COMPLEMENTARY FAMILIES OF THE FIBONACCI-LUCAS RELATIONS

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Abstract

In this paper we present two families of Fibonacci-Lucas identities, with the Sury's identity being the best known representative of one of the families. While these results can be proved by means of the basic identity relating Fibonacci and Lucas sequences we also provide a bijective proof. Both families are then treated by generating functions.

1. Introduction

The Fibonacci and Lucas sequences of numbers are defined as

$$F_{n+2} = F_{n+1} + F_n, \;\; F_0 = 0, \;\; F_1 = 1, \tag{1}$$

$$L_{n+2} = L_{n+1} + L_n, \;\; L_0 = 2, \;\; L_1 = 1, \tag{2}$$

where $n \in \mathbb{N}_0$, and denoted by $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$, respectively. Equivalently, these sequences could be defined as the only solutions (x, y), $x = L_n$, $y = F_n$ of the Diophantine equation

$$x^2 - 5y^2 = 4(-1)^n.$$

For natural numbers k, n we consider the number of ways to tile a $k \times n$ rectangle with certain tiles. The area of a 1×1 rectangle is usually called a *cell*. Tiles that are 1×1 rectangles are called *squares* and tiles of dimension 1×2 are called *dominoes*. In particular, when $k = 1$ a rectangle to tile is called *n-board*.

Among many interpretations of the Fibonacci and Lucas numbers, here we use the fact that the number f_n of n-board tilings with squares and dominoes is equal to F_{n+1}, while the number l_n of the same type of tilings of a labeled circular n-board (labeled in the clockwise direction) is equal to L_n. Namely, an n-board tiling begins either with a square or with a domino, meaning that the numbers f_n obey the same recurrence relation as the Fibonacci numbers. Having in mind that the sequence $(f_n)_{n\geq 0}$ begins with $f_1 = 1$, $f_2 = 2$ we get

$$f_n = F_{n+1}. \tag{3}$$

Similar reasoning proves that the number of tilings of a labeled circular board of the length n is equal to the n-th Lucas number,

$$l_n = L_n. \tag{4}$$

A labeled circular board of length n we shall call an *n-bracelet*. We say that a bracelet is *out of phase* when a single domino covers cells n and 1, and *in phase* otherwise. Consequently there are more ways to tile an n-bracelet than an n-board. More precisely, n-bracelet tilings are equinumerous to the sum of tilings of an n-board and $(n-2)$-board,

$$l_n = f_n + f_{n-2}. \tag{5}$$

This follows from the fact that in phase tilings can be unfolded into n-board tilings while out of phase tilings can be straightened into $(n-2)$-board tilings since a single domino is fixed on cells n and 1. Once having (3) and (4), the statement of Lemma 1 follows immediately from relation (5).

Lemma 1. *The n^{th} Lucas number is equal to the sum of $(n-1)^{\text{st}}$ and $(n+1)^{\text{st}}$ Fibonacci number:*

$$L_n = F_{n-1} + F_{n+1}. \tag{6}$$

Lemma 1 presents the most elementary identity involving both the Fibonacci and Lucas sequence. In what follows we use it in order to prove further identities. It is worth mentioning that there are numerous identities known for these sequences. Many of them one can find in the classic reference [9]. An introduction to Fibonacci polynomials can be seen in [5] while recent results on this subject one can find in [1].

2. Colored Tilings and the Product $m^n F_{n+1}$

A tiling either of a board or bracelet of length n with two kinds of tiles, squares or oriented dominoes, in which each cell can be painted with any of m available colors we shall call an (n, m)-tiling. Thus, the orientation of a domino is defined by colors of cells it covers, meaning that for m available colors we have m^2 types of dominoes. Note that within the (n,m)-tilings one can equivalently use dominoes that paint both cells with the same colors and have m possible directions. The third alternative to define such a tiling is to use dominoes in m^2 colors that are different from the colors of squares (see Figure 1). In what follows we shall use the first definition.

We let $c_1, c_2, \ldots, c_m$ denote colors in such a tiling. In particular, when $m = 2$ we choose white and black as colors c_1 and c_2, respectively. In case $m = 3$ we choose white, black and gray for c_1, c_2, c_3, respectively.

One can easily check that there are 8 tilings of a 2-board in two colors, having two types of squares and four types of dominoes (Figure 1). Furthermore, there are 24 such tilings of a 3-board, 80 tilings of a 4-board, etc. In general, there are $2^n f_n$ such tilings of an n-board. The numbers $1, 3, 18, 81, \ldots, 3^n f_n, \ldots$ also have a combinatorial interpretation. They count the ways to tile an n-board with 3 types of squares and 9 types of dominoes. We generalize these facts in the following lemma.

