MINIMUM COPRIME LABELINGS FOR OPERATIONS ON GRAPHS

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Abstract

A prime labeling of a graph of order n is a labeling of the vertices with the integers 1 to n in which adjacent vertices have relatively prime labels. A coprime labeling maintains the same criterion on adjacent vertices using any set of distinct positive integers. In this paper, we consider several families of graphs or products of graphs that have been shown to not have prime labelings, and answer the natural question of how to label the vertices while minimizing the largest value in its set of labels.

1. Introduction

Consider G to be a simple graph with vertex set V in which $|V| = n$ and edge set E. Throughout this paper, we let p_i be the i^{th} prime number. A *prime labeling* of G is a labeling of V using the distinct integers $\{1, \ldots, n\}$ such that the labels of any pair of adjacent vertices are relatively prime. If such a labeling exists, we call G a *prime graph.* More generally, a *coprime labeling* of G uses distinct labels from the set $\{1, \ldots, m\}$ for some integer $m \geq n$ such that adjacent labels are relatively prime. The minimum value m for which G has a coprime labeling is defined as the *minimum coprime number*, denoted as $\mathfrak{pr}(G)$, and a coprime labeling of G with largest label being $\mathfrak{pr}(G)$ is called a *minimum coprime labeling* of G. A prime graph therefore has $\mathfrak{pr}(G) = n$ as its minimum coprime number.

The concept of a prime labeling of a graph was first developed by Roger Entringer and introduced in [15] by Tout, Dabboucy, and Howalla. While most research has revolved around finding prime labelings for various classes of graphs, our focus is on the problem of determining the minimum coprime number for graphs that have been shown to not be prime, a question that was previously studied for complete bipartite graphs $K_{n,n}$ by Berliner et al. [1]. A dynamic survey of results on the 35 year history of prime labelings is given by Gallian in [4].

It was conjectured by Entringer that all trees have prime labelings, and many classes of trees such as paths, stars, complete binary trees, and spiders have been shown by Fu and Huang in [3] to be prime. Salmasian [10] showed that for every tree T with n vertices ($n \geq 50$), $\mathfrak{pr}(T) \leq 4n$. Pikhurko [7] improved this by showing that for any integer $c > 0$, there is an N such that for any tree T of order $n > N$, $\mathfrak{pr}(T) < (1 + c)n$. Additionally, many graphs that are not trees have been proven to be prime, including cycles, helms, fans, flowers, and books for all sizes; see [2], [12], and [14]. There is a large collection of graphs whose primality depends on the size of its vertex set. The complete graph K_n, for example, is clearly prime only if $n \leq 3$. Additionally, the wheel graph W_n, which consists of a cycle of length n where each vertex on the cycle is adjacent to a central vertex, is prime if and only if n is even. Section 2 examines the minimum coprime number for complete graphs with at least 4 vertices and wheel graphs in which n is odd.

While paths and cycles on n vertices, denoted as P_n and C_n, respectively, are known to be prime, combinations of these through common graph operations often result in a graph that is not prime. Recall the *disjoint union* of graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Deretsky et al. [2] proved that $C_{2k} \cup C_n$ is prime for all integers k and n. However, if both of the cycles are of odd length, their disjoint union is not prime. Determining the minimum coprime number in this case of the union of odd cycles will be our first focus in Section 3. We also consider the union of the complete graph with either a path or the star graph, denoted as S_n where n is the number of degree 1 vertices, also called *pendant* vertices. Youssef and El Sakhawi [16] studied these two union graphs, concluding that $K_m \cup P_n$ is prime if and only if $1 \leq m \leq 3$ or $m = 4$ with $n \geq 1$ being odd. They also determined $K_m \cup S_n$ is prime if and only if the number of primes less than or equal to $m + n + 1$ is at least m. Another graph operation that we will examine is the *corona* operation, which is defined as follows. The corona of a graph G with a graph H, in which $|V(G)| = n$, is denoted by $G \odot H$ and is obtained by combining one copy of G with n copies of H by attaching the i^{th} vertex in G to every vertex within the i^{th} copy of H. In particular, we examine the corona of a complete graph on n vertices with an empty graph on 1 or 2 vertices. We examine the non-prime cases for these unions and coronas to determine their minimum coprime number in Section 3.

Given a graph G, the k^{th} *power* of G, denoted G^k, is defined as the graph with the same vertex set as G but with an edge between each $u, v \in V(G)$ for which $d(u, v) \leq k$ in G. Here the value $d(u, v)$ is the distance between u and v, or the length of the shortest path between the two vertices. The square of paths and cycles, denoted as P_n^2 and C_n^2, were shown not to be prime by Seoud and Youssef in [14]. This implies that higher powers of these are also not prime since G^k is a subgraph of G^ℓ for integers $k \leq \ell$. Section 4 explores the minimum coprime numbers for the square and cube of both the path and cycle graphs.

The *join* of two disjoint graphs G and H, denoted as $G+H$, consists of a vertex set $V(G) \cup V(H)$ with an edge added to connect each vertex in G to those in H, resulting in an edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. Seoud, Diab, and Elsahawi [12] studied the primality of the join of paths with the empty graph $\overline{K}_m$ on m isolated vertices. They proved that $P_n + \overline{K}_2$ is prime if and only $n = 2$ or n is odd, and that the join graph $P_n + \overline{K}_m$ is not prime for all $m \geq 3$. Since these two classes of graphs are subgraphs of $P_n + P_2$ and $P_n + P_m$, respectively, the join of paths follow similar criteria for not being prime. Analogous reasoning applies for the join of two cycles or of a path and a cycle. We will find the minimum coprime number for certain cases of these join graphs depending on the relationship between m and n within Section 5.

