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LOWER BOUNDS FOR NUMBERS WITH THREE PRIME FACTORS

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Abstract

A *k-almost prime number* is a product of k prime numbers, some of which may be repeated. By a 1900 theorem of Landau, the number of k-almost prime numbers not exceeding x is asymptotic to $x(\log\log x)^{k-1}/((k-1)!\log x)$. We prove a numerically explicit lower bound for 3-almost prime numbers which is asymptotic to Landau's formula, and hence to the actual count. It exceeds Landau's formula for all $x \geq 500194$. We prove an analogous lower bound for products of three distinct prime numbers. This expands on previously known results for $k \leq 2$.

1. Introduction

For a natural number n, the arithmetic functions $\omega(n)$ and $\Omega(n)$ denote the number of prime factors of n, counted without (respectively with) repeated prime factors. Thus for n with prime factorization $n = p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}$, we have $\omega(n) = k$ and $\Omega(n) = a_1 + \ldots + a_k$.

Numbers n such that $\Omega(n) = k$ are called *k-almost primes*. Let $\pi_k(x) = |\{n \leq x : \omega(n) = \Omega(n) = k\}|$ denote the counting function of squarefree k-almost primes, and let $\tau_k(x) = |\{n \leq x : \Omega(n) = k\}|$ denote the counting function of k-almost primes.

For $k = 1$ we have $\pi_1(x) = \tau_1(x) = \pi(x)$, the prime counting function. The prime number theorem asserts that $\pi(x)$ is asymptotic to $x/\log x$. In 1900, Landau [6] proved that for each $k \in \mathbb{N}$, the estimate

$$\pi_k(x) = \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\left(1 + O\left(\frac{1}{\log\log x}\right)\right)$$

holds, and that the same estimate also holds for $\tau_k(x)$.

Selberg [11] proved that for any $0 < \delta < 1$, uniformly for all $x \geq 3$ and $1 \leq k \leq (2-\delta)\log\log x$, we have

$$\tau_k(x) = G\left(\frac{k-1}{\log\log x}\right)\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\left(1 + O\left(\frac{k}{(\log\log x)^2}\right)\right),$$

where

$$G(z) = F(1,z)/\Gamma(z+1) \quad \text{and} \quad F(s,z) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z.$$

Classic references [5, 7] provide details on these and similar results. A 1962 paper [10] of Rosser and Schoenfeld and contemporary work [3, 4] of Dusart give explicit bounds for $\pi(x)$, see Lemma 1. A 2018 paper [1] of Bayless et al. established explicit upper bounds for $\pi_k(x)$, $\tau_2(x)$, and $\tau_3(x)$, as well as explicit lower bounds for $\pi_2(x)$ and $\tau_2(x)$. In particular, it is shown [1, Thm. 3.5] that for all $k \geq 2$ and $x \geq 3$, we have

$$\pi_k(x) \leq \frac{1.028x(\log\log x + 0.26153)^{k-1}}{(k-1)!\log x},$$

and [1, Thm. 5.2] that for all $x \geq 10^{12}$, we have

$$\tau_2(x) \geq \pi_2(x) \geq \frac{x(\log\log x + 0.1769)}{\log x}\left(1 + \frac{0.4232}{\log x}\right).$$

Furthermore [1, Thm. 5.3], for all $x \geq 10^{12}$ we have

$$\tau_3(x) \leq \frac{1.028x((\log\log x + 0.26153)^2 + 1.055852)}{2\log x}.$$

A relatively sharp explicit lower bound for $\pi_3(x)$ requires more work, and aside from the explicit results above, the literature consists of implied constants. We prove explicit lower bounds for $\pi_3(x)$ and $\tau_3(x)$ which are asymptotic to Landau's formula, and hence to the actual count. In particular, we prove the following three theorems.

Theorem 1. *For all* $x \geq 500194$*,*

$$\tau_3(x) > \frac{x(\log\log x)^2}{2\log x}.$$

The constant 500194 is optimal, since the inequality is violated at $x = 500194 - \epsilon$ for all sufficiently small $\epsilon > 0$. We also have an analogue of Theorem 1 for squarefree 3-almost primes, also called *sphenic numbers.*

Theorem 2. *For all* $x \geq 10203553$*,*

$$\pi_3(x) > \frac{x((\log\log x)^2 - 1)}{2\log x}.$$

The constant 10203553 is also optimal. We also obtain the following.

Theorem 3. *For all sufficiently large* x*,*

$$\pi_3(x) > \frac{x(\log\log x)^2}{2\log x}.$$

Theorem 3 follows readily from the proof of Theorem 2 by comparing secondary terms, however the reader will see that x must be very large. Without more work, it is unclear where the optimal cutoff c_0 is for Theorem 3 to apply. Lifchitz and Renner [8] computed the values of $\pi_3(10^k)$ for $1 \leq k \leq 19$, and this data shows that $c_0 > 10^{19}$.

2. Notation and Preliminary Lemmas

Throughout the paper we let p, q, and r denote prime numbers and we let $\log x$ denote the natural logarithm. We define $L = \log\log x$, $x_0 = 10^{12}$, and $y_0 = x_0^{1/3} = 10^4$. Furthermore, $\pi(t)$ denotes the prime counting function and $T(t) = \sum_{p \leq t} 1/p$ denotes the sum of reciprocals of prime numbers $p \leq t$. We let $B = 0.2614972128\ldots$ denote the *Mertens constant* and $c = 0.26146521$ (see Lemmas 2 and 3).

