LOWER BOUNDS FOR NUMBERS WITH THREE PRIME FACTORS

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Abstract

A \textit{k-almost prime number} is a product of \( k \) prime numbers, some of which may be repeated. By a 1900 theorem of Landau, the number of \( k \)-almost prime numbers not exceeding \( x \) is asymptotic to \( x (\log \log x)^{k-1}/((k-1)! \log x) \). We prove a numerically explicit lower bound for 3-almost prime numbers which is asymptotic to Landau’s formula, and hence to the actual count. It exceeds Landau’s formula for all \( x \geq 500194 \). We prove an analogous lower bound for products of three distinct prime numbers. This expands on previously known results for \( k \leq 2 \).

1. Introduction

For a natural number \( n \), the arithmetic functions \( \omega(n) \) and \( \Omega(n) \) denote the number of prime factors of \( n \), counted without (respectively with) repeated prime factors. Thus for \( n \) with prime factorization \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \), we have \( \omega(n) = k \) and \( \Omega(n) = a_1 + \ldots + a_k \).

Numbers \( n \) such that \( \Omega(n) = k \) are called \( k \)-almost primes. Let \( \pi_k(n) = |\{n \leq x : \omega(n) = \Omega(n) = k\}| \) denote the counting function of squarefree \( k \)-almost primes, and let \( \tau_k(n) = |\{n \leq x : \Omega(n) = k\}| \) denote the counting function of \( k \)-almost primes.

For \( k = 1 \) we have \( \tau_1(x) = \tau_1(x) = \pi(x) \), the prime counting function. The prime number theorem asserts that \( \pi(x) \) is asymptotic to \( x/\log x \). In 1900, Landau [6] proved that for each \( k \in \mathbb{N} \), the estimate

\[
\pi_k(x) = \frac{x (\log \log x)^{k-1}}{(k-1)! \log x} \left( 1 + O \left( \frac{1}{\log \log x} \right) \right)
\]

holds, and that the same estimate also holds for \( \tau_k(x) \).

Selberg [11] proved that for any \( 0 < \delta < 1 \), uniformly for all \( x \geq 3 \) and \( 1 \leq k \leq (2 - \delta) \log \log x \), we have

\[
\tau_k(x) = G \left( \frac{k}{\log \log x} \right) \frac{x (\log \log x)^{k-1}}{(k-1)! \log x} \left( 1 + O \left( \frac{k}{(\log \log x)^2} \right) \right),
\]
where
\[ G(z) = F(1, z)/\Gamma(z + 1) \quad \text{and} \quad F(s, z) = \prod_p \left( 1 - \frac{z}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^s} \right)^z. \]

Classic references \[5, 7\] provide details on these and similar results. A 1962 paper [10] of Rosser and Schoenfeld and contemporary work \[3, 4\] of Dusart give explicit bounds for \( \pi(x) \), see Lemma 1. A 2018 paper \[1\] of Bayless et al. established explicit upper bounds for \( \pi_k(x) \), \( \tau_2(x) \), and \( \tau_3(x) \), as well as explicit lower bounds for \( \pi_3(x) \) and \( \tau_2(x) \). In particular, it is shown \[1, \text{Thm. 3.5}\] that for all \( k \geq 2 \) and \( x \geq 3 \), we have
\[ \pi_k(x) \leq \frac{1.028x(\log \log x + 0.26153)^{k-1}}{(k-1)! \log x}, \]
and \[1, \text{Thm. 5.2}\] that for all \( x \geq 10^{12} \), we have
\[ \tau_2(x) \geq \frac{x(\log \log x + 0.1769)}{\log x} \left( 1 + \frac{0.4232}{\log x} \right). \]
Furthermore \[1, \text{Thm. 5.3}\], for all \( x \geq 10^{12} \) we have
\[ \tau_3(x) \leq \frac{1.028x((\log \log x + 0.26153)^2 + 1.055852)}{2 \log x}. \]

A relatively sharp explicit lower bound for \( \pi_3(x) \) requires more work, and aside from the explicit results above, the literature consists of implied constants. We prove explicit lower bounds for \( \pi_3(x) \) and \( \tau_3(x) \) which are asymptotic to Landau’s formula, and hence to the actual count. In particular, we prove the following three theorems.

**Theorem 1.** For all \( x \geq 500194 \),
\[ \tau_3(x) > \frac{x(\log \log x)^2}{2 \log x}. \]

The constant 500194 is optimal, since the inequality is violated at \( x = 500194 - \epsilon \) for all sufficiently small \( \epsilon > 0 \). We also have an analogue of Theorem 1 for squarefree 3-almost primes, also called sphenic numbers.

**Theorem 2.** For all \( x \geq 10203553 \),
\[ \pi_3(x) > \frac{x((\log \log x)^2 - 1)}{2 \log x}. \]

The constant 10203553 is also optimal. We also obtain the following.

**Theorem 3.** For all sufficiently large \( x \),
\[ \pi_3(x) > \frac{x(\log \log x)^2}{2 \log x}. \]
Theorem 3 follows readily from the proof of Theorem 2 by comparing secondary terms, however the reader will see that $x$ must be very large. Without more work, it is unclear where the optimal cutoff $c_0$ is for Theorem 3 to apply. Lifchitz and Renner [8] computed the values of $\pi_3(10^k)$ for $1 \leq k \leq 19$, and this data shows that $c_0 > 10^{19}$.

2. Notation and Preliminary Lemmas

Throughout the paper we let $p, q, \text{ and } r$ denote prime numbers and we let $\log x$ denote the natural logarithm. We define $L = \log \log x$, $x_0 = 10^{12}$, and $y_0 = x_0^{1/3} = 10^4$. Furthermore, $\pi(t)$ denotes the prime counting function and $T(t) = \sum_{p \leq t} 1/p$ denotes the sum of reciprocals of prime numbers $p \leq t$. We let $B = 0.2614972128 \ldots$ denote the Mertens constant and $c = 0.26146521$ (see Lemmas 2 and 3).

