PERFECT NUMBERS AND FIBONACCI PRIMES II

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Abstract

We aim to solve the equation $\sigma_2(n) = \ell n^2 + An + B$, where ℓ, A, and B are given integers. We find that this equation has infinitely many solutions only if $\ell = 1$. Then we characterize the solutions to the equation $\sigma_2(n) = n^2 + An + B$. We prove that, except for finitely many computable solutions, all the solutions to this equation with $(A, B) = (L_{2m}, F_{2m}^2 - 1)$ are $n = F_{2k+1}F_{2k+2m+1}$, where both F_{2k+1} and $F_{2k+2m+1}$ are Fibonacci primes. Meanwhile, we show that the twin prime conjecture holds if and only if the equation $\sigma_2(n) - n^2 = 2n + 5$ has infinitely many solutions.

1. Introduction and Main Results

A positive integer is called a *perfect number* if it is equal to the sum of its proper divisors. In other words, a positive integer n is a perfect number if and only if

$$\sum_{\substack{d|n \\ 0<d<n}} d = n. \tag{1}$$

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Let $\sigma_k(n) := \sum_{d|n} d^k$. When $k = 1$, we usually write $\sigma_1(n)$ as $\sigma(n)$. The perfect numbers are exactly those integers n which satisfy

$$\sigma(n) = 2n. \tag{2}$$

Euler [4] proved that all even perfect numbers have the form $n = 2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are primes. Note that a prime of the form $2^p - 1$ is called a *Mersenne prime*. As of December 2018 [8], only 51 Mersenne primes have been found and thus only finitely many even perfect numbers are known. It is unknown whether there are any odd perfect numbers.

Some variants and generalizations of perfect numbers have been studied by many mathematicians including Fermat, Descartes, Mersenne and Euler; see [1, 3, 5] for example.

Recall that the Fibonacci numbers $\{F_n\}_{n\geq 0}$ are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with seed values $F_0 = 0$, $F_1 = 1$. The Lucas numbers $\{L_n\}_{n\geq 0}$ satisfy the same recurrence relation

$$L_n = L_{n-1} + L_{n-2}$$

but with different seed values $L_0 = 2$, $L_1 = 1$. A Fibonacci (resp. Lucas) number F_n (resp. L_n) is called a Fibonacci (resp. Lucas) prime if F_n (resp. L_n) is a prime. In 2015, Cai, Chen and Zhang [1] considered the following arithmetic equation:

$$\sum_{\substack{d|n \\ d<n}} d^2 = 3n. \tag{3}$$

Surprisingly, they found that the structure of its solutions is similar to the structure of perfect numbers. Namely, each solution must be a product of two primes.

Theorem 1. *All the solutions of* (3) *are* $n = F_{2k-1}F_{2k+1}$*, where both* F_{2k-1} *and* F_{2k+1} *are Fibonacci primes.*

Due to the connection with Mersenne primes and Fibonacci primes, the first author called the original perfect numbers M-perfect numbers and the solutions of (3) F-perfect numbers. In the 2013 China-Japan Number Theory Conference, which was held in Fukuoka, Professor Kohji Matsumoto suggested to name the M-perfect numbers (resp. F-perfect numbers) as male (resp. female) perfect numbers.

There are so far only a few Fibonacci primes known and thus only five F-perfect numbers were known and given in [1]. The authors [1] also showed that for any pair of positive integers $(a, b) \neq (2, 3)$ and $a \geq 2$, the equation

$$\sum_{\substack{d|n \\ d<n}} d^a = bn \tag{4}$$

has only finitely many solutions.

In this paper, we extend the work of [1] by considering a more general equation:

$$\sigma_2(n) = \ell n^2 + An + B, \tag{5}$$

where ℓ, A, and B are integers. Our main goal is to find the values of (ℓ, A, B) such that (5) may have infinitely many solutions and give a way to characterize these solutions.

Theorem 2. *If $\ell \neq 1$, then the equation* (5) *has only finitely many solutions.*

This theorem tells us that we only need to discuss the case when $\ell = 1$, and in this case (5) can be rewritten as

$$\sigma_2(n) - n^2 = An + B. \tag{6}$$

Now we characterize the solutions of (6).