In bounding expressions, we will make use of manipulations such as

$$\log\log\frac{x}{a} = \log\log x + \log\left(1 - \frac{\log a}{\log x}\right),$$

the Maclaurin series expansion of the righthand term, and the bounds

$$\frac{1}{\log x}\left(1 + \frac{\log a}{\log x}\right) < \frac{1}{\log\frac{x}{a}} \leq \frac{1}{\log x} + \frac{1}{\log^2 x} \cdot \frac{\log a \log x_0}{\log\frac{x_0}{a}}$$

for $a > 1$ and $x \geq x_0$. We will use the following bounds [10, Thm. 2] of Rosser and Schoenfeld and [4, Cor. 5.2] of Dusart on the prime counting function.

Lemma 1 (Rosser, Schoenfeld, Dusart). *We have*

$$\frac{x}{\log x} < \pi(x)$$

for all $x \geq 17$. *Additionally, we have*

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right),$$

where the lower bound holds for all $x \geq 599$ *and the upper bound holds for all* $x > 1$. *Furthermore, for all* $x > 1$ *we have*

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.53816}{\log^2 x}\right).$$

We also use the following bounds [10, Theorem 5] of Rosser and Schoenfeld and [3, Thm. 6.10], [4, Thm. 5.6] of Dusart which give numerically explicit versions of Mertens' second theorem.

Lemma 2 (Rosser, Schoenfeld, Dusart). *Let* $T(t) = \sum_{p \le t} 1/p$ *denote the reciprocal sum of prime numbers up to* t*. We have*

$$T(t) = \log\log t + B + E(t),$$

where $B = 0.2614972128\ldots$ *denotes the Mertens constant, and*

$$-\frac{1}{2\log^2 t} < E(t) < \frac{1}{\log^2 t}$$

for all $t > 1$*,*

$$|E(t)| < \frac{1}{2\log^2 t}$$

for all $t \ge 286$*,*

$$|E(t)| \le \frac{1}{10\log^2 t} + \frac{4}{15\log^3 t}$$

for all $t \ge 10372$*, and*

$$|E(t)| \le \frac{1}{5\log^3 t}$$

for all $t \ge 2278383$*.*

Combining bounds on the prime number reciprocal sum in [10, 4] readily yields the following lower bound.

Lemma 3. *For* $x > 1$*,*

$$\sum_{p \le x} \frac{1}{p} > \log\log x + 0.26146521.$$

Proof. By Lemma 2, for all $x \ge 2278383$,

$$\sum_{p \le x} \frac{1}{p} \ge \log\log x + B - \frac{0.2}{\log^3 x}, \tag{1}$$

where B is the Mertens constant. By [10, Thm 20],

$$\sum_{p \le x} \frac{1}{p} > \log\log x + B$$

for all $x \le 10^8$. Substituting 10^8 in (1) gives the lemma. □

3. The Proof of Theorem 2

We have

$$\pi_3(x) = |\{n = pqr \le x : p < q < r\}|$$

where p, q, and r denote prime numbers. To determine the count, we note the possible values for each of p, q, r. We have $p < x^{1/3}$, and $q^2 < qr \le x/p$, so that $p < q < \sqrt{x/p}$. Since $pqr \le x$, we also have $q < r \le x/(pq)$. Therefore, $\pi_3(x)$ is equal to the sum of 1 over all possible values of p, q, r, so that

$$\pi_3(x) = \sum_{p<x^{1/3}} \sum_{p<q<\sqrt{\frac{x}{p}}} \sum_{q<r\le\frac{x}{pq}} 1 = \sum_{p\le x^{1/3}} \sum_{p<q\le\sqrt{\frac{x}{p}}} \left(\pi\left(\frac{x}{pq}\right) - \pi(q) \right).$$

Here we have used the fact that the strict inequalities $p < x^{1/3}$ and $q < \sqrt{x/p}$ can be written to include the case of equality, since the quantity vanishes when $q = \sqrt{x/p}$ or when $p = x^{1/3}$. We checked the inequality using the computer program Pari/GP for the interval $10203553 \le x \le 10^{12}$. Thus we may assume $x \ge x_0$. By Lemma 1, since $x/(pq) \ge 599$ we have

$$\pi\left(\frac{x}{pq}\right) \ge \frac{x}{pq\log\frac{x}{pq}}\left(1 + \frac{1}{\log\frac{x}{pq}}\right) \ge \frac{x}{pq\log\frac{x}{pq}}\left(1 + \frac{1}{\log x}\right).$$

We thus have

$$\pi_3(x) \ge \left(1 + \frac{1}{\log x}\right) \sum_{p\le x^{1/3}} S_1 - \sum_{p\le x^{1/3}} S_2,$$

where

$$S_1 = \sum_{p<q\le\sqrt{\frac{x}{p}}} \frac{x}{pq\log\frac{x}{pq}}$$

and

$$S_2 = \sum_{p<q\le\sqrt{\frac{x}{p}}} \pi(q).$$

We determine a lower bound for S_1 and an upper bound for S_2. By Lemma 7 below, we have

$$\sum_{p\le x^{1/3}} S_2 \le x\left(\frac{2L + 0.1436}{\log^2 x} + \frac{10.9113L + 3.1227}{\log^3 x}\right).$$

We next consider S_1. We have

$$S_1 = \frac{x}{p} \sum_{p<q\le\sqrt{y}} \frac{1}{q\log\frac{y}{q}}$$

where $y = x/p > x^{2/3} \geq x_0^{2/3}$. Recall that the function $T(t) = \sum_{p \leq t} 1/p$ denotes the prime reciprocal sum up to t. We bound S_1 below by applying partial summation to obtain

$$\sum_{p<q\leq\sqrt{y}} \frac{1}{q\log\frac{y}{q}} = \frac{T(\sqrt{y})}{\log\sqrt{y}} - \frac{T(p)}{\log\frac{y}{p}} - \int_p^{\sqrt{y}} \frac{T(t)\ dt}{t\log^2\frac{y}{t}}. \tag{2}$$