In bounding expressions, we will make use of manipulations such as

$$\log \log \frac{x}{a} = \log \log x + \log \left(1 - \frac{\log a}{\log x}\right),$$

the Maclaurin series expansion of the righthand term, and the bounds

$$\frac{1}{\log x} \left(1 + \frac{\log a}{\log x}\right) < \frac{1}{\log x} \leq \frac{1}{\log x} + \frac{1}{\log^2 x} \cdot \frac{\log a \log x_0}{\log x},$$

for $a > 1$ and $x \geq x_0$. We will use the following bounds [10, Thm. 2] of Rosser and Schoenfeld and [4, Cor. 5.2] of Dusart on the prime counting function.

**Lemma 1 (Rosser, Schoenfeld, Dusart).** We have

$$\frac{x}{\log x} < \pi(x)$$

for all $x \geq 17$. Additionally, we have

$$\frac{x}{\log x} \left(1 + \frac{1}{\log x}\right) \leq \pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1.2762}{\log x}\right),$$

where the lower bound holds for all $x \geq 599$ and the upper bound holds for all $x > 1$. Furthermore, for all $x > 1$ we have

$$\pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2.53816}{\log^2 x}\right).$$

We also use the following bounds [10, Theorem 5] of Rosser and Schoenfeld and [3, Thm. 6.10], [4, Thm. 5.6] of Dusart which give numerically explicit versions of Mertens’ second theorem.
Lemma 2 (Rosser, Schoenfeld, Dusart). Let \( T(t) = \sum_{p \leq t} 1/p \) denote the reciprocal sum of prime numbers up to \( t \). We have

\[
T(t) = \log \log t + B + E(t),
\]

where \( B = 0.2614972128\ldots \) denotes the Mertens constant, and

\[
- \frac{1}{2 \log^2 t} < E(t) < \frac{1}{\log^2 t}
\]

for all \( t > 1 \),

\[
|E(t)| < \frac{1}{2 \log^2 t}
\]

for all \( t \geq 286 \),

\[
|E(t)| \leq \frac{1}{10 \log^2 t} + \frac{4}{15 \log^3 t}
\]

for all \( t \geq 10372 \), and

\[
|E(t)| \leq \frac{1}{5 \log^3 t}
\]

for all \( t \geq 2278383 \).

Combining bounds on the prime number reciprocal sum in [10, 4] readily yields the following lower bound.

Lemma 3. For \( x > 1 \),

\[
\sum_{p \leq x} \frac{1}{p} > \log \log x + 0.26146521.
\]

Proof. By Lemma 2, for all \( x \geq 2278383 \),

\[
\sum_{p \leq x} \frac{1}{p} > \log \log x + B - \frac{0.2}{\log^2 x}, \tag{1}
\]

where \( B \) is the Mertens constant. By [10, Thm 20],

\[
\sum_{p \leq x} \frac{1}{p} > \log \log x + B
\]

for all \( x \leq 10^8 \). Substituting \( 10^8 \) in (1) gives the lemma. \( \Box \)
3. The Proof of Theorem 2

We have

$$\pi_3(x) = \left| \{ n = pqr : p < q < r \} \right|$$

where $p, q,$ and $r$ denote prime numbers. To determine the count, we note the possible values for each of $p, q, r$. We have $p < x^{1/3}$, and $q^2 < qr \leq x/p$, so that $p < q < \sqrt{x/p}$. Since $pqr \leq x$, we also have $q < r \leq x/(pq)$. Therefore, $\pi_3(x)$ is equal to the sum of 1 over all possible values of $p, q, r$, so that

$$\pi_3(x) = \sum_{p < x^{1/3}} \sum_{p < q < \sqrt{x/p}} \sum_{q < r \leq \frac{x}{pq}} 1 = \sum_{p < x^{1/3}} \sum_{p < q < \sqrt{x/p}} \left( \pi \left( \frac{x}{pq} \right) - \pi(q) \right).$$

Here we have used the fact that the strict inequalities $p < x^{1/3}$ and $q < \sqrt{x/p}$ can be written to include the case of equality, since the quantity vanishes when $q = \sqrt{x/p}$ or when $p = x^{1/3}$. We checked the inequality using the computer program Pari/GP for the interval $10203553 \leq x \leq 10^{12}$. Thus we may assume $x \geq x_0$. By Lemma 1, since $x/(pq) \geq 599$ we have

$$\pi \left( \frac{x}{pq} \right) \geq \frac{x}{pq \log \frac{x}{pq}} \left( 1 + \frac{1}{\log \frac{x}{pq}} \right) \geq \frac{x}{pq \log \frac{x}{pq}} \left( 1 + \frac{1}{\log x} \right).$$

We thus have

$$\pi_3(x) \geq \left( 1 + \frac{1}{\log x} \right) \sum_{p \leq x^{1/3}} S_1 - \sum_{p \leq x^{1/3}} S_2,$$

where

$$S_1 = \sum_{p < q \leq \sqrt{x/p}} \frac{x}{pq \log \frac{x}{pq}}$$

and

$$S_2 = \sum_{p < q \leq \sqrt{x/p}} \pi(q).$$

We determine a lower bound for $S_1$ and an upper bound for $S_2$. By Lemma 7 below, we have

$$\sum_{p \leq x^{1/3}} S_2 \leq x \left( \frac{2L + 0.1436}{\log^2 x} + \frac{10.9113L + 3.1227}{\log^3 x} \right).$$