Lemma 2. *The product $m^n f_n$ represents the number of colored n-board tilings with two kinds of tiles, squares and oriented dominoes.*

Proof. There are f_n uncolored tilings of an n-board with squares and dominoes. According to the definition of oriented dominoes, each cell of a tiling can be painted with only one of the m available colors. Thus, each of f_n tilings gives m^n tilings when we use m types of squares and m^2 types of dominoes. □

The same argument proves that $m^n L_n$ represents the number of colored n-bracelet tilings with squares in m colors and m^2 types of dominoes.

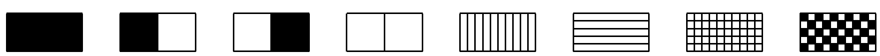

Figure 1: The eight $(2,2)$-tilings of a board of length 2. In general there are $m^n F_{n+1}$ tilings with squares in m colors and dominoes in m^2 colors of a board of length n.

There are a few ways to prove the Fibonacci-Lucas identity (7). It can be proved using Binet's formula [8], and by means of generating functions as well [6]. Here we continue to call it Sury's identity, as it is already done in [6, 7].

Lemma 3. *For the Fibonacci sequence $(F_n)_{n\geq 0}$ and the Lucas sequence $(L_n)_{n\geq 0}$*

$$\sum_{k=0}^{n} 2^k L_k = 2^{n+1} F_{n+1}. \tag{7}$$

Proof. We present an induction proof. We use Lemma 1 and relation (4) to immediately get

$$F_{n+1} + L_{n+1} = 2F_{n+2},$$

which will be used to complete the inductive step. The relation (7) obviously holds true for $n = 1$, since $L_0 + 2L_1 = 4F_2$. Furtermore, we have

$$\begin{aligned}\sum_{k=0}^{n} 2^k L_k + 2^{n+1} L_{n+1} &= 2^{n+1} F_{n+1} + 2^{n+1} L_{n+1} \\ \sum_{k=0}^{n+1} 2^k L_k &= 2^{n+1}(F_{n+1} + L_{n+1}) \\ &= 2^{n+2} F_{n+2}\end{aligned}$$

which completes the proof. □

We let $\mathcal{A}_{n,m}$ denote the set of all (n, m)-board tilings. Let $t_{n,w}$ be the unique all-white squares tiling. Then the other tiling must end with a series of white squares (which could be empty) preceded by a domino or a non-white square. Let $\mathcal{A}^d_{n,m}$ and $\mathcal{A}^s_{n,m}$, resprectively, denote the sets of these tilings. Thus,

$$\mathcal{A}_{n,m} = \mathcal{A}^s_{n,m} \cup \mathcal{A}^d_{n,m} \cup \{t_{n,w}\}.$$

We let $\mathcal{B}_{n,m}$ denote the set of all (n, m)-bracelet tilings. It is worth to emphasize that within $(n, 2)$-bracelet tilings the n-th cell can be tiled by

- a square, in colors c_1, c_2,
- a domino in phase, and a domino out of phase.

We let $\mathcal{B}^{c_i}_{n,m}$ denote the set of (n, m)-bracelet tilings ending with a square of color c_i, the $\mathcal{B}^p_{n,m}$ contains in phase tilings ending with a domino and $\mathcal{B}^o_{n,m}$ contains out of phase tilings. Now we have

$$\mathcal{B}_{n,m} = \bigcup_{i=1}^{m} \mathcal{B}^{c_i}_{n,m} \cup \mathcal{B}^p_{n,m} \cup \mathcal{B}^o_{n,m}.$$

It is worth to note that equalities

$$\begin{aligned}&|\mathcal{B}^{c_1}_{n,m}| = |\mathcal{B}^{c_1}_{n,m}| = \cdots = |\mathcal{B}^{c_m}_{n,m}| \\ &|\mathcal{B}^p_{n,m}| = |\mathcal{B}^o_{n,m}|\end{aligned}$$

obviously holds true.

There is also an elegant bijective proof of Lemma 3 [2]. It is based on a 1-to-2 correspondence between the set $\mathcal{B}_{k,2}$ of bracelet tilings and the set $\mathcal{A}_{k,2}$ of board tilings, $1 \leq k \leq n$. Thus, in both sets tiles are colored in two colors (white and black) meaning that there are two types of squares and four types of dominoes. We let $\mathcal{A}^{c_i}_{n,m}$ denote the set of m-colored n-board tilings ending with a square of color c_i, $i = 1, \ldots, m$.