By Lemma 2 this is bounded below by the expression

$$\frac{2(\log\log\sqrt{y} + B - \frac{1}{2\log^2\sqrt{y}})}{\log y} - \frac{\log\log p + B + \frac{1}{\log^2 p}}{\log\frac{y}{p}}$$
$$- \int_p^{\sqrt{y}} \frac{\log\log t + B + \frac{1}{\log^2 t}}{t\log^2\frac{y}{t}}\ dt.$$

Substituting $u = \log t$ and integrating, this is equal to

$$\frac{-4}{\log^3 y} - \frac{\log\log p}{\log\frac{y}{p}} - \frac{1}{\log^2 p\log\frac{y}{p}} + \frac{\log\log\frac{y}{p}}{\log y} + \frac{\log p\log\log p}{\log y\log\frac{y}{p}}$$
$$+ \frac{1}{\log^2 y\log\frac{y}{p}} - \frac{1}{\log^2 y\log p} - \frac{2\log\log\frac{y}{p}}{\log^3 y} + \frac{2\log\log p}{\log^3 y}.$$

Thus a lower bound for S_1 is

$$\frac{x}{p\log\frac{x}{p}}\left(\log\log\frac{x}{p^2} + \frac{\log p\log\log p}{\log\frac{x}{p^2}} + \frac{1}{\log\frac{x}{p}\log\frac{x}{p^2}} + \frac{2\log\log p}{\log^2\frac{x}{p}}\right)$$
$$- \frac{x}{p\log\frac{x}{p}}\left(\frac{4}{\log^2\frac{x}{p}} + \log\log p\cdot\frac{\log\frac{x}{p}}{\log\frac{x}{p^2}} + \frac{1}{\log^2 p}\cdot\frac{\log\frac{x}{p}}{\log\frac{x}{p^2}} + \frac{1}{\log\frac{x}{p}\log p} + \frac{2\log\log\frac{x}{p^2}}{\log^2\frac{x}{p}}\right). \tag{3}$$

To simplify this expression, we compare the second (respectively, third) term of the first line above to the second (respectively, first) term of the second line. We have

$$\frac{\log p\log\log p}{\log\frac{x}{p^2}} - \log\log p\cdot\frac{\log\frac{x}{p}}{\log\frac{x}{p^2}} = -\log\log p.$$

Also, $1/(\log(x/p)\log(x/p^2)) > 1/\log^2(x/p)$. Therefore, a lower bound for expression (3) is

$$\frac{x}{p\log\frac{x}{p}}\left(\log\log\frac{x}{p^2} + \frac{2\log\log p}{\log^2\frac{x}{p}}\right)$$
$$- \frac{x}{p\log\frac{x}{p}}\left(\log\log p + \frac{3}{\log^2\frac{x}{p}} + \frac{1}{\log^2 p}\cdot\frac{\log\frac{x}{p}}{\log\frac{x}{p^2}} + \frac{1}{\log\frac{x}{p}\log p} + \frac{2\log\log\frac{x}{p^2}}{\log^2\frac{x}{p}}\right). \tag{4}$$

Write the sum of this expression over $p \le x^{1/3}$ as

$$x\left(S_5 + S_6 - (S_7 + S_8 + S_9 + S_{10} + S_{11})\right).$$

Since $2\log\log p - 3$ is negative for $p < 89$ and positive for $p \ge 89$,

$$\begin{aligned} S_6 - S_8 &= \sum_{p \le x^{1/3}} \frac{2\log\log p - 3}{p\log^3 \frac{x}{p}} \\ &\ge \frac{1}{\log^3 x}\sum_{p<89} \frac{2\log\log p - 3}{p}\frac{\log^3 x_0}{\log^3 \frac{x_0}{p}} + \frac{1}{\log^3 x}\sum_{89 \le p \le y_0} \frac{2\log\log p - 3}{p} \\ &\ge \frac{-3.9322}{\log^3 x}. \end{aligned}$$

Note that

$$\begin{aligned} S_{10} = \sum_{p\le x^{1/3}} \frac{1}{p\log p \log^2 \frac{x}{p}} &\le \frac{1}{\log^2 x}\sum_{p \le y_0} \frac{1}{p\log p}\frac{\log^2 x_0}{\log^2 \frac{x_0}{p}} + \frac{2.25}{\log^2 x}\sum_{y_0 < p} \frac{1}{p\log p} \\ &\le \frac{1.7496}{\log^2 x} + \frac{2.25(0.1085)}{\log^2 x} \\ &\le \frac{1.9938}{\log^2 x}. \end{aligned}$$

Here we verified the value $\sum_p 1/(p\log p) = 1.63661\ldots$ found to much higher precision [2, p. 6] by H. Cohen. Similarly,

$$\begin{aligned} S_9 &= \sum_{p \le x^{1/3}} \frac{1}{p\log^2 p \log \frac{x}{p^2}} \\ &\le \sum_{p \le y_0} \frac{1}{p\log^2 p}\left(\frac{1}{\log x} + \frac{1}{\log^2 x}\frac{2\log p\log x_0}{\log\frac{x_0}{p^2}}\right) + \frac{3}{\log x}\sum_{y_0 < p \le x^{1/3}} \frac{1}{p\log^2 p} \\ &\le \frac{1.5151}{\log x} + \frac{3.5585}{\log^2 x} + \frac{3}{\log x}\sum_{y_0<p}\frac{1}{p\log^2 p} \le \frac{1.5328}{\log x} + \frac{3.5585}{\log^2 x}. \end{aligned}$$