We next consider $S_1$. We have

$$S_1 = \frac{x}{p} \sum_{p < q \leq \sqrt{x/p}} \frac{1}{q \log \frac{x}{q}}.$$
where \( y = x/p > x^{2/3} \geq x_0^{2/3} \). Recall that the function \( T(t) = \sum_{p \leq t} 1/p \) denotes the prime reciprocal sum up to \( t \). We bound \( S_1 \) below by applying partial summation to obtain
\[
\sum_{p < q \leq \sqrt{y}} \frac{1}{q \log \frac{y}{q}} = \frac{T(\sqrt{y})}{\log \sqrt{y}} - \frac{T(p)}{\log \frac{y}{p}} \int_p^{\sqrt{y}} \frac{T(t)}{t \log^2 \frac{y}{t}} \, dt.
\] (2)

By Lemma 2 this is bounded below by the expression
\[
\frac{2(\log \log \sqrt{y} + B - \frac{1}{2 \log \sqrt{y}})}{\log y} - \frac{\log \log p + B + \frac{1}{\log y}}{\log \frac{y}{p}} - \int_p^{\sqrt{y}} \frac{\log \log t + B + \frac{1}{\log y}}{t \log^2 \frac{y}{t}} \, dt.
\]

Substituting \( u = \log t \) and integrating, this is equal to
\[
\frac{-4}{\log^3 y} - \frac{\log \log p}{\log y} - \frac{1}{\log^2 p \log \frac{y}{p}} + \frac{\log \log \frac{y}{p}}{\log y} + \frac{\log p \log \log p}{\log y \log \frac{y}{p}} + \frac{1}{\log^2 y \log \frac{y}{p}} - \frac{2 \log \log \frac{y}{p}}{\log^2 y} + \frac{2 \log \log p}{\log^2 y}.
\]

Thus a lower bound for \( S_1 \) is
\[
\frac{x}{p \log \frac{p}{x}} \left( \log \log \frac{x}{p^2} + \frac{\log \log \log p}{\log \frac{p}{x}} + \frac{1}{\log \frac{p}{x} \log \frac{p}{x^2}} + \frac{2 \log \log p}{\log^2 \frac{p}{x}} \right)
\]
\[
- \frac{x}{p \log \frac{x}{p}} \left( \frac{4}{\log^2 \frac{x}{p}} + \log \log p \cdot \frac{\log \frac{x}{p}}{\log \frac{x}{p} \log \frac{p}{x}} + \frac{1}{\log^2 \frac{x}{p}} \log \frac{x}{p} \log \frac{x}{p^2} + \frac{1}{\log \frac{p}{x} \log \frac{x}{p} \log \frac{x}{p^2}} + \frac{2 \log \log \frac{x}{p}}{\log^2 \frac{x}{p^2}} \right).
\] (3)

To simplify this expression, we compare the second (respectively, third) term of the first line above to the second (respectively, first) term of the second line. We have
\[
\frac{\log \log p \log p}{\log \frac{p}{x}} - \log \log p \cdot \frac{\log \frac{x}{p}}{\log \frac{x}{p} \log \frac{p}{x}} = -\log \log p.
\]

Also, \( 1/(\log(x/p) \log(x/p^2)) > 1/\log^2(x/p) \). Therefore, a lower bound for expression (3) is
\[
\frac{x}{p \log \frac{p}{x}} \left( \log \log \frac{x}{p^2} + \frac{2 \log \log p}{\log^2 \frac{p}{x}} \right)
\]
\[
- \frac{x}{p \log \frac{x}{p}} \left( \log \log p + \frac{3}{\log^2 \frac{x}{p}} + \frac{1}{\log^2 \frac{p}{x} \log \frac{x}{p}} + \frac{1}{\log \frac{p}{x} \log \frac{x}{p} \log \frac{x}{p^2}} + \frac{2 \log \log \frac{x}{p}}{\log^2 \frac{x}{p^2}} \right).
\] (4)
Write the sum of this expression over $p \leq x^{1/3}$ as\[ x \left( S_5 + S_6 - (S_7 + S_8 + S_9 + S_{10} + S_{11}) \right). \]

Since $2 \log \log p - 3$ is negative for $p < 89$ and positive for $p \geq 89$,

\[ S_6 - S_8 = \sum_{p \leq x^{1/3}} \frac{2 \log \log p - 3}{p \log^2 \frac{x}{p}} \geq \frac{1}{\log^3 x} \sum_{p < 89} \frac{2 \log \log p - 3 \log^3 x_0}{p \log^3 \frac{x_0}{p}} + \frac{1}{\log^3 x} \sum_{89 \leq p \leq y_0} \frac{2 \log \log p - 3}{p} \geq -3.9322 \frac{\log^3 x}{\log^3 x}. \]

Note that

\[ S_{10} = \sum_{p \leq x^{1/3}} \frac{1}{p \log p \log^2 \frac{x}{p}} \leq \frac{1}{\log^2 x} \sum_{p \leq y_0} \frac{1}{p \log p} \frac{\log^2 x_0}{p \log^2 \frac{x_0}{p}} + \frac{2.25}{\log^2 x} \sum_{y_0 < p} \frac{1}{p \log p} \leq 1.7496 \frac{\log^2 x}{\log^2 x} + \frac{2.25(0.1085)}{\log^2 x} \leq 1.9938 \frac{\log^2 x}{\log^2 x}. \]

Here we verified the value $\sum_{p \leq x^{1/3}} 1/(p \log p) = 1.63661 \ldots$ found to much higher precision [2, p. 6] by H. Cohen. Similarly,

\[ S_9 = \sum_{p \leq x^{1/3}} \frac{1}{p \log^2 p \log \frac{x}{p}} \leq \sum_{p \leq y_0} \frac{1}{p \log^2 p} \left( \frac{1}{\log x} + \frac{1}{\log^2 x} \log \frac{x_0}{p} \right) + \frac{3}{\log x} \sum_{y_0 < p \leq x^{1/3}} \frac{1}{p \log^2 p} \leq 1.5151 \frac{\log x}{\log^2 x} + 3.5585 \frac{\log x}{\log^2 x} \frac{\log^2 x}{\log^2 x} + 3.5585 \frac{\log x}{\log^2 x}. \]