More precisely, we establish a 1-to-1 correspondence between the sets $\mathcal{A}_{k,2}^{c_2}$ and $\mathcal{B}_{k,2}^{c_1}$ as well as between the sets $\mathcal{A}_{k,2}^{d}$ and $\mathcal{B}_{n,2}^{p}$, $1 \leq k \leq n$, by the operations *i)* removing tiles from $k+1$ through n and *ii)* gluing cells k and 1 together.

In the same manner we generate bracelets ending with a black square and out of phase bracelets. Two remaining tilings $t_{n,w}$ consisting from all-white squares are mapped to two 0-bracelets. Thus, twice the number of $(n,2)$-board tilings is needed to establish correspondence with bracelets of length at most n that are tiled with two types of squares and four types of dominoes. Having in mind $|\mathcal{A}_{n,2}| = 2^n f_n$, we have

$$\sum_{k=0}^{n} 2^k L_k = 2 \cdot 2^n f_n = 2^{n+1} F_{n+1},$$

which completes the proof.

3. The Main Result

The next result, Theorem 1, gives an extension of the relation (7). It provides the answer whether there is an identity involving the product $3^n l_n$, respectively $3^n f_n$ (which appears in some other contexts [4]). We prove it combinatorially while a proof by means of Lemma 1 is also possible.

Theorem 1. *For the Fibonacci sequence* $(F_n)_{n\geq 0}$ *and the Lucas sequence* $(L_n)_{n\geq 0}$

$$\sum_{k=0}^{n} 3^k (L_k + F_{k+1}) = 3^{n+1} F_{n+1}. \tag{8}$$

Proof. Note that within $(n,3)$-bracelet tilings the n-th cell can be tiled by

- a square, in one of the colors c_1, c_2, c_3,
- a domino in phase, and a domino out of phase.

According to the previouis notations, the sets $\mathcal{B}_{n,3}^{c_1}$, $\mathcal{B}_{n,3}^{c_2}$ and $\mathcal{B}_{n,3}^{c_3}$ denote those tilings ending with a square in a color c_1, c_2, c_3, respectively, $\mathcal{B}_{n,3}^{p}$ denote tilings ending with a domino in phase and $\mathcal{B}_{n,3}^{o}$ denote tilings with a domino out of phase.

The set $\mathcal{A}_{n,3}$ of $(n,3)$-board tilings we divide into disjoint subsets $\mathcal{A}_{n,3}^{s}$, $\mathcal{A}_{n,3}^{d}$ and $\{t_{n,w}\}$,

$$\mathcal{A}_{n,3} = \mathcal{A}_{n,3}^{s} \cup \mathcal{A}_{n,3}^{d} \cup \{t_{n,w}\}.$$

In the set $\mathcal{A}_{n,3}^{s}$, there are tilings consisting of a nonwhite square covering cell k and white squares on cells $k+1$ through n, $1 \leq k \leq n$. The set $\mathcal{A}_{n,3}^{d}$ contains tilings having a domino on cell k while white squares cover cells $k+1$ through n. Now, removing tiles from $k+1$ through n and gluing cells k and 1 together, from $\mathcal{A}^s_{n,3}$ we obtain $(k,3)$-bracelet tilings ending with gray and those tilings ending with a black square. Thus,

$$|\mathcal{A}^s_{n,3}| = |\mathcal{B}^{c_2}_{n,3}| + |\mathcal{B}^{c_3}_{n,3}|.$$

By the same operations, from the set $\mathcal{A}^d_{n,3}$ we get $(k,3)$-bracelet tilings ending with a domino in phase,

$$|\mathcal{A}^d_{n,3}| = |\mathcal{B}^p_{n,3}|.$$

The board tiling $t_{n,w}$ consisting of all-white squares is mapped to the 0-bracelet.