Here we computed the sum over $y_0 < p \le 10^9$ using Pari/GP and then applied partial summation together with Lemmas 2 and 3 to bound the sum over $p > 10^9$. By Lemma 8,

$$\begin{aligned} S_5 - S_7 \ge{}& \frac{0.5L^2 + 0.121434L - 0.22}{\log x} \\ &+ \frac{-1.5244L + 1.112}{\log^2 x} + \frac{-2L^2 + 2.9067L + 2.5389}{\log^3 x}, \end{aligned}$$

and by Lemma 5,

$$S_{11} \le \frac{2L^2 + 1.3972L}{\log^3 x}.$$

Finally, note that the application of the bounds

$$L + c < T(x) < L + B + \frac{1}{\log^2 x},$$

valid for all $x > 1$, incur error terms which are particularly large for small values of x. We may improve our bounds as follows. In equation (2) we replaced the quantity $-T(p)/(p\log(x/p^2))$ with

$$-\frac{\log\log p + B + \frac{1}{\log^2 p}}{p\log\frac{x}{p^2}}.$$

Thus we may add the following expression to the lower bound:

$$\sum_{p\le x^{1/3}} \frac{\log\log p + B + \frac{1}{\log^2 p} - T(p)}{p\log\frac{x}{p^2}}.$$

A computation gives a lower bound of

$$\frac{1}{\log\frac{x}{4}}\sum_{p\le y_0}\frac{\log\log p + B + \frac{1}{\log^2 p} - T(p)}{p} \ge \frac{0.9}{\log\frac{x}{4}} \ge \frac{0.9}{\log x}\left(1 + \frac{\log 4}{\log x} + \frac{\log^2 4}{\log^2 x}\right).$$

Again considering equation (2) and the following displayed expression, we may add to the lower bound

$$\begin{aligned}
\sum_{p\le x^{1/3}} \int_p^{\sqrt{\frac{x}{p}}} & \frac{\log\log t + B + \frac{1}{\log^2 t} - T(t)}{t\log^2\frac{x}{pt}}dt \\
&\ge \frac{1}{\log^2 x}\sum_{k\le 1228} k\int_{\log p_k}^{\log p_{k+1}}\left(\log u + B + \frac{1}{u^2} - T(p_k)\right)du \\
&\ge \frac{20.4395}{\log^2 x}.
\end{aligned}$$

Here p_k denotes the k-th prime number. Combining all bounds, we obtain

$$\begin{aligned}
\pi_3(x) &\ge \frac{x(0.5L^2 + 0.121434L - 0.8528)}{\log x} + \frac{x(0.5L^2 - 4L + 15)}{\log^2 x} \\
&\qquad - \frac{x(4L^2 + 10.9262L)}{\log^3 x} \\
&\ge \frac{x(0.5L^2 + 0.121434L - 0.8528)}{\log x} + \frac{x(0.5L^2 - 4L + 12)}{\log^2 x} \qquad (5)\\
&\ge \frac{x(0.5L^2 + 0.121434L - 0.8528)}{\log x} + \frac{4x}{\log^2 x} \\
&> \frac{x((\log\log x)^2 - 1)}{2\log x} + \frac{4x}{\log^2 x}
\end{aligned}$$

for all $x \ge x_0$. This completes the proof of Theorem 2.

4. Additional Lemmas

We prove several lemmas used in the proof of Theorem 2.

Lemma 4. *For $x \geq x_0$, we have*

$$\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} \leq \frac{L + 0.071781}{\log^2 x}.$$

Proof. For $x_0 \leq x \leq 10372^3$, we have

$$\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} \leq \frac{1}{\log^2 x} \sum_{p \leq 10372} \frac{\log^2 x_0}{p \log^2 \frac{x_0}{p}} \leq \frac{\log\log x_0 - 0.0279}{\log^2 x} \leq \frac{L}{\log^2 x},$$

by directly computing the sum over $p \leq 10372$. Suppose next that $x > 10372^3$. By Lemmas 2 and 3 and partial summation,

$$\begin{aligned}\sum_{p \leq x^{1/3}} \frac{1}{p \log^2 \frac{x}{p}} &= \frac{T(x^{1/3})}{\log^2(x^{2/3})} - \int_2^{x^{1/3}} \frac{2T(t)\ dt}{t \log^3 \frac{x}{t}} \\ &\leq \frac{\frac{9}{4}(\log\log x^{1/3} + 0.26300402)}{\log^2 x} - \int_2^{x^{1/3}} \frac{2(\log\log t + c)\ dt}{t \log^3 \frac{x}{t}},\end{aligned}$$

recalling that $c = 0.26146521$. The integral is equal to

$$\left[\frac{\log\log t + c}{\log^2 \frac{x}{t}} + \frac{\log\log \frac{x}{t} - \log\log t}{\log^2 x} - \frac{1}{\log x \log \frac{x}{t}}\right]_2^{x^{1/3}}.$$

Subtracting and bounding the resulting expression, we obtain

$$S_3 \leq \frac{\frac{9}{4}(0.26300402 - c) - \log 2 + \frac{1}{2} + c + \log\log \frac{x}{2}}{\log^2 x} \leq \frac{L + 0.071781}{\log^2 x}$$

for all $x \geq 10372^3$, and therefore for all $x \geq x_0$. □

Lemma 5. *For $x \geq x_0$, we have*

$$\sum_{p \leq x^{1/3}} \frac{1}{p \log^3 \frac{x}{p}} \leq \frac{L + 0.6986}{\log^3 x}.$$