Theorem 3. (i) *If $A = 0$ and $B = 1$, then all the solutions of* (6) *are $n = p$, where p is a prime.*
(ii) *If $A = 1$ and $B = 1$, then all the solutions of* (6) *are $n = p^2$, where p is a prime.*
(iii) *If $(A, B) \neq (0, 1)$ and $(A, B) \neq (1, 1)$, then, except for finitely many computable solutions in the range $n \leq (|A| + |B|)^3$, all the solutions of* (6) *are $n = pq$, where $p < q$ are primes which satisfy the following equation:*

$$p^2 + q^2 + (1 - B) = Apq. \tag{7}$$

For some special pairs (A, B), the equation (7) has solutions which have interesting structures.

Theorem 4. *Let m be a positive integer. Except for finitely many computable solutions in the range $n \leq (L_{2m} + F_{2m}^2 - 1)^3$, all the solutions of*

$$\sigma_2(n) - n^2 = L_{2m}n - (F_{2m}^2 - 1) \tag{8}$$

are
(i) *$n = F_{2k+1}F_{2k+2m+1} (k \geq 0)$, where both F_{2k+1} and $F_{2k+2m+1}$ are Fibonacci primes;*
(ii) *$n = F_{2k+1}F_{2m-2k-1} (0 \leq k < m, k \neq \frac{m-1}{2})$, where both F_{2k+1} and $F_{2m-2k-1}$ are Fibonacci primes.*

We call those solutions of (8) which are not in the form of (i) or (ii) exceptional solutions. For $1 \leq m \leq 5$, by using Mathematica, we find that there are no exceptional solutions. In particular, if $m = 1$, (8) becomes (3), and Theorem 4 reduces to Theorem 1. For $1 \leq m \leq 3$, according to the list of known Fibonacci

m	$L_{2m}n - (F_{2m}^2 - 1)$	n
1	$3n$	$F_3F_5, F_5F_7, F_{11}F_{13}, F_{431}F_{433}, F_{569}F_{571}$
2	$7n - 8$	$F_3F_7, F_7F_{11}, F_{13}F_{17}, F_{43}F_{47}$
3	$18n - 63$	$F_5F_{11}, F_7F_{13}, F_{11}F_{17},$ $F_{17}F_{23}, F_{23}F_{29}, F_{131}F_{137}$

Table 1: Solutions of (8) for $1 \leq m \leq 3$.

primes [7], we present the known solutions in Table 1. The two solutions $F_{431}F_{433}$ and $F_{569}F_{571}$ in Table 1 have 180 and 238 digits, respectively.

The exceptional solutions may exist for larger m. For example, if $m = 6$, (8) becomes

$$\sigma_2(n) - n^2 = 322n - 20735.$$

For this equation, there is exactly one exceptional solution given by $n = 1755 = 3^3 \cdot 5 \cdot 13$. For other m, exceptional solutions may exist as well. It would be difficult to find all the possible values of m and n such that (8) has no exceptional solutions.

Meanwhile, we have a companion result concerning the Lucas sequence.

Theorem 5. *Except for finitely many computable solutions in the range* $n \leq \left(L_{2m} + L_{2m}^2 - 3\right)^3$, *all the solutions of*

$$\sigma_2(n) - n^2 = L_{2m}n + (L_{2m}^2 - 3) \tag{9}$$

are $n = L_{2k-1}L_{2k+2m-1}$, *where both* L_{2k-1} *and* $L_{2k+2m-1}$ *are Lucas primes.*

Theorem 6. *Except for finitely many computable solutions in the range* $n \leq (L_{2m} + L_{2m}^2 - 5)^3$, *all the solutions of*

$$\sigma_2(n) - n^2 = L_{2m}n - (L_{2m}^2 - 5) \tag{10}$$

are
(i) $n = L_{2k}L_{2k+2m} (k \geq 0)$, *where both* L_{2k} *and* L_{2k+2m} *are Lucas primes.*
(ii) $n = L_{2k}L_{2m-2k} (0 \leq k \leq m, k \neq \frac{m}{2})$, *where both* L_{2k} *and* L_{2m-2k} *are Lucas primes.*