Our final section concludes with open problems for further research. While there are still many unanswered questions about the primality of graphs such as conjectures on trees and unicyclic graphs being prime, we focus on open questions regarding the minimum coprime number of particular classes of graphs.

2. Complete Graphs and Wheels

Consider the complete graph K_n on n vertices. It is easy to see that K_n is prime if and only if $n \leq 3$. The clearest way to determine when a graph is not prime is based on the *independence number* of the graph, defined as the size of the largest set of vertices S (called an independent set) in which no pair of these vertices are adjacent. In order to possibly have a prime labeling, a graph needs an independence number of at least $\left\lfloor \frac{|V|}{2} \right\rfloor$ in order for an independent set of vertices to exist for the even labels (a fact first noted in [3]). The graph K_n has an independence number of 1 since all vertices are adjacent to each other, hence we examine the minimum coprime number for the complete graph with 4 or more vertices.

Proposition 1. *Let $n \geq 4$. The minimum coprime number of K_n is $\mathfrak{pr}(K_n) = p_{n-1}$.*

Proof. Since each vertex is adjacent to every other vertex, only prime numbers and 1 can be used as vertex labels to keep each pair of vertices relatively prime. Thus, we can label the graph with a minimum coprime labeling by using the first $n-1$ primes along with 1. □

Next we consider the wheel graph W_n. We name the vertices $v_1, \ldots, v_n$ as shown in Figure 1 with v_1 representing the center vertex and the remaining v_i listed in clockwise order with v_2 being adjacent to v_{n+1}. Lee, Wui, and Yeh [5] demonstrated

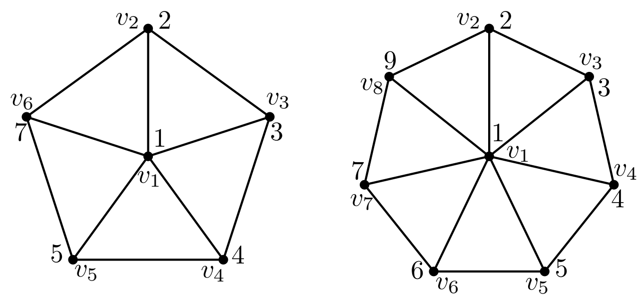

Figure 1: The graph W_n

that W_n is prime if and only if n is even. The following result determines the minimum coprime number for the odd case.

Proposition 2. *Let n be odd. Then the minimum coprime number of W_n is $\mathfrak{pr}(W_n) = n + 2$.*

Proof. We label the vertices v_i as i for each $i \in \{1, \ldots, n\}$ and v_{n+1} as $n + 2$. This labeling is coprime since each adjacent pair of labels falls into one of four cases: the center label 1 is in the pair, the labels are consecutive integers, the labels are n and $n + 2$ which are consecutive odd numbers, or the labels are 2 and the odd integer $n + 2$. The labeling is a minimum coprime labeling since it was proven in [5] to be not prime, hence a labeling with maximum label being the number of vertices, $n + 1$, is impossible to achieve. □

3. Disjoint Union and Corona Operations

The following observation is straightforward from the definitions of prime and coprime labelings and will be useful in some of our upcoming proofs. We first recall that if G and H are graphs with $V(G) = V(H)$ and $E(G) \subseteq E(H)$, then we say H is a *spanning supergraph* of G, and G is a *spanning subgraph* of H.

Observation 1. *If G is not prime, then a spanning supergraph of G is not prime. If G is prime, then any spanning subgraph of G is also prime.*

The disjoint union of two graphs has been shown to be prime for a variety of graphs under certain conditions. In [2], the disjoint union of cycles was shown to be prime when at least one of the cycles has an even number of vertices; that is, $C_{2k} \cup C_n$ is prime for all positive $k, n \in \mathbb{Z}$. Examining the case in which both cycles are odd, we see that, similarly to the wheel graph in Proposition 2, we only need the largest label to be $|V(G)| + 1$ to achieve a minimum coprime labeling.

Theorem 2. *For all $k, \ell \geq 1$, the minimum coprime number of the disjoint union of two odd-length cycles is*

$$\mathfrak{pr}(C_{2k+1} \cup C_{2\ell+1}) = 2(k+\ell) + 3.$$

Proof. We label the vertices of the graph, as shown in Figure 2, using the labels $1, 3, 4, \ldots, 2k+1, 2k+2$ in this order on the cycle C_{2k+1} and the labels $2, 2k+3, 2k+4, \ldots, 2(k+\ell)+1, 2(k+\ell)+3$ on the cycle $C_{2\ell+1}$. Most of the edges have endpoints with labels of the form $\{m, m+1\}$, which are relatively prime as consecutive integers. The remaining edges either include 1 as an endpoint, connect 2 to an odd label, or have consecutive odd labels $2(k+\ell)+1$ and $2(k+\ell)+3$ assigned to its vertices, which are all relatively prime pairs.