Proof. Note that for $x_0 \leq x \leq 10372^3$, we have

$$\sum_{p \leq x^{1/3}} \frac{1}{p \log^3 \frac{x}{p}} \leq \sum_{p \leq 10372} \frac{1}{p \log^3 x} \frac{\log^3 x_0}{\log^3 \frac{x_0}{p}} \leq \frac{L + 0.5411}{\log^3 x},$$

by directly computing the sum over $p \le 10372$. Suppose next that $x \ge 10372^3$. By partial summation,

$$\sum_{p \le x^{1/3}} \frac{1}{p \log^3 \frac{x}{p}} = \frac{T(x^{1/3})}{\log^3 x^{2/3}} - \int_2^{x^{1/3}} \frac{3T(t)\ dt}{t \log^4 \frac{x}{t}}$$
$$\le \frac{\log\log x^{1/3} + 0.26300402}{\log^3 x^{2/3}} - \int_2^{x^{1/3}} \frac{3(\log\log t + c)\ dt}{t \log^4 \frac{x}{t}},$$

using Lemmas 2 and 3. The integral is equal to

$$\left[\frac{\log\log t + c}{\log^3 \frac{x}{t}} + \frac{\log\log \frac{x}{t} - \log\log t}{\log^3 x} - \frac{1}{\log^2 x \log \frac{x}{t}} - \frac{1}{2\log x \log^2 \frac{x}{t}}\right]_2^{x^{1/3}}.$$

Subtracting and bounding the resulting expression, we find that

$$\sum_{p \le x^{1/3}} \frac{1}{p \log^3 \frac{x}{p}} \le \frac{\frac{27}{8}(0.26300402 - c) - \log 2 + \frac{9}{8} + \log\log \frac{x}{2} + c}{\log^3 x},$$

from which we obtain the lemma. □

Lemma 6. *For $x \ge x_0$ we have*

$$\sum_{p \le x^{1/3}} \frac{1}{\sqrt{p} \log \frac{x}{p}} \le \frac{0.4383 x^{1/6}}{\log x}.$$

Proof. Let $x \ge x_0$. By partial summation,

$$\sum_{p \le x^{1/3}} \frac{1}{\sqrt{p} \log \frac{x}{p}} \le \frac{1.5}{\log x} \sum_{p \le x^{1/3}} \frac{1}{\sqrt{p}}$$
$$= \frac{1.5}{\log x}\left(\frac{\pi(x^{1/3})}{x^{1/6}} + \frac{1}{2}\int_2^{x^{1/3}} \frac{\pi(t)\ dt}{t^{3/2}}\right)$$
$$\le \frac{1.5}{\log x}\left(\frac{\pi(x^{1/3})}{x^{1/6}} + 16.85461 + \frac{1}{2}\int_{y_0}^{x^{1/3}} \frac{\pi(t)\ dt}{t^{3/2}}\right).$$

By Lemma 1, an upper bound for this expression is therefore

$$\frac{1.5}{\log x}\left(\frac{3.4154822 x^{1/6}}{\log x} + 16.85461 + 0.06180521 \int_{y_0}^{x^{1/3}} \frac{dt}{t^{1/2}}\right),$$

from which the lemma readily follows. □

Lemma 7. *For $x \geq x_0$ we have*

$$\sum_{p \leq x^{1/3}} S_2 \leq x \left(\frac{2L + 0.1436}{\log^2 x} + \frac{10.9113L + 3.1227}{\log^3 x} \right).$$

Proof. We have the general formula

$$\sum_{q \leq t} \pi(q) = 1 + 2 + \ldots + \pi(t) = \frac{1}{2}(\pi(t)^2 + \pi(t)).$$

Therefore,

$$S_2 = \frac{1}{2} \left(\pi \left(\sqrt{\frac{x}{p}} \right)^2 + \pi \left(\sqrt{\frac{x}{p}} \right) - \pi(p)^2 - \pi(p) \right).$$

We will address the third term below and drop the fourth term. Let $x \geq x_0$. By Lemma 1, we have

$$\frac{1}{2}\pi \left(\sqrt{\frac{x}{p}} \right) \leq \frac{1.1385\sqrt{x}}{\sqrt{p} \log \frac{x}{p}}$$

and

$$\frac{1}{2}\pi \left(\sqrt{\frac{x}{p}} \right)^2 \leq \frac{2x}{p \log^2 \frac{x}{p}} \left(1 + \frac{2.551155}{\log \frac{x}{p}} \right)^2 \leq \frac{2x}{p \log^2 \frac{x}{p}} + \frac{10.9113x}{p \log^3 \frac{x}{p}}.$$

We thus have

$$\sum_{p \leq x^{1/3}} S_2 \leq x \sum_{p \leq x^{1/3}} \left(\frac{2}{p \log^2 \frac{x}{p}} + \frac{10.9113}{p \log^3 \frac{x}{p}} + \frac{1.1385}{\sqrt{x}} \frac{1}{\sqrt{p} \log \frac{x}{p}} \right).$$

Combining Lemmas 4, 5, and 6, we obtain

$$\sum_{p \leq x^{1/3}} S_2 \leq x \left(\frac{2L + 0.1436}{\log^2 x} + \frac{10.9113L + 7.6227}{\log^3 x} + \frac{1.1}{\log^4 x} \right).$$

We now return to consider the remaining term:

$$\begin{aligned}
\frac{1}{2} \sum_{p \leq x^{1/3}} \pi(p)^2 =& \frac{1}{2} \cdot \frac{\pi(x^{1/3})(\pi(x^{1/3}) + 1)(2\pi(x^{1/3}) + 1)}{6} \\
&\geq \frac{\pi(x^{1/3})^3}{6} \\
&\geq \frac{1}{6} \left(\frac{x^{1/3}}{\log x^{1/3}} \left(1 + \frac{1}{\log x^{1/3}} \right) \right)^3 \\
&\geq \frac{4.5x}{\log^3 x} \left(1 + \frac{9}{\log x} \right) \\
&= \frac{4.5x}{\log^3 x} + \frac{40.5x}{\log^4 x}.
\end{aligned}$$

Here we have used Lemma 1. Combining these bounds, we obtain the lemma. □

Lemma 8. *For all $x \ge x_0$, we have*

$$S_5 - S_7 \ge \frac{0.5L^2 + 0.121434L - 0.22}{\log x} + \frac{-1.5244L + 1.112}{\log^2 x} + \frac{-2L^2 + 2.9067L + 2.5389}{\log^3 x}.$$

