SHORT PROOFS OF EULER-TYPE IDENTITIES FOR COMPOSITIONS

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Abstract

In this paper we present two short proofs of the Euler-type identities for compositions of positive integers. We also obtain several identities for the number of compositions into parts greater than a given integer m. In addition, we obtain identities for the number of palindromic compositions into parts greater than m.

1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers called parts which sum to n. For example, the compositions of 4 are: (4), $(3,1)$, $(1,3)$, $(2,2)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, $(1,1,1,1)$. A palindromic composition [9] of n is one that remains unchanged when the order of its parts is reversed. For example, there are four palindromic compositions of 4, namely, $(4), (1,2,1), (2,2), (1,1,1,1)$.

A composition may be represented graphically by means of the MacMahon zig-zag graph [5]. It is similar to the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. For instance, the zig-zag graph of the composition (6,3,1,2,2) is shown in Figure 1.

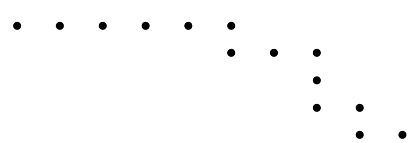

Figure 1. Zig-Zag Graph

The conjugate of a composition is obtained by reading its graph by columns from left to right. We see that the figure demonstrates that the conjugate of the composition $(6,3,1,2,2)$ is $(1,1,1,1,1,2,1,3,2,1)$.

The following well-known partition identity is Euler's Theorem [1, 2].

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Theorem 1.1 (Euler's Theorem). *The number of partitions of n into odd parts equals the number of partitions of n in which no part is repeated.*

Inspired by Euler's Theorem, Andrew Sills [9] published a bijective proof of the following identity for compositions.

Theorem 1.2 (Sills). *The number of compositions of n into odd parts equals the number of compositions of $n+1$ into parts greater than 1.*

Sills' proof demonstrates a nice application of the MacMahon conjugate of a composition. In the same spirit, Munagi [6] also obtained the following identities. Here, the letter m denotes a positive integer greater than 1 in the following theorems.

Theorem 1.3 ([6]). *The following sets of compositions are equinumerous:*
(i) Compositions of n using only the parts 1 and m;
(ii) Compositions of $n+1$ into parts congruent to 1 (mod m);
(iii) Compositions of $n+m$ into parts greater than $m-1$.

Theorem 1.4 ([6]). *The number of compositions of n into parts congruent to m (mod $m+1$) equals the number of compositions of $n+1$ without 1's into parts congruent to 1 (mod m).*

Theorem 1.5 ([6]). *The number of compositions of n into parts greater than m ($m \neq 1$) equals the number of compositions of $n-2m$ into 1's and 2's with no consecutive 2's.*

Making use of the zig-zag graph and the conjugate of compositions, Munagi [6] gave bijective proofs of the above theorems. However, his proofs were relatively longer. In this paper, we will present shorter and direct proofs of Theorems 1.3 and 1.4 in Section 2. In particular, we do not use the conjugate of a composition in our proofs. In Section 3, we obtain several identities for the number of compositions with parts greater than m. In addition, some identities for palindromic compositions of n into parts greater than m are obtained.

2. Two Shorter Proofs

2.1 Proof of Theorem 1.3

Proof. $(i) \Longleftrightarrow (ii)$: Let $C = (c_1, c_2, ..., c_k)$ be a composition of n using only the parts 1 or m. We first append 1 to the right end of C to obtain a composition B of $n+1$, then add 1 to all adjacent m's on the left of it to form new parts from right to left in B. Therefore, we get a composition of $n+1$ into parts congruent to $1 \ (mod\ m)$.

Conversely, let A be a composition of $n+1$ into parts congruent to $1 \ (mod\ m)$. We first replace every part $mk+1$ by $\underbrace{m+m+...+m}_{k}+1$ to obtain a composition D of $n+1$ with parts of size 1 or m in which the last part is 1. Next, delete the part 1 on the right end of D to obtain a composition of n. Thus, we obtain a composition of n using only parts 1 or m.

$(i) \Longleftrightarrow (iii)$: Let $C = (c_1, c_2, ..., c_k)$ be a composition of n using only the parts 1 or m. Appending m to the right end of C, we have a composition F of $n+m$ using only parts 1 or m in which the last part is m. Then we add m to all adjacent 1's on the left of it to produce new parts from right to left in F. In this way, we obtain a composition of $n+m$ into parts greater than $m-1$.

For example, let $m=4$, then the process that the composition $(1,4,1,1,1)$ of 8 transforms to the composition $(5,7)$ of 12 is as follows.

$$(1,4,1,1,1) \to (1,4,1,1,1,4) \to (5,7).$$

Conversely, let A be a composition of $n+m$ into parts greater than $m-1$, and the last part is d, where $d \geq m$. We directly delete d if $d=m$; if $d>m$, we first replace d by $(d-m)$, and then replace $(d-m)$ with $1,1,\cdots,1$ ($d-m$ times). So we obtain a composition H of n. Next, we replace every part greater than m with $1,1,...,1,m$ in H. In this way, we obtain a composition of n with parts only 1's and m's.