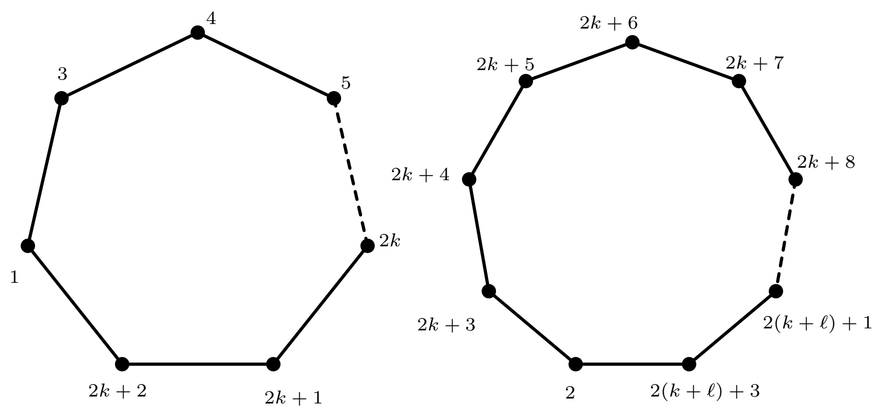

Figure 2: Minimum coprime labeling of $C_{2k+1} \cup C_{2\ell+1}$

It remains to show that this labeling is minimum, which is accomplished by demonstrating that the graph does not have a prime labeling since $2(k+\ell)+3$ is one larger than the size of the vertex set V of $C_{2k+1} \cup C_{2\ell+1}$. No prime labeling exists because the independence number for the graph is $k+\ell$, which is smaller than the required $\left\lfloor \frac{|V|}{2} \right\rfloor = k+\ell+1$ independent vertices. Therefore, our labeling is a minimum coprime labeling. □

We now find a minimum coprime labeling for the union of the complete graph with a path or a star graph. These classes of graphs, $K_m \cup P_n$ and $K_m \cup S_n$, were investigated by Youssef and El Sakhawi in [16]. They proved that $K_m \cup P_n$ is prime if and only if $1 \leq m \leq 3$ or $m = 4$ and $n \geq 1$ is odd. See Figure 3 for an example of such a graph with a minimum coprime labeling.

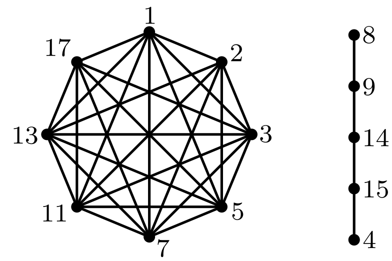

Figure 3: Minimum coprime labeling of $K_8 \cup P_5$ when $n \leq r$

Theorem 3. *Let m and n be positive integers, and if $m \geq 5$, we let $r = p_{m-1} - 2m + 4$. The minimum coprime number for $K_m \cup P_n$ is*

$$\mathfrak{pr}(K_m \cup P_n) = \begin{cases} m+n & \text{if } 1 \leq m \leq 3 \text{ or } m = 4 \text{ and } n \text{ is odd} \\ m+n+1 & \text{if } m = 4 \text{ and } n \text{ is even} \\ p_{m-1} & \text{if } m \geq 5 \text{ and } n \leq r \\ 2m+n-4 & \text{if } m \geq 5,\ n > r, \text{ and } n \text{ is odd} \\ 2m+n-3 & \text{if } m \geq 5,\ n > r, \text{ and } n \text{ is even.} \end{cases}$$

Proof. The case when $1 \leq m \leq 3$ or $m = 4$ and n is odd was already shown in [16] since $K_m \cup P_n$ is prime under those conditions. When $m = 4$ and n is even, we can label K_4 with $1, 2, 3, 5$ and use the integers $6, 7, \ldots, n+5$ to label P_n. Hence $\mathfrak{pr}(K_m \cup P_n) = m + n + 1$ since the graph is not prime.

Suppose that $m \geq 5$ and $n \leq r$. It is clear by our discussion on K_m that $\mathfrak{pr}(K_m \cup P_n) \geq p_{m-1}$ since K_m is a subgraph of $K_m \cup P_n$. As was done in Proposition 1, label the vertices of K_m with 1 and the first $m-1$ primes. Consider the vertices on the path as $V(P_n) = \{v_1, \ldots, v_n\}$, which we will label using the following sequence $X = \{x_i\}_{i=1}^{\infty}$. When n is even, we assign the value x_i to the vertex v_i for $i = 1, \ldots n$. When n is odd, we alter this slightly by assigning x_i to v_i for $i = 1, \ldots n-1$, and we label v_n by 4:

$$X = 8, 9, 14, 15, 16, 21, 22, 25, 26, 27, 28, 33, 34, 35, 38, 39, 44, 45, 46, 49, \ldots$$

The sequence is defined after the initial two terms by including each integer starting with 14 except we skip each prime p_i for all $i \geq 7$ along with the following three cases, determined partly by whether p_i and p_{i+1} are twin primes (i.e., $p_i + 2 = p_{i+1}$): we skip $p_i - 1$ if p_i is not the first of two twin primes and $3|(p_i - 1)$, as in the case of $p_{12} = 37$; skip $p_i - 1$ and $p_{i+1} - 1$ if p_i and p_{i+1} are twin primes and $5|(p_i - 1)$, such as the case of $p_{13} = 41$ and $p_{14} = 43$; and skip $p_i + 1$ for all other prime numbers.