Here we computed the sum over $y_0 < p \leq 10^9$ using Pari/GP and then applied partial summation together with Lemmas 2 and 3 to bound the sum over $p > 10^9$. By Lemma 8,

\[ S_6 - S_7 \geq \frac{0.5L^2 + 1.21434L - 0.22}{\log x} + \frac{-1.5244L + 1.112}{\log^2 x} + \frac{-2L^2 + 2.9067L + 2.5389}{\log^3 x}, \]

and by Lemma 5,

\[ S_{11} \leq \frac{2L^2 + 1.3972L}{\log^3 x}. \]
Finally, note that the application of the bounds

$$L + c < T(x) < L + B + \frac{1}{\log^2 x},$$

valid for all $x > 1$, incur error terms which are particularly large for small values of $x$. We may improve our bounds as follows. In equation (2) we replaced the quantity $-T(p)/(p \log(x/p^2))$ with

$$- \frac{\log \log p + B + \frac{1}{\log^2 p}}{p \log \frac{x}{p^2}}.$$

Thus we may add the following expression to the lower bound:

$$\sum_{p \leq x^{1/3}} \frac{\log \log p + B + \frac{1}{\log^2 p} - T(p)}{p \log \frac{x}{p^2}}.$$

A computation gives a lower bound of

$$\frac{1}{\log \frac{x}{y_0}} \sum_{p \leq x^{1/3}} \frac{\log \log p + B + \frac{1}{\log^2 p} - T(p)}{p} \geq \frac{0.9}{\log \frac{x}{4}} \geq \frac{0.9}{\log x} \left(1 + \frac{\log 4}{\log x} + \frac{\log^2 4}{\log x}\right).$$

Again considering equation (2) and the following displayed expression, we may add to the lower bound

$$\sum_{p \leq x^{1/3}} \int_{t} \log \log t + B + \frac{1}{\log^2 t} - T(t) \frac{t \log^2 \frac{x}{pt}}{p} dt \geq \frac{1}{\log \frac{x}{4}} \sum_{k \leq 128} k \int_{\log p_k + 1} \left(\log u + B + \frac{1}{u^2} - T(p_k)\right) du \geq 20.4395 \frac{\log^2 x}{\log^2 x}.$$ 

Here $p_k$ denotes the $k$-th prime number. Combining all bounds, we obtain

$$\pi_3(x) \geq \frac{x(0.5L^2 + 0.121434L - 0.8528)}{\log x} + \frac{x(0.5L^2 - 4L + 15)}{\log^2 x} - \frac{x(4L^2 + 10.9262L)}{\log^4 x} \geq \frac{x(0.5L^2 + 0.121434L - 0.8528)}{\log x} + \frac{x(0.5L^2 - 4L + 12)}{\log^2 x} \geq \frac{x(0.5L^2 + 0.121434L - 0.8528)}{\log x} + \frac{4x}{\log^2 x} \geq \frac{x((\log \log x)^2 - 1)}{2 \log x} + \frac{4x}{\log^2 x}$$

for all $x \geq x_0$. This completes the proof of Theorem 2.
4. Additional Lemmas

We prove several lemmas used in the proof of Theorem 2.

**Lemma 4.** For $x \geq x_0$, we have

$$
\sum_{p \leq x^{1/3}} \frac{1}{p^{\log^2 x / p}} \leq \frac{L + 0.071781}{\log^2 x}.
$$

**Proof.** For $x_0 \leq x \leq 10372^3$, we have

$$
\sum_{p \leq x^{1/3}} \frac{1}{p^{\log^2 x / p}} \leq \frac{1}{\log^2 x} \sum_{p \leq 10372} \log^2 x_0 \frac{p^{\log^2 x / p}}{p} \leq \frac{\log \log x_0 - 0.0279}{\log^2 x} \leq \frac{L}{\log^2 x},
$$

by directly computing the sum over $p \leq 10372$. Suppose next that $x > 10372^3$. By Lemmas 2 and 3 and partial summation,

$$
\sum_{p \leq x^{1/3}} \frac{1}{p^{\log^2 x / p}} = \frac{T(x^{1/3})}{\log^2(x^{2/3})} - \int_{2}^{x^{1/3}} \frac{2T(t) \, dt}{t \log^3 \frac{t}{x}}
$$

$$
\leq \frac{2}{3} \left( \log \log x^{1/3} + 0.26300402 \right) \frac{\log^2 x}{\log^2 x} - \int_{2}^{x^{1/3}} \frac{2(\log \log t + c) \, dt}{t \log^3 \frac{t}{x}},
$$

recalling that $c = 0.26146521$. The integral is equal to

$$
\left[ \frac{\log \log t + c + \log \log \frac{t}{x} - \log \log t}{\log^2 x} \frac{1}{\log x \log \frac{t}{x}} \right]_{2}^{x^{1/3}}.
$$

Subtracting and bounding the resulting expression, we obtain

$$
S_3 \leq \frac{\frac{2}{3} (0.26300402 - c) - \log 2 + \frac{1}{2} + c + \log \log \frac{2}{x}}{\log^2 x} \leq \frac{L + 0.071781}{\log^2 x},
$$

for all $x \geq 10372^3$, and therefore for all $x \geq x_0$. \qed

**Lemma 5.** For $x \geq x_0$, we have

$$
\sum_{p \leq x^{1/3}} \frac{1}{p^{\log^3 x / p}} \leq \frac{L + 0.6986}{\log^3 x}.
$$