For example, let $m=4$, then the process that the composition $(5,7)$ of 12 transforms to the composition $(1,4,1,1,1)$ of 8 is as follows.

$$(5,7) \to (5,3) \to (5,1,1,1) \to (1,4,1,1,1).$$

This completes the proof. □

2.2 Proof of Theorem 1.4

In order to prove the theorem, Munagi gave the following lemma in [6].

Lemma 2.1. *The number of compositions of n into parts congruent to $r \ (mod\ m)$ equals the number of compositions of n into parts r and m in which the first part is r.*

Proof of Theorem 1.4. From Lemma 2.1, we need to proof that the number of compositions of $n+1$ without 1's into parts congruent to $1 \ (mod\ m)$ equals the number of compositions of n with parts m and $m+1$ in which the first part is m.

Let α be a composition of $n+1$ without 1's into parts congruent to 1 $(mod\ m)$ in which the first part is d. We replace d by $d-1$, and replace $(d-1)$ with $m, m, ..., m$. Then the other parts of α are replaced by $m+1, m, ..., m$. Therefore, we obtain a composition of n with parts m and $m+1$ in which the first part is m.

For example, when $m = 3$, the process that the composition $(4, 4, 7)$ of 15 produces the composition of 14 is as follows.

$$(4, 4, 7) \longrightarrow (3, 4, 7) \longrightarrow (3, 4, 4, 3).$$

Conversely, let γ be a composition of n into parts m and $m+1$ in which the first part is m. We first append 1 to the left end of γ to obtain a composition δ of $n+1$. Next, we add 1 to all adjacent m's to the right of it to produce a new part as the first part of δ. Meanwhile, we adjoin $m+1$ and all adjacent m's to the right of it to produce new parts from left to right in δ. Consequently, we obtain a composition of $n+1$ without 1's into parts congruent to 1 $(mod\ m)$.

For example, when $m = 3$, the process that the composition $(3, 4, 3, 4)$ of 14 produces the composition of 15 is as follows.

$$(3, 4, 3, 4) \longrightarrow (1, 3, 4, 3, 4) \longrightarrow (4, 7, 4).$$

This completes the proof. □

3. Several Identities for Special Compositions

We state a stronger version of Theorem 1.5 which will account for the number of 1's separating two 2's as follows.

Theorem 3.1. *The number of compositions of* n *into parts greater than* m $(m \neq 1)$ *equals the number of compositions of* $n-2m$ *into* 1*'s and* 2*'s such that there are at least* $m-1$ *ones between every pair of consecutive* 2*'s.*

Proof. Without loss of generality, Munagi has given the proof of the case $m = 2$ in [6]. We now consider the case $m = 3$. The zig-zag graph of a composition into parts greater than 3, say $(4, 4, 5, 6)$, is shown in Figure 2 below.

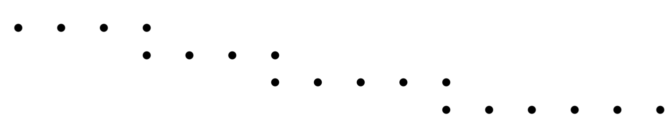

Figure 2. Zig-Zag Graph of $(4, 4, 5, 6)$

Notice that such graph always has at least 3 nodes before the first stack of vertical nodes, and at least 3 nodes after the last stack. Also the large sizes (≥ 4) of the parts insures that each stack contains exactly two vertical nodes, with at least two nodes between successive pairs of vertical nodes. Thus on deleting the first 3 nodes, and the last 3 nodes, we find that the conjugate of the remaining graph is a composition of the second type, $(2, 1, 1, 2, 1, 1, 1, 2, 1, 1)$, is shown in Figure 3 below.

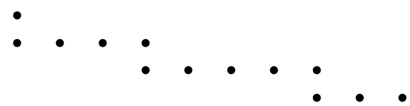

Figure 3. Zig-Zag Graph of $(1, 4, 5, 3)$

Therefore, we know that there are at least $m - 1$ ones between every pair of consecutive 2's in a composition of the second type. This completes the proof. $\square$

And we obtain the following results for the compositions into parts greater than m $(m \neq 1)$.

Theorem 3.2. *Let $m > 1$ be an integer. Then the number of compositions of n into parts greater than m equals the number of compositions of $n - 2m + 1$ into parts of size 1 or 3 such that there are at least $m - 2$ ones between every pair of consecutive 3's.*

Proof. For any composition C of n into parts greater than m, we first obtain a composition C_1 of $n - 2m$ into 1's or 2's such that there are at least $m - 1$ ones between every pair of consecutive 2's by Theorem 3.1. Next, we append 1 to the right end of C_1 to obtain a composition C_2 of $n - 2m + 1$ into 1's or 2's such that there are at least $m - 1$ ones between every pair of consecutive 2's, and the last part of C_2 is 1. Finally, we add 1 to 2 on the left of it to form new parts from right to left in C_2. As a result, we obtain a composition of $n - 2m + 1$ into parts of size 1 or 3 such that there are at least $m - 2$ ones between every pair of consecutive 3's.