It is clear that each pair of adjacent labels in K_m are relatively prime. The same is true for the labels on P_n since the distance between any such pair is at most 5, and we ensure, through the first two cases of values that were skipped, that no pair of adjacent labels are divisible by 3 or 5. This is because if $p_i - 1$ is divisible by 3 for p_i being a non-twin prime, then the vertices with labels $p_i - 2$ and $p_i + 1$ will be adjacent with $\gcd(p_i - 2, p_i + 1) = 1$. Likewise, if $p_i - 1$ is divisible by 5 when p_i is the first of a twin prime pair, then the vertices with labels $p_i - 2$ and $p_i + 3$ will be adjacent with $\gcd(p_i - 2, p_i + 3) = 1$. Therefore, this is a coprime labeling.

Notice that the largest label on P_n will be odd since the sequence X is increasing, and we use 4 as the final label if n is odd. There are $\frac{p_{m-1}+1}{2}$ odd numbers less than or equal to p_{m-1}. Of these, $m-1$ of them are odd prime numbers that we used as labels on K_m, leaving $\frac{p_{m-1}+1}{2} - m + 1$ odd integers that are smaller than p_{m-1}as labels for P_n. Then our path can include up to $2\left(\frac{p_{m-1}+1}{2} - m + 1\right) + 1 = p_{m-1} - 2m + 4 = r$ vertices while maintaining that all of its labels are smaller than p_{m-1}. By our assumption of $n \leq r$, $\mathfrak{pr}(K_m \cup P_n) = p_{m-1}$ in this case.

Next suppose that $m \geq 5$ and $n > r$. We label K_m as before, label the first $r-1$ vertices of P_n using $x_1, \ldots, x_{r-1}$, and label v_r by 4. For the remaining vertices, if n is even, we label $v_{r+1}, \ldots, v_n$ by the sequence

$$p_{m-1} + 2, p_{m-1} + 3, \ldots, p_{m-1} + n - r + 1.$$

If n is odd, we label $v_{r+1}, \ldots, v_{n-1}$ by the above sequence up to $p_{m-1} + n - r$, but we label v_n by 8 and reassign the label for v_1 as 10 if $m > 5$. Note that if $m = 5$, this reassignment for n odd is not needed because $r = 1$ and hence the first label in the path is 4. The labeling is coprime based on the discussion in the last case about labeling the path using the sequence X, and the fact that any newly adjacent pairs of labels are consecutive, an odd integer adjacent to 4 or 8, or potentially the labels 10 and 9 at the beginning of the path. It is a minimum coprime labeling since our maximum label is an odd integer, either $p_{m-1}+n-r+1$ if n is even or $p_{m-1}+n-r$ if n is odd, and our labeling includes every odd value up to this maximum while using as many even labels as possible on the path. Our result for $\mathfrak{pr}(K_m \cup P_n)$ follows since $p_{m-1} + n - r + 1 = 2m + n - 3$ and $p_{m-1} + n - r = 2m + n - 4$. □

The last class of disjoint union graphs we will investigate is the union of a complete graph and a star, $K_m \cup S_n$, where S_n has n pendant vertices. See Figure 4 for an example of the union of K_9 and the star S_6 with a minimum coprime labeling. By [16], $K_m \cup S_n$ is prime when $\pi(m+n+1) \geq m$ where $\pi(m+n+1)$ is the number of prime numbers less than or equal to $m + n + 1$. To find the minimum coprime labeling of such a graph, we first define three numbers and provide examples of each.

Definition 4. For positive integers t and b, we let $\varphi(t, b)$ be the number of composite integers not equal to 1 that are less than or equal to b and relatively prime with t.

Example 5. We observe that $\varphi(25, 31) = 14$, since there are eleven composite numbers less than 25 that are relatively prime with 25, and three such numbers that are between 25 and 31.

Next, we define a number k_b that will potentially be used in our minimum coprime labeling as the center of the star S_n.

Definition 6. Given positive integers m and n, and a composite integer b in the interval $[p_{m-1}, p_m]$, we let $k_b = \min(\{q^2 : q^2 < b, \varphi(q^2, b) \geq n, \text{ and } q \text{ is prime}\})$. In the case that no such q exists in which q^2 fits the required inequalities, we define $k_b = \infty$.

Example 7. When $m = 10$ and $n = 10$, the smallest prime squares that fit the required inequalities for $b = 27$ is $k_{27} = 25$ yet when $b = 28$, we see that $k_{28} = 9$. On the other hand, when $m = 4$ and n is any positive integer, we find that when $b = 6$, $k_6 = \infty$ since the only $q^2 < b$ is 4 and $\varphi(4, 6) = 0 < n$.

In the upcoming labeling, the integer b will represent the largest label assigned to a pendant vertex of the star S_n. The center of the star will be labeled by the integer k_b, resulting in the largest label among the vertices of S_n being the following value.

Definition 8. For $m, n \in \mathbb{N}$, we let $\alpha = \min(\{\max(k_b, b) : b \in (p_{m-1}, p_m)\})$.

Again, note it is possible for $k_b = \infty$ for each b value, particularly if n is sufficiently large compared to b, and in this case $\alpha = \infty$ as well. We now formalize our labeling in the following result.