In the same way we establish a correspondence between $(n,3)$-board tilings and out of phase $(k,3)$-bracelet tilings together with twice the number of $(k,3)$-bracelets ending with a white square,

$$|\mathcal{A}^s_{n,3}| = |\mathcal{B}^{c_1}_{n,3}| + |\mathcal{B}^{c_3}_{n,3}|,$$
$$|\mathcal{A}^d_{n,3}| = |\mathcal{B}^o_{n,3}|.$$

The extra set $\mathcal{B}^{c_3}_{n,3}$ of $(k,3)$-bracelets ending with black square can be unfolded into $(k-1,3)$-board tilings (according to the arguments we use when proving Lemma 1). This means that relation

$$\sum_{i=1}^{3} |\mathcal{B}^{c_i}_{n,3}| + |\mathcal{B}^p_{n,3}| + |\mathcal{B}^o_{n,3}| + 1 + \sum_{k=0}^{n-1} 3^k F_{k+1} = 2|\mathcal{A}_{n,3}|$$

holds true. Furthermore, we have

$$\sum_{k=0}^{n} 3^k L_k + \sum_{k=0}^{n-1} 3^k F_{k+1} = 2 \cdot 3^n F_{n+1}$$

and finally

$$\sum_{k=0}^{n} 3^k L_k + \sum_{k=0}^{n} 3^k F_{k+1} = 3^{n+1} F_{n+1},$$

which completes the proof. □

For example, when $n = 3$ then

$$|\mathcal{B}^{c_i}_{n,3}| = 18 + 3 + 1 = 22, \quad i = 1,2,3,$$
$$|\mathcal{B}^p_{n,3}| = |\mathcal{B}^o_{n,3}| = 27 + 9 = 36$$

and

$$|\mathcal{A}^s_{n,3}| = 36 + 6 + 2 = 44,$$
$$|\mathcal{A}^d_{n,3}| = 27 + 9 = 36.$$

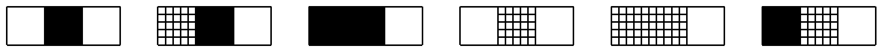

Figure 2: The six tilings in the set $\mathcal{A}^s_{3,3}$, for the case $k = 2$.

Figure 2 illustrates this instance showing $(3, 3)$-board tilings in the set $\mathcal{A}^s_{3,3}$ for which $k = 2$.

Clearly, there is a 1-to-1 correspondence between (n, m)-boards with a domino as the last nonwhite tile and either in phase tilings or out of phase tilings. There is also a 1-to-1 correspondence between boards with a square as the last nonwhite tile and (k, m)-bracelets, $1 \leq k \leq n$ ending with a square in color c_i, $1 \leq i \leq m-1$. In order to establish a 1-to-2 correspondence we have to add $m-2$ sets of $(k-1, m)$-boards to the set of (k, m)-bracelets. This reasoning proves Theorem 2. The identity for the next case $m = 4$ is as follows:

$$\sum_{k=0}^{n} 4^k (L_k + 2F_{k+1}) = 4^{n+1} F_{n+1}. \tag{9}$$

Theorem 2. *For the Fibonacci sequence $(F_n)_{n\geq 0}$ and the Lucas sequence $(L_n)_{n\geq 0}$ of numbers, with $m \geq 2$*

$$\sum_{k=0}^{n} m^k [L_k + (m-2)F_{k+1}] = m^{n+1} F_{n+1}. \tag{10}$$

In Theorem 3 we give a further extension of Theorem 3, to any m. We prove it by applying Lemma 1.

Theorem 3. *The alternating sum of products m^{n-k}, $0 \leq k \leq n$ with Fibonacci and Lucas numbers is equal to either positive or negative value of $(n+1)$-th Fibonacci number,*

$$\sum_{k=0}^{n} (-m)^{n-k} [L_{k+1} + (m-2)F_k] = F_{n+1}. \tag{11}$$

where $m \geq 2$.

Proof. We have

$$\begin{aligned}
&m^n L_1 - m^{n-1} L_2 + m^{n-2} L_3 - \cdots + (-1)^n L_{n+1} \\
&\quad = m^n (F_0 + F_2) - m^{n-1}(F_1 + F_3) + m^{n-2}(F_2 + F_4) - \cdots + (-1)^n (F_n + F_{n+2}) \\
&\quad = m^n F_1 - m^{n-1}(mF_1 + F_2) + m^{n-2}(mF_2 + F_3) - \cdots + (-1)^n (mF_n + F_{n+1}) \\
&\quad = m^n F_1 - m^n F_1 - m^{n-1} F_2 + m^{n-1} F_2 + m^{n-2} F_3 - m^{n-2} F_3 - \cdots + (-1)^n F_{n+1} \\
&\quad = (-1)^n F_{n+1}.
\end{aligned}$$

□

Corollary 1. *The alternating sum of products $2^{n-k}L_{k+1}$, $0 \le k \le n$ is equal to either a positive or negative value of the $(n+1)^{\mathrm{st}}$ Fibonacci number:*

$$\sum_{k=0}^{n}(-1)^k 2^{n-k} L_{k+1} = (-1)^n F_{n+1}. \tag{12}$$

Thus, Theorem 2 and Theorem 3 give complementary families of the Fibonacci-Lucas identities, with the Sury identity as the best known representative of (10).