**Proof.** Note that for $x_0 \leq x \leq 10372^3$, we have

$$
\sum_{p \leq x^{1/3}} \frac{1}{p^{\log^3 x / p}} \leq \sum_{p \leq 10372} \frac{1}{p^{\log^3 x / p}} \leq \frac{L + 0.5411}{\log^3 x},
$$

and so on.
by directly computing the sum over \( p \leq 10372 \). Suppose next that \( x \geq 10372^3 \). By partial summation,

\[
\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} = \frac{T(x^{1/3})}{\log^3 x^{2/3}} - \int_2^{x^{1/3}} \frac{3T(t) \, dt}{t \log^4 \frac{x}{t}} \\
\leq \frac{\log \log x^{1/3} + 0.26300402}{\log^3 x^{2/3}} - \int_2^{x^{1/3}} \frac{3(\log \log t + c) \, dt}{t \log^4 \frac{x}{t}},
\]

using Lemmas 2 and 3. The integral is equal to

\[
\left[ \frac{\log \log t + c}{\log^3 \frac{x}{t}} + \frac{\log \frac{t}{x} - \log \log t}{\log^3 x} - \frac{1}{\log^2 x \log \frac{x}{t}} - \frac{1}{2 \log x \log^2 \frac{x}{t}} \right] \frac{x^{1/3}}{2}.
\]

Subtracting and bounding the resulting expression, we find that

\[
\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} \leq 27 (0.26300402 - c) - \log 2 + \frac{9}{x} + \log \log \frac{x}{2} + c, \frac{1}{\log^3 x},
\]

from which we obtain the lemma.

\[ \square \]

**Lemma 6.** For \( x \geq x_0 \) we have

\[
\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} \leq \frac{0.4383 x^{1/6}}{\log x}.
\]

**Proof.** Let \( x \geq x_0 \). By partial summation,

\[
\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} \leq \frac{1.5}{\log x} \sum_{p \leq x^{1/3}} \frac{1}{\sqrt{p}} \\
= \frac{1.5}{\log x} \left( \frac{\pi(x^{1/3})}{x^{1/6}} + \frac{1}{2} \int_2^{x^{1/3}} \frac{\pi(t) \, dt}{t^{3/2}} \right) \\
\leq \frac{1.5}{\log x} \left( \frac{\pi(x^{1/3})}{x^{1/6}} + 16.85461 + \frac{1}{2} \int_{y_0}^{x^{1/3}} \frac{\pi(t) \, dt}{t^{3/2}} \right).
\]

By Lemma 1, an upper bound for this expression is therefore

\[
\frac{1.5}{\log x} \left( \frac{3.4154822 x^{1/6}}{\log x} + 16.85461 + 0.06180521 \int_{y_0}^{x^{1/3}} \frac{dt}{t^{1/2}} \right),
\]

from which the lemma readily follows.

\[ \square \]
Lemma 7. For \( x \geq x_0 \) we have

\[
\sum_{p \leq x^{1/3}} S_2 \leq x \left( \frac{2L + 0.1436}{\log^2 x} + \frac{10.9113L + 3.1227}{\log^4 x} \right).
\]

Proof. We have the general formula

\[
\sum_{q \leq t} \pi(q) = 1 + 2 + \ldots + \pi(t) = \frac{1}{2} (\pi(t)^2 + \pi(t)).
\]

Therefore,

\[
S_2 = \frac{1}{2} \left( \pi \left( \sqrt{\frac{x}{p}} \right)^2 - \pi \left( \sqrt{\frac{x}{p}} \right)^2 - \pi(p)^2 - \pi(p) \right).
\]

We will address the third term below and drop the fourth term. Let \( x \geq x_0 \). By Lemma 1, we have

\[
\frac{1}{2} \pi \left( \sqrt{\frac{x}{p}} \right) \leq \frac{1.1385 \sqrt{x}}{\sqrt{p} \log \frac{x}{p}}
\]

and

\[
\frac{1}{2} \pi \left( \sqrt{\frac{x}{p}} \right)^2 \leq \frac{2x}{p \log^2 \frac{x}{p}} \left( 1 + \frac{2.551155}{\log \frac{x}{p}} \right)^2 \leq \frac{2x}{p \log^2 \frac{x}{p}} + \frac{10.9113x}{p \log^4 \frac{x}{p}}.
\]

We thus have

\[
\sum_{p \leq x^{1/3}} S_2 \leq x \sum_{p \leq x^{1/3}} \left( \frac{2}{p \log^2 \frac{x}{p}} + \frac{10.9113}{p \log^4 \frac{x}{p}} + \frac{1.1385}{\sqrt{x}} \sqrt{\frac{x}{p}} \right).
\]

Combining Lemmas 4, 5, and 6, we obtain

\[
\sum_{p \leq x^{1/3}} S_2 \leq x \left( \frac{2L + 0.1436}{\log^2 x} + \frac{10.9113L + 7.6227}{\log^4 x} + \frac{1.1}{\log^4 x} \right).
\]

We now return to consider the remaining term:

\[
\frac{1}{2} \sum_{p \leq x^{1/3}} \pi(p)^2 = \frac{1}{2} \cdot \frac{\pi(x^{1/3})(\pi(x^{1/3}) + 1)(2\pi(x^{1/3}) + 1)}{6}
\]

\[
\geq \frac{\pi(x^{1/3})^3}{6}
\]

\[
\geq \frac{1}{6} \left( \frac{x^{1/3}}{\log x^{1/3}} \left( 1 + \frac{1}{\log x^{1/3}} \right) \right)^3
\]

\[
\geq \frac{4.5x}{\log^3 x} \left( 1 + \frac{9}{\log x} \right)
\]

\[
= \frac{4.5x}{\log^3 x} + 40.5x
\]

Here we have used Lemma 1. Combining these bounds, we obtain the lemma. \( \square \)
Lemma 8. For all $x \geq x_0$, we have
\[
S_5 - S_7 \geq \frac{0.5L^2 + 0.121434L - 0.22}{\log x} + \frac{-1.5244L + 1.112}{\log^2 x} + \frac{-2L^2 + 2.9067L + 2.5389}{\log^3 x}.
\]