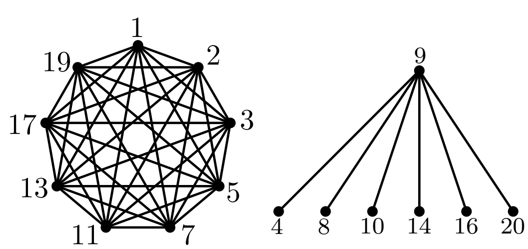

Figure 4: Minimum coprime labeling of $K_9 \cup S_6$ with $\pi(m + n + 1) < m$, $n > r$, and $\alpha \leq p_m$

Theorem 9. *Let m, n be positive integers, p be the largest prime number such that $p^2 < p_{m-1}$, and $r = p_{m-1} - m - \left\lfloor \frac{p_{m-1}}{p} \right\rfloor + 1$. The minimum coprime number for*

$K_m \cup S_n$ is

$$\mathfrak{pr}(K_m \cup S_n) = \begin{cases} m+n+1 & \text{if } \pi(m+n+1) \geq m, \\ p_{m-1} & \text{if } \pi(m+n+1) < m \text{ and } n \leq r \\ \alpha & \text{if } \pi(m+n+1) < m,\ n > r, \text{ and } \alpha \leq p_m \\ p_m & \text{otherwise.} \end{cases}$$

Proof. By the work in [16], if $\pi(m+n+1) \geq m$ then $\mathfrak{pr}(K_m \cup S_n) = m+n+1$. Now suppose that $\pi(m+n+1) < m$. Since $\mathfrak{pr}(K_m) = p_{m-1}$ by Proposition 1, $\mathfrak{pr}(K_m \cup S_n) \geq p_{m-1}$. It is clear that $K_m \cup S_n$ will not be prime in this case. We first suppose that $n \leq r$. Again, label the vertices of K_m using 1 and the first $m-1$ primes. Let the center of the star be labelled as p^2. There are $p_{m-1} - m$ positive integers that are not used on the labels of K_m. Since the center of the star is labeled p^2, there are at most $\left\lfloor \frac{p_{m-1}}{p} \right\rfloor - 1$ positive integers less than p_{m-1} that we can use to label the pendant vertices of S_n. Since we assumed for this case that $n \leq r = p_{m-1} - m - \left\lfloor \frac{p_{m-1}}{p} \right\rfloor + 1$, we have $\mathfrak{pr}(K_m \cup S_n) = p_{m-1}$.

Now suppose that $n > r$. There are many options for the label of the center of the star, so we aim to use the smallest composite number less than p_m that is relatively prime with at least n composite numbers less than some fixed value $b \in (p_{m-1}, p_m)$. Since q^2 (q prime) is relatively prime with at least the same number of composite numbers as qt for some integer $t > 1$ with $t \neq q$, we would choose q^2 over qt as the label for the center of the star. Thus k_b is defined to be a candidate for the center of the star for each integer $b \in (p_{m-1}, p_m)$. If this were a minimum coprime labeling then its minimum coprime number is $\max(k_b, b)$. Hence we look for the minimum among all of these maximums and denote such a value as α. By construction, there are more than n relatively prime composite numbers less than α that are not 1, so the pendant vertices of the star can be labeled by these. Thus, this is a coprime labeling. We choose α to be minimum and thus the result follows if $\alpha \leq p_m$.

If we find that $\alpha > p_m$, then we instead use the first m primes on K_m, and use 1 as the label of the center of the star, in which case $\mathfrak{pr}(K_m \cup S_n) = p_m$ as long as the assumption $\pi(m+n+1) < m$ still holds. □

Next we consider the corona of a complete graph with the empty graph of one or two vertices. In [16], it was shown that $K_n \odot K_1$ and $K_n \odot \overline{K}_2$ are prime under certain conditions, particularly if $n \leq 7$ for K_1 and if $n \leq 16$ for $\overline{K}_2$. Later, El Sonbaty, Mahran, and Seoud [13] showed that $K_n \odot \overline{K}_m$ is not prime if $n > \pi(n(m+1)) + 1$ and conjectured that $K_n \odot \overline{K}_m$ is prime if $n < \pi(n(m+1)) + 1$. We give the minimum coprime number for $K_n \odot K_1$ and $K_n \odot \overline{K}_2$ below. An example of $K_8 \odot K_1$ is given in Figure 5.

Theorem 10. *If n is an integer with $n > 7$, then $\mathfrak{pr}(K_n \odot K_1) = p_{n-1}$.*

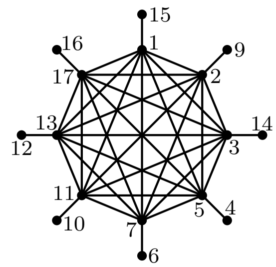

Figure 5: Minimum coprime labeling of $K_8 \odot K_1$

Proof. By Proposition 1, $\mathfrak{pr}(K_n \odot K_1) \geq p_{n-1} \geq 17$. Let $u_1, u_2, \ldots, u_n$ be the vertices in K_n, and for each $i \in \{1, \ldots, n\}$, let v_i be the vertices adjacent to u_i for the n copies of K_1. Label the vertices u_i with p_{i-1} for $i \in \{2, \ldots, n\}$ and label u_1 with 1. Then label v_i with $p_{i-1} - 1$ for $i \in \{4, \ldots, n\}$. We label v_1, v_2, v_3 with $15, 9, 14$, respectively. Since these three labels are specifically chosen to be relatively prime with their neighbors $1, 2$, and 3, and all other pendant edges connect labels that are consecutive, this labeling is coprime. This results in $\mathfrak{pr}(K_n \odot K_1) \leq p_{n-1}$, combining with the previous inequality to prove our claim of $\mathfrak{pr}(K_n \odot K_1) = p_{n-1}$. □