Proof. Recall the definitions

$$S_5 = \sum_{p \le x^{1/3}} \frac{\log\log \frac{x}{p^2}}{p \log \frac{x}{p}}, \qquad S_7 = \sum_{p \le x^{1/3}} \frac{\log\log p}{p \log \frac{x}{p}}.$$

We apply partial summation with $f(t) = \log\log \frac{x}{t^2} / \log \frac{x}{t}$ to obtain

$$S_5 = \frac{T(x^{1/3}) \log\log x^{1/3}}{\log x^{2/3}} - \int_2^{x^{1/3}} \frac{T(t) \log\log \frac{x}{t^2}\ dt}{t \log^2 \frac{x}{t}} + \int_2^{x^{1/3}} \frac{2T(t)\ dt}{t \log \frac{x}{t} \log \frac{x}{t^2}}.$$

We next apply partial summation with $g(t) = \log\log t / \log \frac{x}{t}$ to obtain

$$S_7 = \frac{T(x^{1/3}) \log\log x^{1/3}}{\log x^{2/3}} - \int_2^{x^{1/3}} \frac{T(t) \log\log t\ dt}{t \log^2 \frac{x}{t}} - \int_2^{x^{1/3}} \frac{T(t)\ dt}{t \log t \log \frac{x}{t}}.$$

Subtracting,

$$S_5 - S_7 = -I_1 + I_2 + I_3 + I_4, \tag{6}$$

where

$$I_1 = \int_2^{x^{1/3}} \frac{T(t) \log\log \frac{x}{t^2}\ dt}{t \log^2 \frac{x}{t}},$$

$$I_2 = \int_2^{x^{1/3}} \frac{2T(t)\ dt}{t \log \frac{x}{t} \log \frac{x}{t^2}},$$

$$I_3 = \int_2^{x^{1/3}} \frac{T(t) \log\log t\ dt}{t \log^2 \frac{x}{t}},$$

and

$$I_4 = \int_2^{x^{1/3}} \frac{T(t)\ dt}{t \log t \log \frac{x}{t}}.$$

Let $x \ge x_0$. Splitting the interval at $x^{1/6}$,

$$\begin{aligned}
I_1 &\le \left(L - \log\frac{3}{2}\right)\int_{x^{1/6}}^{x^{1/3}} \frac{T(t)}{t\log^2\frac{x}{t}}dt + L\int_2^{x^{1/6}} \frac{T(t)}{t\log^2\frac{x}{t}}dt \\
&\le \left(L - \log\frac{3}{2}\right)\int_{x^{1/6}}^{x^{1/3}} \frac{\log\log t + B + \frac{1}{\log^2 t}}{t\log^2\frac{x}{t}}dt + L\int_2^{x^{1/6}} \frac{\log\log t + B + \frac{1}{\log^2 t}}{t\log^2\frac{x}{t}}dt \\
&= \left(L - \log\frac{3}{2}\right)\left(F(x^{1/3}) - F(x^{1/6})\right) + L\left(F(x^{1/6}) - F(2)\right) \\
&= L\left(F(x^{1/3}) - F(2)\right) - \left(\log\frac{3}{2}\right)\left(F(x^{1/3}) - F(x^{1/6})\right),
\end{aligned}$$

where the antiderivative F is given by

$$\begin{aligned}
&\frac{\log\log\frac{x}{t}}{\log x} + \frac{\log t\log\log t}{\log x\log\frac{x}{t}} + \frac{B}{\log\frac{x}{t}} + \frac{1}{\log^2 x\log\frac{x}{t}} \\
&\quad - \frac{1}{\log^2 x\log t} - \frac{2\log\log\frac{x}{t}}{\log^3 x} + \frac{2\log\log t}{\log^3 x}.
\end{aligned}$$

Bounding the resulting expression gives

$$I_1 \le \frac{0.5L^2 - 0.94565L + 0.1361}{\log x} + \frac{2.2153L}{\log^2 x} + \frac{2L^2 - 2.9067L - 2.0810}{\log^3 x},$$

where we made use of the identity

$$\log\log\frac{x}{2} = \log\log x + \log\left(1 - \frac{\log 2}{\log x}\right)$$

and bounded the Maclaurin series of the right term by comparison to a geometric series. We next bound

$$I_2 \ge \int_2^{x^{1/3}} \frac{2(\log\log t + c)\ dt}{t\log\frac{x}{t}\log\frac{x}{t^2}} = \int_{\log 2}^{\log x^{1/3}} \frac{2(\log u + c)\ du}{(\log x - u)(\log x - 2u)}.$$

Integrating, a lower bound for I_2 is

$$\begin{aligned}
&\frac{2}{\log x}\left[\mathrm{Li}_2\left(\frac{\log t}{\log x}\right) - \mathrm{Li}_2\left(\frac{\log t^2}{\log x}\right) + (\log\log t + c)\left(\log\log\frac{x}{t} - \log\log\frac{x}{t^2}\right)\right]_2^{x^{1/3}} \\
&= \frac{2}{\log x}\left(\mathrm{Li}_2\left(\frac{1}{3}\right) - \mathrm{Li}_2\left(\frac{2}{3}\right) + (\log 2)(L - \log 3 + c)\right) \\
&+ \frac{2}{\log x}\left(-\mathrm{Li}_2\left(\frac{\log 2}{\log x}\right) + \mathrm{Li}_2\left(\frac{\log 4}{\log x}\right) + (\log\log 2 + c)\log\left(1 - \frac{\log 2}{\log\frac{x}{2}}\right)\right) \\
&\ge \frac{(\log 4)L - 2.0947}{\log x} + \frac{(\log 4)(1 - \log\log 2 - c)}{\log^2 x} + \frac{\log^2 2}{2\log^3 x}.
\end{aligned}$$