Proof. Recall the definitions
\[
S_5 = \sum_{p \leq x^{1/3}} \frac{\log \log x}{p \log \frac{x}{p}}, \quad S_7 = \sum_{p \leq x^{1/3}} \frac{\log x}{p \log \frac{x}{p}}.
\]

We apply partial summation with $f(t) = \log \log \frac{x}{t^2} / \log \frac{x}{t}$ to obtain
\[
S_5 = \frac{T(x^{1/3}) \log \log x^{1/3}}{\log x^{2/3}} - \int_2^{x^{1/3}} \frac{T(t) \log \log \frac{x}{t}}{t \log^2 \frac{x}{t}} \, dt + \int_2^{x^{1/3}} \frac{2T(t) \, dt}{t \log \frac{x}{t} \log \frac{x}{t^2}}.
\]

We next apply partial summation with $g(t) = \log \log \frac{x}{t} / \log \frac{x}{t}$ to obtain
\[
S_7 = \frac{T(x^{1/3}) \log \log x^{1/3}}{\log x^{2/3}} - \int_2^{x^{1/3}} \frac{T(t) \log \log t \, dt}{t \log^2 \frac{x}{t}} - \int_2^{x^{1/3}} \frac{T(t) \, dt}{t \log t \log \frac{x}{t}}.
\]

Subtracting,
\[
S_5 - S_7 = -I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 = \int_2^{x^{1/3}} \frac{T(t) \log \log \frac{x}{t} \, dt}{t \log^2 \frac{x}{t}},
\]
\[
I_2 = \int_2^{x^{1/3}} \frac{2T(t) \, dt}{t \log \frac{x}{t} \log \frac{x}{t^2}},
\]
\[
I_3 = \int_2^{x^{1/3}} \frac{T(t) \log \log t \, dt}{t \log^2 \frac{x}{t}},
\]
and
\[
I_4 = \int_2^{x^{1/3}} \frac{T(t) \, dt}{t \log t \log \frac{x}{t}}.
\]
Let $x \geq x_0$. Splitting the interval at $x^{1/6}$,

\[ I_1 \leq \left( L - \log \frac{3}{2} \right) \int_{x^{1/6}}^{x^{1/3}} \frac{T(t)}{t \log^2 \frac{x}{t}} \, dt + L \int_{2}^{x^{1/6}} \frac{T(t)}{t \log^2 \frac{x}{t}} \, dt \]

\[ \leq \left( L - \log \frac{3}{2} \right) \int_{x^{1/6}}^{x^{1/3}} \log \log t + B + \frac{1}{\log^2 t} \, dt + L \int_{2}^{x^{1/6}} \log \log t + B + \frac{1}{\log^2 t} \, dt \]

\[ = \left( L - \log \frac{3}{2} \right) \left( F(x^{1/3}) - F(x^{1/6}) \right) + L \left( F(x^{1/6}) - F(2) \right) \]

\[ = L \left( F(x^{1/3}) - F(2) \right) - \left( \log \frac{3}{2} \right) \left( F(x^{1/3}) - F(x^{1/6}) \right) , \]

where the antiderivative $F$ is given by

\[ \frac{\log \log \frac{x}{t}}{\log x} + \frac{\log t \log \log t}{\log x \log \frac{x}{t}} + \frac{B}{\log^2 x \log \frac{x}{t}} + \frac{1}{\log^2 x \log \frac{x}{t}} \]

\[ - \frac{1}{\log^2 x \log t} - \frac{2 \log \log \frac{x}{t}}{\log^3 x} + \frac{2 \log \log t}{\log^3 x} . \]

Bounding the resulting expression gives

\[ I_1 \leq \frac{0.5L^2 - 0.94565L + 0.1361}{\log x} + \frac{2.2153L}{\log^2 x} + \frac{2L^2 - 2.9067L - 2.0810}{\log^3 x} , \]

where we made use of the identity

\[ \log \log \frac{x}{2} = \log \log x + \log \left( 1 - \frac{2}{\log x} \right) \]

and bounded the Maclaurin series of the right term by comparison to a geometric series. We next bound

\[ I_2 \geq \int_{2}^{x^{1/3}} \frac{2(\log \log t + c)}{t \log^2 \frac{x}{t}} \, dt = \int_{\log 2}^{\log x^{1/3}} \frac{2(\log u + c)}{\log x - u}(\log x - 2u) \, du \]

Integrating, a lower bound for $I_2$ is

\[ \frac{2}{\log x} \left[ \text{Li}_2 \left( \frac{\log t}{\log x} \right) - \text{Li}_2 \left( \frac{\log^2 t}{\log x} \right) + (\log \log t + c) \left( \log \log \frac{x}{t} - \log \log \frac{x}{t^2} \right) \right]_{2}^{x^{1/3}} \]

\[ = \frac{2}{\log x} \left( \text{Li}_2 \left( \frac{1}{3} \right) - \text{Li}_2 \left( \frac{2}{3} \right) + (\log 2)(L - \log 3 + c) \right) \]

\[ + \frac{2}{\log x} \left( -\text{Li}_2 \left( \frac{\log 2}{\log x} \right) + \text{Li}_2 \left( \frac{\log 4}{\log x} \right) + (\log \log 2 + c) \log \left( 1 - \frac{2}{\log x^2} \right) \right) \]

\[ \geq \frac{(\log 4)L - 2.0947}{\log x} + \frac{(\log 4)(1 - \log \log 2 - c)}{\log^2 x} + \frac{\log^2 2}{2 \log^3 x} . \]
Here \( \text{Li}_2 \) denotes the dilogarithm, defined by the improper integral

\[
\text{Li}_2(z) = - \int_0^z \frac{\log(1-t)}{t} \, dt.
\]

For the terms \( \log 4 / \log^2 x + (\log^2 2) / (2 \log^3 x) \) above, we have used the fact that the dilogarithm is concave up, together with the expansion

\[
\text{Li}_2(z) = \sum_{k \geq 1} \frac{z^k}{k^2}
\]

for \(|z| < 1\), putting \( z = \log 2 / \log x \), to bound

\[
\text{Li}_2 \left( \log \frac{4}{\log x} \right) - \text{Li}_2 \left( \log \frac{2}{\log x} \right) \geq \text{Li}_2 \left( \frac{\log 4 - \log 2}{\log x} \right) - \text{Li}_2(0)
\]

\[
= \text{Li}_2 \left( \frac{\log 2}{\log x} \right) \geq \frac{\log 2}{\log x} + \frac{\log^2 2}{4 \log^2 x}.
\]

We have also used the bound \(- \log(1-z) \geq z\), putting \( z = \log 2 / \log(x/2) \) and noting that \( \log 2 + c < 0 \).