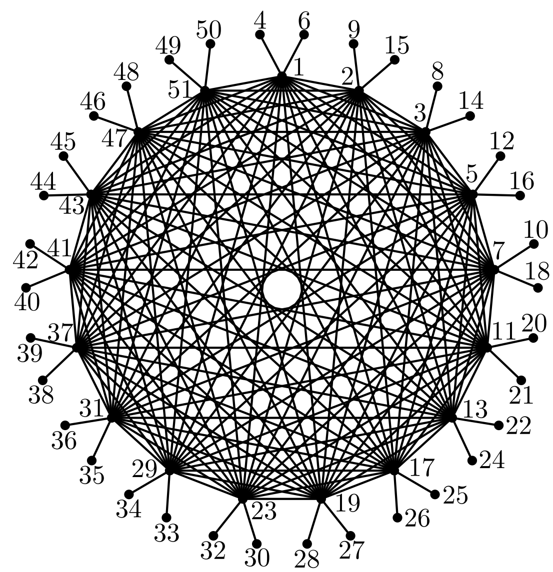

Figure 6: Minimum coprime labeling of $K_{17} \odot \overline{K}_2$

An example of $K_{17} \odot \overline{K}_2$ is given in Figure 6.

Theorem 11. *If n is an integer with $n > 16$, then $\mathfrak{pr}(K_n \odot \overline{K}_2) = p_{n-1}$.*

Proof. By Proposition 1, $\mathfrak{pr}(K_n \odot K_1) \geq p_{n-1} \geq 53$. We will label the vertices in K_n and the first 16 corresponding copies of $\overline{K}_2$, and then we will label the remaining vertices in a more structured manner. Let $u_1, u_2, \ldots, u_n$ be the vertices in K_n and $v_{i,1}, v_{i,2}$ be the vertices for the n copies of $\overline{K}_2$. Label the vertices u_i for $i \in \{2, 3, \ldots, n\}$ with p_{i-1} and label u_1 with 1. We label the sequence of vertices $v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}, v_{3,1}, v_{3,2}, \ldots, v_{16,2}$, respectively, with the following sequence of labels:

$$4, 6, 9, 15, 8, 14, 12, 16, 10, 18, 20, 21, 22, 24, 25, 26, 27, 28,$$
$$30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48.$$

For $i > 16$, if p_{i-1} is not a twin prime, we label $v_{i,1}, v_{i,2}$ with $p_{i-1} - 2, p_{i-1} - 1$. If instead p_{i-1} and p_i are twin primes, we label $v_{i,1}, v_{i,2}$ with $p_{i-1} - 3, p_{i-1} - 2$ and label $v_{i+1,1}, v_{i+1,2}$ with $p_{i-1} - 1, p_{i-1} + 1$. One can see by inspection that the labels on the first 16 vertices in K_n are relatively prime with the corresponding pendant labels from the sequence, and the coprime condition is also upheld for pendant edges from the other vertices v_i with $i > 16$ since each label p_{i-1} is within distance 3 from each of its adjacent labels. Therefore, we have obtained our result since this is a minimum coprime labeling with largest label being p_{n-1}. □

As long as n is sufficiently large compared to m, the authors believe that the minimum prime labeling of $K_n \odot \overline{K}_m$ is p_{n-1}. It is likely that several subsequent cases beyond $m = 2$ can be proven in a similar manner to Theorem 11, but a generalization for all m eludes discovery. As such, we leave this as an open problem in Section 6 and a conjecture below.

Conjecture 1. For all $m > 0$, there exists an $M > m$ such that for all $n > M$, $\mathfrak{pr}(K_n \odot \overline{K}_m) = p_{n-1}$.

4. Powers of Paths and Cycles

We next consider the graph P_n^2 with $n \geq 6$ vertices. Seoud and Youssef [14] proved that this graph is not prime when $n = 6$ and $n \geq 8$. We will construct a minimum coprime labeling of P_n^2 for these cases. A lower bound for the minimum coprime number for the graph would be obtained by using the maximum amount of even labels that can be used based on the independence number of P_n^2, shown in [14] to be $\left\lceil \frac{n}{3} \right\rceil$, along with the smallest possible odd labels. Figure 7 shows a minimum coprime labeling of the graph P_6^2 and P_{10}^2, and in the former case, the path is represented as $v_1, v_2, \ldots, v_6$ with the horizontal edges connecting vertices of distance 2. Since the independence number of P_6^2 is 2, we can only use two even labels, which prevents a prime labeling. Instead, a minimum coprime labeling can be achieved with $\mathfrak{pr}(P_6^2) = 7$.

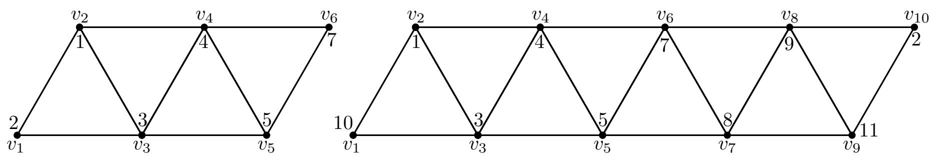

Figure 7: Minimum coprime labeling of P_6^2(left) and P_{10}^2 (right)

The following theorem regarding the minimum coprime number of the path squared was verified for the $n = 6$ case through the labeling in Figure 7, and the general case of $n \geq 8$ will be proven by a series of lemmas.

