below a given ordered partition B in $\mathcal{P}_n$ are the ordered partitions formed by taking the union of two consecutive parts in B. We label in $\mathcal{P}_n$ as follows: "1st label " consists of all the singleton partitions, "2nd label" consists of all the partitions lying just below the "1st label" partitions, and so on.

Now to prove the theorem combinatorially, we just need to prove that if f is bijective then $c_f = sign(f)$, and if f is not bijective, then $c_f = 0$. Now, for the case when f is not bijective, it suffices to prove combinatorially that $c_f = 0$ whenever f is acyclic(i.e., $f^k(i) = i$ implies $k = 1$ for all $i \in [n]$). IJS [1] showed how to reduce the problem of calculating coefficients c_f, when f is not bijective, to that of calculating the coefficients $c_{\overline{f}}$ corresponding to acyclic functions $\overline{f} : [n] \to [n]$ (see Lemma 2, in [1]). Hence, to address Question 1 we need only to prove combinatorially the following result.

Proposition 1. *Let $f : [n] \to [n]$ be a function. Then*

1. $c_f = \sum_{B \in S_f} (-1)^{n-|B|} = 0$ *if f is an acyclic non-identity function.*
2. $c_f = sign(f)$, *if f is bijective.*

Proof. 1. Since f is acyclic, f can be very naturally viewed as a rooted forest as described in [1], where the fixed points of f are the roots and each component tree contains a single root. We also call the rooted forest associated to f as f. Now since f is nonidentity, f has a component having more than one element. Without loss of generality let 1 be the root of this component.

Let $\acute{S}_f = \{B \in S_f : \beta_1$, the first part of B, contains $1\}$. If a singleton partition belongs to $\acute{S}_f$, then every partition below it must belong to $\acute{S}_f$. Conversely, let $B \in \acute{S}_f$ and $B = (\beta_1, \beta_2, \ldots, \beta_r)$, where $1 \in \beta_1$. We prove that there exists a singleton partition $\acute{B} \in \acute{S}_f$ such that $\acute{B} \geq B$ in $\mathcal{P}_n$. To do this, first place the 1 at the beginning. Now look at the remaining elements of β_1. Let $\beta_1 \setminus \{1\} = A_1 \bigcup A_2 \bigcup \cdots \bigcup A_k$, where for any particular i, each element of A_i belongs to the same component of f and $A_i \bigcap A_j = \emptyset$ for $i \neq j$. In other words, A_i's are the branches of the tree rooted at 1. Now order each element of A_1 by their distance from the root of their mother component (elements having the same distance from the root can be put in any order). Do this for A_2, A_3 and so on. Continue the same method for β_2, β_3 and so on and we get our desired $\acute{B}$. So $\acute{S}_f$ consists of all the singleton partitions in S_f with 1 at the beginning and all the partitions below them.

Now we pair up all the partitions of $\acute{S}_f$ in an interesting way. Let $1, x_2, x_3, \ldots, x_n$ be a typical 1st label partition in $\acute{S}_f$. This is an abuse of notation. By $1, x_2, x_3, \ldots, x_n$ we mean the ordered tuple $(1, x_2, x_3, \ldots, x_n)$. We pair this element with a 2nd label partition $\{1, x_2\}, x_3, x_4, \ldots, x_n$. Do this for all 1st label partitions in $\acute{S}_f$.

Now, which 2nd label partitions in $\acute{S}_f$ are still unpaired? A typical partition of this type is of the form $1, y_2, y_3, \ldots, \{y_i, y_{i+1}\} \ldots, y_n$. We pair these partitions with the 3rd label partitions as follows: $1, y_2, y_3, \ldots, \{y_i, y_{i+1}\} \ldots, y_n$ is paired with $\{1, y_2\}, y_3, y_4, \ldots, \{y_i, y_{i+1}\}, \ldots, y_n$.

Now we ask the same question, i.e., which 3rd label partitions are still unpaired? The answer is the partitions with singleton one as their first part, i.e., of the form

$1, \alpha_1, \alpha_2, \ldots, \alpha_k$ and we pair these partitions with the 4th label partitions as follows: $1, \alpha_1, \alpha_2, \ldots, \alpha_k$ is paired with $\{1\} \bigcup \alpha_1, \alpha_2, \ldots, \alpha_k$. We proceed this way until there would be no unpaired partitions. The way we paired up the partitions in $\acute{S}_f$ immediately tells us that the contribution of them in c_f is zero, i.e., $\sum_{B \in \acute{S}_f} (-1)^{n-|B|} = 0$.

Now let $k \geq 1$ and $\beta = \beta_1, \beta_2, \ldots, \beta_k, \beta_{k+1}, \beta_{k+2}, \ldots, \beta_r$ be a partition in S_f, where $1 \in \beta_{k+1}$. We fix this partition. Let $S_f^\beta = \{\gamma \in S_f : \gamma = \beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_s$ and $1 \in \gamma_1\}$. Note that $\beta \in S_f^\beta$. Let $M = [n] \setminus (\beta_1 \bigcup \beta_2 \bigcup \cdots \bigcup \beta_k)$. Since 1 is a root of f, $|M| > 1$. Let us denote $\mathcal{P}^1(M)$ to be the set of all ordered partitions of the set M, with 1 belonging to the first part. So $S_f^\beta = \{\gamma \in S_f : \gamma = \beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_s$, where $\gamma_1, \gamma_2, \ldots, \gamma_s \in \mathcal{P}^1(M)\}$. So each partition of S_f^β is of the form $(\beta_1, \beta_2, \ldots, \beta_k)$ followed by an element of $\mathcal{P}^1(M)$. Now since $|M| > 1$, we can apply our previous argument to pair up each element of S_f^β just forgetting the part $(\beta_1, \beta_2, \ldots, \beta_k)$ of each element, so that their contribution in c_f is zero, i.e., $\sum_{B \in S_f^\beta} (-1)^{n-|B|} = 0$. Since β is arbitrary, we conclude that $c_f = 0$.