Here Li_2 denotes the dilogarithm, defined by the improper integral

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt.$$

For the terms $(\log 4)/\log^2 x + (\log^2 2)/(2\log^3 x)$ above, we have used the fact that the dilogarithm is concave up, together with the expansion

$$\mathrm{Li}_2(z) = \sum_{k\geq 1} \frac{z^k}{k^2}$$

for $|z| < 1$, putting $z = \log 2/\log x$, to bound

$$\begin{aligned}\mathrm{Li}_2\left(\frac{\log 4}{\log x}\right) - \mathrm{Li}_2\left(\frac{\log 2}{\log x}\right) &\geq \mathrm{Li}_2\left(\frac{\log 4 - \log 2}{\log x}\right) - \mathrm{Li}_2(0)\\ &= \mathrm{Li}_2\left(\frac{\log 2}{\log x}\right) \geq \frac{\log 2}{\log x} + \frac{\log^2 2}{4\log^2 x}.\end{aligned}$$

We have also used the bound $-\log(1-z) \geq z$, here putting $z = \log 2/\log(x/2)$ and noting that $\log\log 2 + c < 0$.

We now turn to I_3. In order to bound I_3 in the right direction, we write

$$I_3 = \int_2^e \frac{T(t)\log\log t\ dt}{t\log^2 \frac{x}{t}} + \int_e^{x^{1/3}} \frac{T(t)\log\log t\ dt}{t\log^2 \frac{x}{t}}.$$

The first integral is equal to

$$\frac{1}{2\log x}\left(\log\left(1 - \frac{1-\log 2}{\log \frac{x}{2}}\right) - \frac{\log 2\log\log 2}{\log\frac{x}{2}}\right) \geq \frac{-0.0304}{\log^2 x}.$$

A lower bound for the second integral is

$$\begin{aligned}\int_e^{x^{1/3}} \frac{(\log\log t + c)\log\log t\ dt}{t\log^2\frac{x}{t}} &= \int_1^{\log x^{1/3}} \frac{(\log u + c)\log u\ du}{(\log x - u)^2}\\ &= \frac{1}{\log x}\left[c\log\left(1-\frac{u}{\log x}\right) + \frac{cu\log u}{\log x - u} + 2\mathrm{Li}_2\left(\frac{u}{\log x}\right)\right]_1^{\log x^{1/3}}\\ &\quad + \frac{1}{\log x}\left[2(\log u)\log\left(1-\frac{u}{\log x}\right) + \frac{u\log^2 u}{\log x - u}\right]_1^{\log x^{1/3}}\\ &\geq \frac{0.5L^2 - 1.77881L + 1.9771}{\log x} + \frac{c-2}{\log^2 x} - \frac{0.5188}{\log^3 x}.\end{aligned}$$

Next, note that the use of the inequality $T(t) > \log\log t + c$ allows us to add to the lower bound

$$\int_3^{x^{1/3}} \frac{(T(t) - (\log\log t + c)) \log\log t \; dt}{t \log^2 \frac{x}{t}}$$
$$\geq \frac{1}{\log^2 \frac{x}{3}} \sum_{2 \leq k \leq 1228} \int_{\log p_k}^{\log p_{k+1}} (T(p_k) - c - \log u) \log u \; du$$
$$\geq \frac{0.3099}{\log^2 x} \left(1 + \frac{\log 9}{\log x}\right).$$

We therefore have

$$I_3 \geq \frac{0.5L^2 - 1.77881L + 1.9771}{\log x} - \frac{1.4591}{\log^2 x} + \frac{0.1621}{\log^3 x}.$$

We now consider

$$I_4 = \int_2^{x^{1/3}} \frac{T(t) \; dt}{t \log t \log \frac{x}{t}} \geq \int_2^{x^{1/3}} \frac{(\log\log t + c) \; dt}{t \log t \log \frac{x}{t}} = \int_{\log 2}^{\log x^{1/3}} \frac{(\log u + c) \; du}{u(\log x - u)}.$$

Integrating, we obtain the expression

$$\frac{1}{\log x}\left[c \log u - (\log u) \log\left(1 - \frac{u}{\log x}\right) - c \log(\log x - u) - \mathrm{Li}_2\left(\frac{u}{\log x}\right) + \frac{\log^2 u}{2}\right]_{\log 2}^{\log x^{1/3}}.$$

We bound this expression to obtain

$$I_4 \geq \frac{0.5L^2 - 0.4317L - 0.3608}{\log x} + \frac{0.7658}{\log^2 x} + \frac{0.0556}{\log^3 x}.$$

Note that the inequality $T(t) > \log\log t + c$ was used to bound I_4. We may therefore add the following expression to the lower bound:

$$\int_2^{x^{1/3}} \frac{T(t) - (\log\log t + c)}{t \log t \log \frac{x}{t}} dt \geq \frac{1}{\log \frac{x}{2}} \int_2^{x^{1/3}} \frac{T(t) - (\log\log t + c)}{t \log t} dt.$$

Splitting the interval $[2, y_0]$ into subintervals between primes $[p_k, p_{k+1}]$, $k = 1, 2, \ldots$ 1228, and using the numerical value of $T(p_k)$ in each subinterval, we find by computation in Pari/GP that

$$\frac{1}{\log \frac{x}{2}} \int_2^{x^{1/3}} \frac{T(t) - (\log\log t + c)}{t \log t} dt \geq \frac{0.3945}{\log \frac{x}{2}} \geq \frac{0.3945}{\log x} \left(1 + \frac{\log 2}{\log x}\right).$$

A similar argument allows us to improve our upper bound on the integral I_1 in equation (6). Here we used the upper bound

$$L \int_2^{x^{1/6}} \frac{T(t)}{t \log^2 \frac{x}{t}} dt \leq L \int_2^{x^{1/6}} \frac{\log\log t + B + \frac{1}{\log^2 t}}{t \log^2 \frac{x}{t}} dt.$$