For example, let $m = 3$, then the process that the composition $(5, 4, 6)$ of 15 produces the composition $(1, 3, 1, 3, 1, 1)$ of 10 is as follows.

$$(5, 4, 6) \longrightarrow (1, 2, 1, 1, 2, 1, 1) \longrightarrow (1, 2, 1, 1, 2, 1, 1, 1) \longrightarrow (1, 3, 1, 3, 1, 1).$$

Obviously, this correspondence is one-to-one. This completes the proof. $\square$

If $m = 2$ in Theorem 3.2, we have the following corollary.

Corollary 3.1. *The number of compositions of n into parts greater than 2 equals the number of compositions of $n - 3$ into parts of size 1 or 3.*

Now we consider a special case of Theorem 1.3. If $m = 3$, we know that the number of compositions of n using only parts 1 or 3 equals the number of compositions of $n + 1$ into parts congruent to 1 $(mod\ 3)$. Hence we naturally obtain the following identity by Corollary 3.1.

Corollary 3.2. *The number of compositions of n into parts greater than 2 equals the number of compositions of $n-2$ into parts congruent to 1 (mod 3).*

We may see that Corollary 3.2 explains $(ii) \iff (iii)$ of Theorem 1.3 in another way when $m = 3$.

Similar to proof of Theorem 3.2, we take the following theorem.

Theorem 3.3. *Let $m > 2$ be an integer. Then the number of compositions of n into parts greater than m equals the number of compositions of $n - 2m + 2$ into parts of size 1 or 4 such that there are at least $m - 3$ ones between every pair of consecutive 4's.*

Further, we present the following general conclusion.

Theorem 3.4. *Let m, k be positive integers, $m > 1, m \geq k - 2 \geq 0$. Then the number of compositions of n into parts greater than m equals the number of compositions of $n - 2m + (k - 1)$ into parts of size 1 or $k + 1$ such that there are at least $m - k$ ones between every pair of consecutive $(k + 1)$'s.*

When $m = k$ in Theorem 3.4, we have the following corollary.

Corollary 3.3. *The number of compositions of n into parts greater than m equals the number of compositions of $n - m - 1$ into parts of size 1 or $m + 1$.*

We know that Corollary 3.3 explains $(i) \iff (iii)$ of Theorem 1.3 in another way.

Furthermore, we study the palindromic compositions, and obtain the following results for the palindromic compositions.

Theorem 3.5. *The number of palindromic compositions of n into parts greater than m $(m \neq 1)$ equals the number of palindromic compositions of $n - 2m$ into 1's or 2's such that there are at least $m - 1$ ones between every pair of consecutive 2's.*

Theorem 3.6. *The number of palindromic compositions of n into parts greater than 2 equals the number of palindromic compositions of $n - 2$ into parts congruent to 1 (mod 3).*

Theorem 3.7. *The number of palindromic compositions of n into parts greater than m $(m \neq 1)$ equals the number of palindromic compositions of $n - 2m + 1$ into parts of size 1 or 3 such that there are at least $m - 2$ ones between every pair of consecutive 3's.*

Theorem 3.8. *The number of palindromic compositions of n into parts greater than m $(m > 2)$ equals the number of palindromic compositions of $n - 2m + 2$ into parts of size 1 or 4 such that there are at least $m - 3$ ones between every pair of consecutive 4's.*

Theorem 3.9. *Let m, k be positive integers, $m > 1, m \geq k-2 \geq 0$. Then the number of palindromic compositions of n into parts greater than m equals the number of palindromic compositions of $n-2m+(k-1)$ into parts of size 1 or $k+1$ such that there are at least $m-k$ ones between every pair of consecutive $(k+1)$'s.*

From Munagi's proof of Theorem 1.5 in [6], we can easily know that the palindromic compositions of n into parts greater than m correspond to the palindromic compositions of $n-2m$ into 1's or 2's such that there are at least $m-1$ ones between every pair of consecutive 2's. This is Theorem 3.5. Using the proof of Theorem 3.2 we assert Theorem 3.7 is true. Furthermore, the proofs of all other theorems are true. Therefore, we omitted the proofs of the above theorems. We only offer the following example to illustrate Theorem 3.8.

Example 3.1. *If $n = 14, m = 3$, then there are 4 palindromic compositions of 14 into parts greater than 3 as follows: $(14), (7, 7), (5, 4, 5), (4, 6, 4)$. Similarly, there are 4 palindromic compositions of 10 into parts of size 1 or 4 as follows.*

$$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1), (4, 1, 1, 4), (1, 4, 4, 1), (1, 1, 1, 4, 1, 1, 1).$$

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