Similarly, we call those solutions of (9) or (10) which are not a product of two Lucas primes exceptional solutions. From these two theorems and by using Mathematica, we find that the equations (9) and (10) have no exceptional solutions for $1 \leq m \leq 5$. We present the known solutions of (9) with $1 \leq m \leq 5$ in Table 2. The two solutions $n = L_{613}L_{617}$ and $L_{4787}L_{4793}$ in Table 2 have 258 and 2003 digits, respectively.

Since we do not know whether there exist infinitely many Fibonacci primes or Lucas primes, it is not clear whether there are infinitely many solutions described in Theorems 4–6.

m	$L_{2m}n + (L_{2m}^2 - 3)$	n
1	$3n + 6$	$L_5L_7, L_{11}L_{13}, L_{17}L_{19}$
2	$7n + 46$	$L_7L_{11}, L_{13}L_{17}, L_{37}L_{41}, L_{613}L_{617}$
3	$18n + 321$	$L_5L_{11}, L_7L_{13}, L_{11}L_{17}, L_{13}L_{19}, L_{31}L_{37}, L_{41}L_{47}, L_{47}L_{53}, L_{4787}L_{4793}$

Table 2: Solutions of (9) for $1 \leq m \leq 3$.

Finally, we consider the following equation:

$$\sigma_2(n) - n^2 = An + (k^2 + 1). \tag{11}$$

Theorem 7. *Let A be a positive integer and k an integer.*
(i) *If $A \neq 2$ or k is odd, then* (11) *has only finitely many solutions;*
(ii) *If $A = 2$ and k is even, then except for finitely many computable solutions in the range $n < (|A| + k^2 + 1)^3$, all the solutions of* (11) *are $n = p(p + k)$, where both p and $p + k$ are primes.*

This theorem leads to an interesting equivalent statement of the Polignac's conjecture.

Corollary 1. *For any even integer k, there are infinitely many prime pairs $(p, p+k)$ if and only if the equation*

$$\sigma_2(n) - n^2 = 2n + (k^2 + 1)$$

has infinitely many solutions.

In particular, the twin prime conjecture holds if and only if the equation

$$\sigma_2(n) - n^2 = 2n + 5$$

has infinitely many solutions.

2. Proofs of the Theorems

Proof of Theorem 2. Suppose that for some ℓ, (5) has infinitely many solutions n_i with $n_1 < n_2 < \cdots < n_m < \cdots$. Then we have $\lim_{m\to\infty} n_m = \infty$. Thus

$$\lim_{m\to\infty} \frac{\sigma_2(n_m)}{n_m^2} = \lim_{m\to\infty} \left(\ell + \frac{A}{n_m} + \frac{B}{n_m^2}\right) = \ell. \tag{12}$$

However, for any positive integer n, we have

$$1 < \frac{\sigma_2(n)}{n^2} = \sum_{d|n} \frac{1}{d^2} < \sum_{d=1}^{\infty} \frac{1}{d^2} = \frac{\pi^2}{6}. \tag{13}$$

Combining (12) with (13), we conclude that $\ell = 1$. □

In order to prove other theorems, we need the following two lemmas.

Lemma 1 ([6]). *Let m, n be nonnegative integers and $n \geq m$. We have*
(i) $5F_n^2 + 4(-1)^n = L_n^2$,
(ii) $L_m L_n + 5F_m F_n = 2L_{m+n}$,
(iii) $F_n L_m = F_{n+m} + (-1)^m F_{n-m}$,
(iv) $L_n F_m = F_{n+m} - (-1)^m F_{n-m}$,
(v) $5F_m F_n = L_{m+n} - (-1)^m L_{n-m}$.

Lemma 2 ([2, Corollary 7). *] All nonnegative integer solutions of the equations $x^2 - 5y^2 = -4$ and $x^2 - 5y^2 = 4$ are given by $(x, y) = (L_{2n+1}, F_{2n+1})$ and $(x, y) = (L_{2n}, F_{2n})$ with $n \geq 0$, respectively.*

Proof of Theorem 3. (i) Since $\sigma_2(n) = n^2 + 1$ if and only if n is a prime, the statement is clearly true.