Bounding the difference in the interval $[2, 100]$, we add $0.6909L/\log^2 x$ to the lower bound. Combining the bounds for I_1, I_2, I_3, and I_4 completes the proof of Lemma 8. $\square$

5. The Proof of Theorem 1

We now prove Theorem 1. Note that $\tau_3(x) = \pi_3(x) + N(x)$, where $N(x) = |\{n \leq x : n = p^2q\}|$, and p and q denote (possibly equal) primes. For each such p, the number of possibilities for q is $\pi(x/p^2)$, and we thus have

$$N(x) = \sum_{p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right).$$

The range $500194 \leq x \leq 10^{12}$ was checked using the computer program Pari/GP. Thus we may assume $x \geq x_0$. Now, a lower bound for $\pi_3(x)$ is given in Theorem 2. The contribution from $N(x)$ is given by the following lemma, from which Theorem 1 follows by an argument analogous to that of inequality (5) in the proof of Theorem 2.

Lemma 9. *For all $x \geq x_0$ we have*

$$\sum_{p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right) \geq x\left(\frac{\alpha}{\log x} + \frac{0.9861}{\log^2 x} + \frac{1.2861}{\log^3 x}\right),$$

where $\alpha = \sum_p 1/p^2 = 0.4522474\ldots$ denotes the reciprocal sum of squares of primes.

Proof. Let $x \geq x_0$. We wish to apply the lower bound $\pi(t) > t/\log t$, valid for $t \geq 17$, in Lemma 1. Thus we write

$$\sum_{p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right) = \sum_{p \leq \sqrt{\frac{x}{17}}} \pi\left(\frac{x}{p^2}\right) + \sum_{\sqrt{\frac{x}{17}} < p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right).$$

The right sum is

$$\sum_{\sqrt{\frac{x}{17}} < p \leq \sqrt{\frac{x}{2}}} \pi\left(\frac{x}{p^2}\right) = -6 \cdot \pi\left(\sqrt{\frac{x}{17}}\right) + \pi\left(\sqrt{\frac{x}{13}}\right) + \ldots + \pi\left(\sqrt{\frac{x}{2}}\right).$$

Also by Lemma 1, this expression is bounded below by

$$2\sqrt{x}\left(-6 \cdot \frac{1.0972}{\sqrt{17}\log\frac{x}{17}} + \sum_{k=1}^{6} \frac{1}{\sqrt{p_k}\log\frac{x}{p_k}}\right) \geq \frac{1.8188\sqrt{x}}{\log x},$$

where p_k denotes the k-th prime number. For the remaining term, let $S(t) = \sum_{p\le t}(1/p^2)$. By partial summation,

$$\sum_{p\le\sqrt{\frac{x}{17}}}\frac{1}{p^2\log\frac{x}{p^2}} = \frac{S(\sqrt{\frac{x}{17}})}{\log 17} - \int_2^{\sqrt{\frac{x}{17}}}\frac{2S(t)\ dt}{t\log^2\frac{x}{t^2}}$$

$$= \frac{S(\sqrt{\frac{x}{17}})}{\log 17} - \int_2^{\sqrt{\frac{x}{17}}}\frac{2\alpha\ dt}{t\log^2\frac{x}{t^2}} + \int_2^{\sqrt{\frac{x}{17}}}\frac{2(\alpha - S(t))\ dt}{t\log^2\frac{x}{t^2}}.$$

Now,

$$\int_2^{\sqrt{\frac{x}{17}}}\frac{2\alpha\ dt}{t\log^2\frac{x}{t^2}} = \frac{\alpha}{\log 17} - \frac{\alpha}{\log\frac{x}{4}} \le \frac{\alpha}{\log 17} - \frac{\alpha}{\log x}\left(1+\frac{\log 4}{\log x}\right).$$

Also,

$$\int_2^{\sqrt{\frac{x}{17}}}\frac{2(\alpha - S(t))\ dt}{t\log^2\frac{x}{t^2}} \ge \sum_{k\le 21433}(\alpha - S(p_k))\left[\frac{1}{\log\frac{x}{t^2}}\right]_{p_k}^{p_{k+1}}$$

$$= \sum_{k\le 21433}(\alpha - S(p_k))\frac{\log\frac{p_{k+1}^2}{p_k^2}}{\log\frac{x}{p_{k+1}^2}\log\frac{x}{p_k^2}}$$

$$\ge \frac{0.3592}{\log\frac{x}{9}\log\frac{x}{4}}$$

$$\ge \frac{0.3592}{\log^2 x}\left(1+\frac{\log 9}{\log x}\right)\left(1+\frac{\log 4}{\log x}\right)$$

$$\ge \frac{0.3592}{\log^2 x} + \frac{1.2872}{\log^3 x}.$$

Thus,

$$\sum_{p\le\sqrt{\frac{x}{17}}}\frac{1}{p^2\log(\frac{x}{p^2})} \ge \frac{\alpha}{\log x}\left(1+\frac{\log 4}{\log x}\right) + \frac{0.3592}{\log^2 x} + \frac{1.2872}{\log^3 x} - \frac{1}{\log 17}\sum_{p>\sqrt{\frac{x}{17}}}\frac{1}{p^2}.$$

By [9, Lem. 2.7], we have

$$\frac{1}{\log 17}\sum_{p>\sqrt{\frac{x}{17}}}\frac{1}{p^2} < \frac{1/\log 17}{\sqrt{\frac{x}{17}}\log\sqrt{\frac{x}{17}}} = \frac{2\sqrt{17}/\log 17}{\sqrt{x}\log\frac{x}{17}} \le \frac{3.2431}{\sqrt{x}\log x}.$$

Combining these bounds, we obtain Lemma 9. □

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