We now turn to \( I_3 \). In order to bound \( I_3 \) in the right direction, we write

\[
I_3 = \int_2^c \frac{T(t) \log \log t \, dt}{t \log^2 \frac{x}{t}} + \int_c^{x^{1/3}} \frac{T(t) \log \log t \, dt}{t \log^2 \frac{x}{t}}.
\]

The first integral is equal to

\[
\frac{1}{2 \log x} \left( \log \left( 1 - \frac{1 - \log 2}{\log \frac{x}{2}} \right) - \frac{\log 2 \log \log 2}{\log \frac{x}{2}} \right) \geq -0.0304 \frac{1}{\log^2 x}.
\]

A lower bound for the second integral is

\[
\int_c^{x^{1/3}} \frac{(\log \log t + c) \log \log t \, dt}{t \log^2 \frac{x}{t}} = \int_1^{\log x^{1/3}} \frac{(\log u + c) \log u \, du}{(\log x - u)^2}
\]

\[
= \frac{1}{\log x} \left[ c \log \left( 1 - \frac{u}{\log x} \right) + \frac{cu \log u}{\log x - u} + 2 \text{Li}_2 \left( \frac{u}{\log x} \right) \right]_{\log x^{1/3}} + \frac{1}{\log x} \left[ 2(\log u) \log \left( 1 - \frac{u}{\log x} \right) + \frac{u \log^2 u}{\log x - u} \right]_{\log x^{1/3}}
\]

\[
\geq 0.5L^2 - 1.77881L + 1.9771 + \frac{c - 2}{\log^2 x} - 0.5188 \frac{1}{\log^3 x}.
\]

Next, note that the use of the inequality \( T(t) > \log \log t + c \) allows us to add to the
lower bound
\[
\int_3^{x^{1/3}} \frac{(T(t) - (\log \log t + c)) \log \log t}{t \log^2 \frac{x}{t}} dt \\
\geq \frac{1}{\log^2 \frac{x}{3}} \sum_{2 \leq k \leq 1228} \int_{\log p_k}^{\log p_{k+1}} (T(p_k) - c - \log u) \log u \\du \\
\geq \frac{0.3099}{\log^2 x} \left( 1 + \frac{\log 9}{\log x} \right).
\]
We therefore have
\[
I_3 \geq \frac{0.5L^2 - 1.77881L + 1.9771}{\log x} - \frac{1.4591}{\log^2 x} + \frac{0.1621}{\log^3 x}.
\]
We now consider
\[
I_4 = \int_2^{x^{1/3}} \frac{T(t)}{t \log t \log \frac{x}{t}} dt \geq \int_2^{x^{1/3}} \frac{(\log \log t + c)}{t \log t \log \frac{x}{t}} dt = \int_{\log 2}^{\log x^{1/3}} \frac{(\log u + c) du}{u(\log x - u)}.
\]
Integrating, we obtain the expression
\[
\frac{1}{\log x} \left[ c \log u - (\log u) \log \left( 1 - \frac{u}{\log x} \right) - c \log(\log x - u) - \text{Li}_2 \left( \frac{u}{\log x} \right) + \frac{\log^2 u}{2} \right]_{\log 2}^{\log x^{1/3}}.
\]
We bound this expression to obtain
\[
I_4 \geq \frac{0.5L^2 - 0.4317L - 0.3608}{\log x} + \frac{0.7658}{\log^2 x} + \frac{0.0556}{\log^3 x}.
\]
Note that the inequality \(T(t) > \log \log t + c\) was used to bound \(I_4\). We may therefore add the following expression to the lower bound:
\[
\int_2^{x^{1/3}} \frac{T(t) - (\log \log t + c)}{t \log t \log \frac{x}{t}} dt \geq \frac{1}{\log^2 \frac{x}{2}} \int_2^{x^{1/3}} \frac{T(t)}{t \log t} dt.
\]
Splitting the interval \([2, y_0]\) into subintervals between primes \([p_k, p_{k+1}], k = 1, 2, \ldots, 1228\), and using the numerical value of \(T(p_k)\) in each subinterval, we find by computation in Pari/GP that
\[
\frac{1}{\log^2 \frac{x}{2}} \int_2^{x^{1/3}} \frac{T(t) - (\log \log t + c)}{t \log t} dt \geq \frac{0.3945}{\log \frac{x}{2}} \geq \frac{0.3945}{\log x} \left( 1 + \frac{\log 2}{\log x} \right).
\]
A similar argument allows us to improve our upper bound on the integral \(I_1\) in equation (6). Here we used the upper bound
\[
L \int_2^{x^{1/6}} \frac{T(t)}{t \log^2 \frac{x}{t}} dt \leq L \int_2^{x^{1/6}} \frac{\log \log t + B + \frac{1}{\log^2 \frac{x}{t}}}{t \log^2 \frac{x}{t}} dt.
\]
Bounding the difference in the interval $[2, 100]$, we add $0.6909L/\log^2 x$ to the lower bound. Combining the bounds for $I_1, I_2, I_3$, and $I_4$ completes the proof of Lemma 8.

\[ \Box \]

5. The Proof of Theorem 1

We now prove Theorem 1. Note that $\tau_3(x) = \pi_3(x) + N(x)$, where $N(x) = |\{n \leq x : n = p^2q\}|$, and $p$ and $q$ denote (possibly equal) primes. For each such $p$, the number of possibilities for $q$ is $\pi(x/p^2)$, and we thus have

\[ N(x) = \sum_{p \leq \sqrt{x}} \pi \left( \frac{x}{p^2} \right). \]

The range $500194 \leq x \leq 10^{12}$ was checked using the computer program Pari/GP. Thus we may assume $x \geq x_0$. Now, a lower bound for $\pi_3(x)$ is given in Theorem 2. The contribution from $N(x)$ is given by the following lemma, from which Theorem 1 follows by an argument analogous to that of inequality (5) in the proof of Theorem 2.