Theorem 12. *Let $n = 6$ or $n \geq 8$. The minimum coprime number of P_n^2 is given by*

$$\mathfrak{pr}(P_n^2) = \begin{cases} 4k - 1 & \text{if } n = 3k \text{ or } 3k + 1 \\ 4k + 1 & \text{if } n = 3k + 2. \end{cases}$$

Assume that $n \geq 8$ for the following lemmas that will prove the general case of Theorem 12. To construct a minimum coprime labeling of P_n^2, we define a sequence $X = \{x_i\}_{i=1}^{\infty}$ of integers for which the first n numbers will be used as labels for the vertices $\{v_1, \ldots, v_n\}$. The sequence X consists of a length 45 segment with the repeated pattern of even, odd, and odd integers, with the subsequent terms of the sequence defined by shifting the initial entries by multiples of 60. The first 30 odd numbers are included in the initial segment along with the first 15 even numbers that are not multiples of 3 or 5. The definition of the sequence is the following:

$$\begin{aligned}\{x_1, \ldots, x_{45}\} = \{&2, 1, 3, 4, 5, 7, 8, 9, 11, 14, 13, 15, 16, 17, 19, 22, 21, 23, 26, 25, 27, 28,\\ &29, 31, 32, 33, 35, 34, 37, 39, 38, 41, 43, 44, 45, 47, 46, 49, 51, 52, 53,\\ &55, 56, 57, 59\},\end{aligned}$$

and for $i > 45$, we recursively define $x_i = x_{i-45} + 60$.

In order to examine whether adjacent vertices will have relatively prime labels, the following fact about the distance between the labels of such vertices will be quite useful.

Lemma 1. *Given adjacent vertices v_i and v_j in P_n^2 for some $1 \leq i, j \leq n$, the labels in the sequence $\{x_1, \ldots, x_n\}$ satisfy $|x_i - x_j| \leq 5$.*

Proof. Based on the structure of P_n^2, the neighborhood of a vertex v_i with $i = 3, \ldots, n-2$ is the set $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$. By inspection of the 45 labels in the initial segment of the sequence, we can verify that x_i is within distance 5 of the label of any adjacent vertex when $i \leq 43$.

For adjacent vertices v_i and v_j where $v_i \in \{v_1, \ldots, v_{45}\}$ and $v_j \in \{v_{46}, \ldots, v_{90}\}$, we only need to consider the labels $x_{44} = 57$, $x_{45} = 59$, $x_{46} = 62$, and $x_{47} = 61$, where the latter two labels are the result of shifting x_1 and x_2 by 60. We see that our desired inequality $|x_i - x_j| \leq 5$ is satisfied by each adjacent pair such as $|x_{44} - x_{46}| = 5$.

Adjacent vertices with indices that are both larger than 45 will also satisfy $|x_i - x_j| \leq 5$ because $x_i = x_a + 60m$ and $x_j = x_b + 60m$ for some integers $1 \leq a, b \leq 47$ and positive integer m. Since x_a and x_b satisfy the inequality as shown above, the shifted values x_i and x_j also maintain a distance of 5 or less, which covers the remaining possible cases of indices i and j. □

Note that if we continued to define $x_{46}, \ldots, x_{54}$ in the manner of the first 45 terms by including all even numbers that are not multiples of 3 or 5, then we would have $x_{52} = 64$ and $x_{54} = 71$. This would have contradicted Lemma 1 and eventually would result in the sequence containing adjacent labels that are both multiples of 7. Shifting the $x_1, \ldots, x_{45}$ segment by 60 to have $x_{46} = 62$ skips the even number 58 and maintains that the distance between adjacent labels remain small enough to guarantee a coprime labeling, as shown in the following lemma.

Lemma 2. *The labels* $\{x_1, \ldots, x_n\}$ *are a coprime labeling of* P_n^2.

Proof. By Lemma 1, the distance between the labels of adjacent vertices is at most 5; hence, no two adjacent vertices will have labels with a factor of 6 or higher in common. By design, even labels are not adjacent, eliminating the possibility of labels sharing a factor of 2. None of the even labels are divisible by 3 or 5, and any two odd labels both divisible by 3 or 5 are spaced out enough to avoid being adjacent, resulting in no adjacent labels having a 3 or 5 as a common factor. Thus our labeling is coprime. □

Lemma 3. *If* $\max(x_1, \ldots, x_n)$ *is odd, then the labels* $\{x_1, \ldots, x_n\}$ *are a minimum coprime labeling of* P_n^2.

Proof. Notice that even integer labels cannot be 1 or 2 indices apart, which corresponds to the independence number of our graph being $\left\lceil \frac{n}{3} \right\rceil$. Hence, we have used as few odd labels as possible, while also using all of the odd numbers from 1 to $\max(x_1, \ldots, x_n)$. Thus we cannot make $\mathfrak{pr}(P_n^2)$ any smaller. □

For the case of the maximum label within $\{x_1, \ldots, x_n\}$ being even, we alter the sequence of labels to achieve a minimum labeling. Through examination of the first 45 terms of the sequence and the fact that remaining terms are simply shifted from this initial segment, we see that this situation can only occur when $n = 3k+1$ or $3k+2$. Additionally, we observe that the largest even label in either case is in position $3k+1$. We create a new sequence $\{x_i^*\}_{i=1}^n$ by defining $x_1^* = 10$, $x_{3k+1}^* = 2$, and $x_i^* = x_i$ for $i \in \{2, \ldots, n\} \setminus \{3k+1\}$.