4. A Generating Functions Approach

Sums as the one in (8) or (10) can be systematically evaluated using *generating functions*. We recall that

$$\sum_{n\ge 0} F_n z^n = \frac{z}{1-z-z^2} \quad \text{and} \quad \sum_{n\ge 0} L_n z^n = \frac{2-z}{1-z-z^2}.$$

Consequently we have

$$\begin{aligned}\sum_{n\ge 0}\Bigg(\sum_{0\le k\le n} m^k F_k\Bigg) z^n &= \frac{1}{1-z}\frac{mz}{1-mz-m^2z^2}\\ &= \frac{m}{m^2+m-1}\frac{1+m^2z}{1-mz-m^2z^2} - \frac{m}{m^2+m-1}\frac{1}{1-z}.\end{aligned}$$

Comparing coefficients, this leads to the explicit formula

$$\sum_{0\le k\le n} m^k F_k = \frac{m^{n+1}}{m^2+m-1}[F_{n+1} + mF_n] - \frac{m}{m^2+m-1}.$$

Likewise,

$$\begin{aligned}\sum_{n\ge 0}\Bigg(\sum_{0\le k\le n} m^k L_k\Bigg) z^n &= \frac{1}{1-z}\frac{2-mz}{1-mz-m^2z^2}\\ &= \frac{m}{m^2+m-1}\frac{(2m+1)-m(m-2)z}{1-mz-m^2z^2} + \frac{m-2}{m^2+m-1}\frac{1}{1-z}\end{aligned}$$

and

$$\sum_{0\le k\le n} m^k L_k = \frac{m^{n+1}}{m^2+m-1}[(2m+1)F_{n+1} - (m-2)F_n] + \frac{m-2}{m^2+m-1}.$$

This evaluates the formula (10):

$$\sum_{0\le k\le n} m^k[L_k+(m-2)F_{k+1}] = \sum_{0\le k\le n} m^k L_k + \frac{m-2}{m}\sum_{1\le k\le n+1} m^k F_k$$
$$= \frac{m^{n+1}}{m^2+m-1}[(2m+1)F_{n+1}-(m-2)F_n] + \frac{m-2}{m^2+m-1}$$
$$+ \frac{m-2}{m}\frac{m^{n+1}}{m^2+m-1}[F_{n+1}+mF_n] - \frac{m-2}{m}\frac{m}{m^2+m-1} + \frac{m-2}{m}m^{n+1}F_{n+1}$$
$$= m^{n+1}F_{n+1}.$$

Alternating sums, as in (11), can be handled as well: Since $L_{k+1} = F_k + F_{k+2} = 2F_k + F_{k+1}$ we find

$$L_{k+1} + (m-2)F_k = F_{k+1} + mF_k.$$

It follows that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-m)^{n-k}[L_{k+1}+(m-2)F_k]\right)z^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-m)^{n-k}[F_{k+1}+mF_k]\right)z^n$$
$$= \left[\sum_{i=0}^{\infty}(-mz)^i\right]\left[\sum_{j=0}^{\infty}(F_{j+1}+mF_j)z^j\right]$$
$$= \frac{1}{1+mz}\left(\frac{1}{1-z-z^2}+\frac{mz}{1-z-z^2}\right)$$
$$= \frac{1}{1-z-z^2}$$
$$= \sum_{n=0}^{\infty}F_{n+1}z^n.$$

Similarly,

$$\sum_{n\ge 0}\left(\sum_{0\le k\le n}(-1)^k m^{n-k}L_k\right)z^n = \sum_{k,l\ge 0}(-1)^k L_k z^k m^l z^l$$
$$= \frac{1}{1-mz}\frac{2+z}{1+z-z^2}$$
$$= \frac{1}{m^2+m-1}\frac{(m-2)-z(2m+1)}{1+z-z^2} - \frac{m(2m+1)}{m^2+m-1}\frac{1}{1-mz},$$

which leads to

$$\sum_{0\le k\le n}(-1)^k m^{n-k}L_k = \frac{(-1)^{n+1}}{m^2+m-1}[(m-2)F_{n+1}-(2m+1)F_n] + \frac{m(2m+1)}{m^2+m-1}m^n.$$

Formulæ (11) and (12) follow from these two in a straightforward way.

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