(ii) Let n have at least two distinct prime factors. Then we can write $n = p^\alpha c$, where p is a prime number, $c > 1$, $p \nmid c$ and $\alpha \geq 1$. Thus

$$n^2 + n + 1 = \sigma_2(n) \geq n^2 + 1 + c^2 + (p^\alpha)^2,$$

which implies that $p^\alpha c = n \geq c^2 + (p^\alpha)^2$. This shows that $(p^\alpha)^2 - p^\alpha c + c^2 \leq 0$. But this is impossible. Therefore, we get $n = p^\alpha$ for some prime p and integer $\alpha \geq 1$. The case $\alpha = 1$ is impossible and therefore $\alpha \geq 2$. Assume that $\alpha \geq 3$. Then $n = p \cdot p^{\alpha-1}$ and so

$$n^2 + n + 1 = \sigma_2(n) \geq n^2 + 1 + p^2 + p^{2\alpha-2}.$$

This implies that

$$p^\alpha = n \geq p^2 + p^{2\alpha-2} = p^2 + p^\alpha p^{\alpha-2} > p^\alpha,$$

which is impossible. Hence all the solutions of (6) are $n = p^2$, where p is a prime.

(iii) We assume that $(A, B) \neq (0, 1)$ and $(A, B) \neq (1, 1)$. Suppose $n > (|A| + |B|)^3$ is a solution of (6).

If $n = abc$ where $1 < a < b < c$ are positive integers. By the arithmetic-geometric mean inequality, we have

$$\sigma_2(n) - n^2 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq 3(a^4b^4c^4)^{1/3} = 3n^{\frac{4}{3}}.$$

Thus

$$An + B = \sigma_2(n) - n^2 \geq 3n^{4/3} > n \cdot n^{1/3} > n(|A| + |B|) \geq An + B.$$

This is a contradiction. Hence n cannot be a product of three distinct positive integers which are greater than 1.

Let $\omega(n)$ denote the number of distinct prime factors of n. Then we have $\omega(n) \leq 2$, since otherwise we can write $n = abc$ where $1 < a < b < c$, which leads to a contradiction.

Case 1: If $\omega(n) = 1$, we write $n = p^\alpha$ where p is a prime. Since $n = p \cdot p^2 \cdot p^{\alpha-3}$, we must have $\alpha \leq 5$.

If $\alpha = 1$ then, from (6) we get $Ap + B = 1$. Note that $A \neq 0$ since otherwise $B = 1$, contradicting the assumption $(A, B) \neq (0, 1)$. Now we have $n = p = (1 - B)/A \leq 1 + |B| < (|A| + |B|)^3$, which is a contradiction.

If $\alpha = 2$, from (6) we deduce that $p^2(1 - A) = B - 1$. Note that $A \neq 1$ since otherwise $(A, B) = (1, 1)$, contradicting our assumption. Therefore, $|B| \geq p^2 - 1 \geq 3$. We see that (6) has at most one solution $n = p^2 = (B - 1)/(1 - A)$. But then $n \leq |B| + 1 \leq (|A| + |B|)^3$, which is a contradiction.

If $3 \leq \alpha \leq 5$, from (6) we deduce that

$$p(p + p^3 + \cdots + p^{2\alpha-3} - Ap^{\alpha-1}) = B - 1. \tag{14}$$

Note that

$$p + p^3 + \cdots + p^{2\alpha-3} - Ap^{\alpha-1} \equiv p \pmod{p^2}.$$

Hence $p + p^3 + \cdots + p^{2\alpha-3} - Ap^{\alpha-1} \neq 0$, $p^2 | B - 1$ and $B \neq 1$. If $A \neq 0$, we have $n = p^\alpha \leq p^5 \leq (1 + |B|)^{5/2} \leq (|A| + |B|)^3$, which is a contradiction. If $A = 0$, then (14) implies $B \geq p^2 + 1 \geq 5$. We still have $n = p^\alpha \leq p^5 \leq (1 + |B|)^{5/2} \leq (|A| + |B|)^3$, which is again a contradiction.