**Lemma 9.** For all $x \geq x_0$ we have

\[ \sum_{p \leq \sqrt{x}} \pi \left( \frac{x}{p^2} \right) \geq x \left( \frac{\alpha}{\log x} + \frac{0.9861}{\log^2 x} + \frac{1.2861}{\log^3 x} \right), \]

where $\alpha = \sum_p 1/p^2 = 0.4522474\ldots$ denotes the reciprocal sum of squares of primes.

**Proof.** Let $x \geq x_0$. We wish to apply the lower bound $\pi(t) > t/\log t$, valid for $t \geq 17$, in Lemma 1. Thus we write

\[ \sum_{p \leq \sqrt{x}} \pi \left( \frac{x}{p^2} \right) = \sum_{p \leq \sqrt{x}} \pi \left( \frac{x}{p^2} \right) + \sum_{\sqrt{x} < p \leq \sqrt{x}} \pi \left( \frac{x}{p^2} \right). \]

The right sum is

\[ \sum_{\sqrt{x} < p \leq \sqrt{x}} \pi \left( \frac{x}{p^2} \right) = -6 \cdot \pi \left( \sqrt{\frac{x}{17}} \right) + \pi \left( \sqrt{\frac{x}{13}} \right) + \ldots + \pi \left( \sqrt{\frac{x}{2}} \right). \]

Also by Lemma 1, this expression is bounded below by

\[ 2\sqrt{x} \left( -6 \cdot \frac{1.0972}{\sqrt{17} \log \frac{x}{17}} + \sum_{k=1}^{6} \frac{1}{\sqrt{pk \log \frac{x}{pk}}} \right) \geq \frac{1.8188 \sqrt{x}}{\log x}, \]
where $p_k$ denotes the $k$-th prime number. For the remaining term, let $S(t) = \sum_{p \leq t} (1/p^2)$. By partial summation,

$$
\sum_{p \leq \sqrt{\frac{t}{17}}} \frac{1}{p^2 \log \frac{p}{17}} = \frac{S(\sqrt{\frac{t}{17}})}{\log 17} - \int_2^{\sqrt{\frac{t}{17}}} \frac{2S(t) \, dt}{t \log^2 \frac{t}{17}}
$$

$$
= \frac{S(\sqrt{\frac{t}{17}})}{\log 17} - \int_2^{\sqrt{\frac{t}{17}}} \frac{2 \alpha \, dt}{t \log^2 \frac{t}{17}} + \int_2^{\sqrt{\frac{t}{17}}} \frac{2(\alpha - S(t)) \, dt}{t \log^2 \frac{t}{17}}.
$$

Now,

$$
\int_2^{\sqrt{\frac{t}{17}}} \frac{2 \alpha \, dt}{t \log^2 \frac{t}{17}} = \frac{\alpha}{\log 17} - \frac{\alpha}{\log \frac{t}{17}} \leq \frac{\alpha}{\log 17} - \frac{\alpha}{\log x} \left(1 + \frac{\log 4}{\log x}\right).
$$

Also,

$$
\int_2^{\sqrt{\frac{t}{17}}} \frac{2(\alpha - S(t)) \, dt}{t \log^2 \frac{t}{17}} \geq \sum_{k \leq 21433} (\alpha - S(p_k)) \left[ \frac{1}{\log \frac{p_k+1}{p_k}} \right] \left[ \frac{\log \frac{p_k+1}{p_k}}{\log \frac{p_k}{p_k+1}} \right] \left[ \frac{\log \frac{p_k}{p_k+1}}{\log \frac{x}{p_k+1}} \right]
$$

$$
\geq \sum_{k \leq 21433} \left( \frac{0.3592}{\log \frac{3}{4} \log \frac{t}{17}} \right)
$$

$$
\geq \frac{0.3592}{\log^2 x} \left(1 + \frac{\log 9}{\log x}\right) \left(1 + \frac{\log 4}{\log x}\right)
$$

$$
\geq \frac{0.3592}{\log^2 x} + \frac{1.2872}{\log^3 x} .
$$

Thus,

$$
\sum_{p \leq \sqrt{\frac{t}{17}}} \frac{1}{p^2 \log(p/17)} \geq \frac{\alpha}{\log x} \left(1 + \frac{\log 4}{\log x}\right) + \frac{0.3592}{\log^2 x} + \frac{1.2872}{\log^3 x} - \frac{1}{\log 17} \sum_{p > \sqrt{\frac{t}{17}}} \frac{1}{p^2}.
$$

By [9, Lem. 2.7], we have

$$
\frac{1}{\log 17} \sum_{p > \sqrt{\frac{t}{17}}} \frac{1}{p^2} < \frac{1}{\sqrt{\frac{t}{17}}} \log \frac{\sqrt{t/17}}{17} = \frac{2\sqrt{17}/\log 17}{\sqrt{x} \log \frac{\sqrt{t}{17}}} \leq \frac{3.2431}{\sqrt{x} \log x}.
$$

Combining these bounds, we obtain Lemma 9.

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References