2. Assume that f is bijective. In this case, Corollary 1 of [1] proved that $c_f = sgn(f)$ by a direct consequence of the identity $\sum_{k=0}^n (-1)^{n-k} k! S(n,k) = 1$, where $S(n,k)$ is the Stirling number of 2nd kind. Here we give a combinatorial proof of this identity almost identical to that of the acyclic case. Actually we have $\sum_{k=0}^n (-1)^{n-k} k! S(n,k) = \sum_{B \in \mathcal{P}_n} (-1)^{n-|B|}$. Note that $S_I = \mathcal{P}_n$, where I is the identity mapping from $[n] \to [n]$. Let us consider the set of all partitions in $\mathcal{P}_n$ with singleton $\{n\}$ as the last part and call this set $\acute{\mathcal{P}_n}$. Since the elements of $\mathcal{P}_n \setminus \acute{\mathcal{P}_n}$ can be paired up exactly the same way as the previous argument(replacing 1 by n), we have $\sum_{B \in \mathcal{P}_n} (-1)^{n-|B|} = \sum_{B \in \acute{\mathcal{P}_n}} (-1)^{n-|B|}$. But elements of $\acute{\mathcal{P}_n}$ are just the elements of $\mathcal{P}_{n-1}$ followed by a singleton $\{n\}$. Hence we have $\sum_{B \in \acute{\mathcal{P}_n}} (-1)^{n-|B|} = -\sum_{B \in \mathcal{P}_{n-1}} (-1)^{n-|B|} = \sum_{B \in \mathcal{P}_{n-1}} (-1)^{(n-1)-|B|} = 1$(by induction, since initially $(-1)^{1-1} 1! S(1,1) = 1$), where we assume $B = \acute{B} \cup \{n\}$. $\square$

3. Topological Proof of Ryser's Formula

Recall that Ryser's formula for the permanent of a square matrix $A = (a_{ij})_{n \times n}$ is $perm(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^n \sum_{j \in S} a_{ij}$, which can be restated as $perm(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} (\sum_{f:[n] \to S} a_f)$. In the following result we give a topological proof of Ryser's formula for permanents of a square matrix.

Proposition 2. *If $A = (a_{ij})_{n \times n}$ is square matrix, then*

$$perm(A) = \sum_{S \subseteq [n]} (-1)^{n-|S|} (\sum_{f:[n] \to S} a_f).$$

Proof. Consider the expression $\sum_{S \subseteq [n]} (-1)^{n-|S|} (\sum_{f:[n] \to S} a_f)$. Upon expansion this would be of the form $\sum_{f:[n] \to [n]} d_f a_f$. If f is surjective then it is immediate that $d_f = 1$. So to prove Ryser's formula, it suffices to prove that $d_f = 0$, if f is not surjective. Suppose that range$(f) = S \subset [n]$. Then $d_f = \sum_{\acute{S} \supseteq S} (-1)^{n-|\acute{S}|}$. Now $\sum_{\acute{S} \subseteq [n]} (-1)^{n-|\acute{S}|} = \sum_{\acute{S} \supseteq S} (-1)^{n-|\acute{S}|} + \sum_{\acute{S}} (-1)^{n-|\acute{S}|}$, where the 2nd summation on the R.H.S of the above equation runs through all subsets of $[n]$ not containing S. Now $\sum_{\acute{S} \subseteq [n]} (-1)^{n-|\acute{S}|} = (-1)^n \sum_{\acute{S} \subseteq [n]} (-1)^{|\acute{S}|} = (-1)^n \chi(\Delta^{n-1}) = (-1)^n$, where Δ^{n-1} is the standard simplex and $\chi(\Delta^{n-1})$ is its Euler characteristic, which is equal to 1. Now, the subsets of $[n]$, not containing S form the faces of an abstract simplicial complex Δ with vertex set $[n]$. But how does Δ look like? In fact, $\Delta = x_1 * x_2 * \cdots * x_{n-|S|} * (\partial \Delta^{|S|-1})$, where $\Delta^{|S|-1}$ is the simplex with vertex set S and $\partial \Delta^{|S|-1}$ is its boundary sphere, $'*'$ is the usual simplicial join operation(see [2], page 12) and $[n] \setminus S = \{x_1, x_2, \ldots x_{n-|S|}\}$. Since Δ is an iterated cone, its geometric realization $|\Delta|$ (see [2], page 16) is contractible and so $\chi(|\Delta|) = 1$. Hence $d_f = \sum_{\acute{S} \supseteq S} (-1)^{n-|\acute{S}|} = (-1)^n - (-1)^n = 0$. $\square$

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