Case 2: If $\omega(n) = 2$, we write $n = p^\alpha q^\beta$. If $\alpha \geq 3$, then $n = p \cdot p^{\alpha-1} \cdot q^\beta$, which is a contradiction. Therefore $\alpha \leq 2$ and similarly $\beta \leq 2$.

If $(\alpha, \beta) = (2, 2)$, we can write $n = p \cdot q \cdot pq$, which is a contradiction.

If $(\alpha, \beta) = (1, 2)$, then

$$\sigma_2(n) - n^2 = 1 + p^2 + p^2q^2 + q^2 + q^4 = Apq^2 + B \leq (|A| + |B|)pq^2.$$

This implies $p^2q^2 < (|A| + |B|)pq^2$ and $q^4 < (|A| + |B|)pq^2$. Therefore, we have $p < |A| + |B|$ and $q^2 < (|A| + |B|)p$. Hence

$$n = pq^2 < (|A| + |B|)^2 p < (|A| + |B|)^3,$$

which is a contradiction.

Similarly if $(\alpha, \beta) = (2, 1)$, we have $n < (|A| + |B|)^3$.

Finally, if $(\alpha, \beta) = (1, 1)$, from (6) we see that p and q must satisfy (7), and this proves (iii). □

To prove Theorems 4-6, we first observe that they correspond to different assignments of (A, B) in (6). By Theorem 3 we know that, except for those computable solutions $n \le (|A|+|B|)^3$, all the solutions are of the form $n = pq$, where p, q are distinct primes satisfying (7). Note that (7) is equivalent to

$$(2p - Aq)^2 - (A^2-4)q^2 = 4(B-1). \tag{15}$$

Proof of Theorem 4. Let $(A,B) = (L_{2m}, -F_{2m}^2+1)$. Then (15) becomes

$$(2p - L_{2m}q)^2 - (L_{2m}^2-4)q^2 = -4F_{2m}^2.$$

By Lemma 1 (i), the above equation becomes $(2p-L_{2m}q)^2 - 5F_{2m}^2q^2 = -4F_{2m}^2$. This implies $F_{2m}|2p-L_{2m}q$. Writing $2p - L_{2m}q = uF_{2m}$, where u is an integer, we deduce that $u^2-5q^2=-4$. By Lemma 2, we have $(u,q) = (\pm L_{2k+1}, F_{2k+1})$ for some nonnegative integer k.

`Case 1:` Let $(u,q) = (L_{2k+1}, F_{2k+1})$. Then $p = \frac{1}{2}(L_{2m}F_{2k+1} + L_{2k+1}F_{2m})$. By (iii) and (iv) of Lemma 1, we have $p = F_{2k+1+2m}$. Hence $n = F_{2k+1}F_{2k+2m+1}$, where both F_{2k+1} and $F_{2k+2m+1}$ are primes.

`Case 2:` Let $(u,q) = (-L_{2k+1}, F_{2k+1})$. Then $p = \frac{1}{2}(L_{2m}F_{2k+1} - L_{2k+1}F_{2m})$.

If $2k+1 > 2m$, by (iii) and (iv) of Lemma 1, we have $p = F_{2k+1-2m}$. Hence $n = F_{2k+1}F_{2k+1-2m}$.

If $2k+1 < 2m$, by (iii) and (iv) of Lemma 1, we have $p = F_{2m-2k-1}$. Hence $n = F_{2k+1}F_{2m-2k-1}$. □

Proof of Theorem 5. Let $(A,B) = (L_{2m}, L_{2m}^2-3)$. Then (15) becomes

$$(2p - L_{2m}q)^2 - (L_{2m}^2-4)q^2 = 4(L_{2m}^2-4). \tag{16}$$

By Lemma 1 (i), we have $L_{2m}^2 - 4 = 5F_{2m}^2$. From (16) we deduce that $2p - L_{2m}q = 5F_{2m}u$ for some integer u. Now (16) becomes $q^2-5u^2=-4$. By Lemma 2, we have $(q,u) = (L_{2k+1}, \pm F_{2k+1})$ for some integer $k \ge 0$.

`Case 1:` Let $(q,u) = (L_{2k+1}, F_{2k+1})$. Then by Lemma 1 (ii), we have $p = \frac{1}{2}(L_{2m}L_{2k+1} + 5F_{2m}F_{2k+1}) = L_{2m+2k+1}$. Hence, $n = L_{2k+1}L_{2k+2m+1}$.

`Case 2:` Let $(q,u) = (L_{2k+1}, -F_{2k+1})$. Then

$$p = \frac{1}{2}(L_{2m}L_{2k+1} - 5F_{2m}F_{2k+1}) = \frac{1}{2}(L_{2m}L_{2k+1} + 5F_{2m}F_{2k+1}) - 5F_{2m}F_{2k+1}.$$

If $2m > 2k+1$ then, by (ii) and (v) of Lemma 1, we have $p = -L_{2m-2k-1}$, which is impossible.

If $2m < 2k+1$ then, by (ii) and (v) of Lemma 1, we have $p = L_{2k+1-2m}$. Thus we have $n = L_{2k+1}L_{2k+1-2m}$, which is the same as Case 1. □

Proof of Theorem 6. Let $(A,B) = (L_{2m}, 5 - L_{2m}^2)$. Then (15) becomes

$$(2p - L_{2m}q)^2 - (L_{2m}^2-4)q^2 = 4(4 - L_{2m}^2).$$

By Lemma 1 (ii), we have $L_{2m}^2 - 4 = 5F_{2m}^2$. Thus we can write $2p - L_{2m}q = 5F_{2m}u$ for some integer u. The above equation becomes $q^2 - 5u^2 = 4$. By Lemma 2, we deduce that $(q, u) = (L_{2k}, \pm F_{2k})$ for some integer $k \geq 0$.

Case 1: $(q, u) = (L_{2k}, F_{2k})$. By Lemma 1, we have $p = \frac{1}{2}(L_{2m}L_{2k} + 5F_{2m}F_{2k}) = L_{2m+2k}$. Hence $n = L_{2k}L_{2k+2m}$.

Case 2: $(q, u) = (L_{2k}, -F_{2k})$. We have

$$p = \frac{1}{2}(L_{2m}L_{2k} - 5F_{2m}F_{2k}) = \frac{1}{2}(L_{2m}L_{2k} + 5F_{2m}F_{2k}) - 5F_{2m}F_{2k}.$$

If $m > k$ then, by Lemma 1 we have $p = F_{2m-2k}$, and hence $n = F_{2m-2k}F_{2k}$. Since p and q are distinct primes, we have $k \neq \frac{m}{2}$.

If $m \leq k$ then, by Lemma 1 we have $p = F_{2k-2m}$. Hence $n = F_{2k-2m}F_{2k}$. □

Proof of Theorem 7. Without loss of generality, we assume that $k \geq 0$. Let n be a solution of (11) such that $n > (|A| + k^2 + 1)^3$. It follows from Theorem 3 (iii) that $n = pq$, where p and q are distinct primes satisfying $p^2 + q^2 - k^2 = Apq$. Let $p < q$. Note that $q|(p - k)(p + k)$.

If $p = k$, then we have $q = Ak$ and $n = Ak^2 < (|A| + k^2 + 1)^3$, which is a contradiction.

If $p < k$, then $p + k < 2k$. Since $q|p - k$ or $q|p + k$, we have $q < 2k$ and hence $n < 2k^2 < (|A| + k^2 + 1)^3$, which is a contradiction.

If $p > k$, then $q|p + k$. Note that $2q > 2p > p + k$. We must have $q = p + k$. Thus $A = (p^2 + q^2 - k^2)/pq = 2$ and $n = p(p + k)$. For $p \geq 3$, k must be even.

Conversely, if $A = 2$, k is even and both p and $p + k$ are primes, then it is easy to see that $n = p(p + k)$ is a solution of (11). □

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