Lemma 4. *If* $\max(x_1, \ldots, x_n)$ *is even, then the labels* $\{x_1^*, \ldots, x_n^*\}$ *are a minimum coprime labeling of* P_n^2*.*

Proof. As described above, to be in the case of the maximum label being even, we know that $n = 3k+1$ or $3k+2$. By inspection of the initial segment of our sequence of labels, the first such $n \geq 8$ with an even label being the largest in the sequence $\{x_1, \ldots, x_n\}$ would be $n = 10$ in which $x_{10} = 14$. For this case, or for any larger such n value, replacing x_1 and x_{3k+1} (which we note is x_n or x_{n-1}) with $x_1^* = 10$ and $x_{3k+1}^* = 2$ would result in the maximum of the labels falling into the case of Lemma 3 since one can observe that the second largest label in $\{x_1, \ldots, x_n\}$ will always be odd if the largest one is even.

Following the reasoning of the previous lemma, the labeling $\{x_1^*, \ldots, x_n^*\}$ is a minimum coprime labeling as long as we show it is still a coprime labeling. This sequence matches the original sequence except for two values; hence, the only adjacent vertex pairs that need to be checked as still having relatively prime labels are the vertices v_1 and v_{3k+1} with their respective neighbors. In the case of $n = 3k+1$, $v_{3k+1} = v_n$ is only adjacent to v_{n-1} and v_{n-2}. Since both labels for these vertices are odd, the label $x_n^* = 2$ is relatively prime with its adjacent labels. Similarly if $n = 3k+2$, then $v_{3k+1} = v_{n-1}$ is only adjacent to v_{n-3}, v_{n-2}, and v_n. Again, all of these vertices will be labeled by odd numbers, which are relatively prime to 2. Likewise, the label $x_1^* = 10$ is only adjacent to the second and third vertices with labels $x_2^* = 1$ and $x_3^* = 3$, so the labels are once again relatively prime. Thus, the labeling $\{x_1^*, \ldots, x_n^*\}$ is a minimum coprime labeling of P_n^2. □

Proof of Theorem 12. The previous lemmas in this section have shown that the sequence $\{x_1, \ldots, x_n\}$ or $\{x_1^*, \ldots, x_n^*\}$ provides a minimum coprime labeling of P_n^2. It only remains to show that the maximum value in the labeling sequence, which was shown to be odd, is in fact $4k-1$ or $4k+1$ depending on the value of $n \pmod 3$. In the case of $n = 3k$, there are $\left\lceil \frac{n}{3} \right\rceil = k$ even labels used, so $\mathfrak{pr}(P_n^2)$ is the $(n-k)^{\text{th}}$ odd number, which is $2(n-k) - 1 = 4k - 1$. The other two cases follow similarly to give us the value of $\mathfrak{pr}(P_n^2)$. □

We next consider the square of the cycle, C_n^2, for $n \geq 4$. Seoud and Youssef [14] showed this graph is not prime when $n \geq 4$ and that its independence number is $\left\lfloor \frac{n}{3} \right\rfloor$, which will be the maximum allowable number of even labels. Note the only difference compared to the squared path graph is the additional edges $v_1 v_{n-1}, v_1 v_n$, and v_2v_n. These do require some alterations to our labeling of P_n^2 in some cases to maintain the coprime property, which results in the following for the minimum coprime number. See Figure 8 for an example of a minimum coprime labeling of C_8^2.

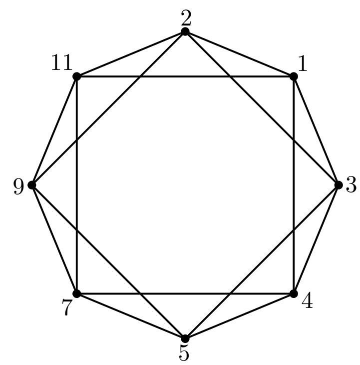

Figure 8: Minimum coprime labeling of C_8^2

Theorem 13. *Let $n \geq 4$. The minimum coprime number of C_n^2 is given by*

$$\mathfrak{pr}(C_n^2) = \begin{cases} 4k-1 & \text{if } n = 3k \\ 4k+1 & \text{if } n = 3k+1 \\ 4k+3 & \text{if } n = 3k+2. \end{cases}$$

Proof. For the case of $n = 3k$, the labeling of vertices $v_1, \ldots, v_n$ using the sequence $\{x_1, \ldots, x_n\}$ still provides a minimum coprime labeling as it did with the P_n^2. This is because $x_1 = 2$ and $x_2 = 1$ while the final two labels x_{n-1} and x_n are odd, so the additional three new edges maintain our coprime property.

The other two cases cannot simply use the same labeling as the squared path graph since the independence number is 1 smaller for C_n^2 compared to P_n^2. Also note that we will only need to adjust the sequence $\{x_1, \ldots x_n\}$ instead of $\{x_1^*, \ldots, x_n^*\}$ even in the cases of $n = 3k+1$ and $3k+2$ that needed the largest even label reassigned as 2 in the path squared labeling. This is because this largest even label will instead be reassigned as an odd number in the following cases.

When $n = 3k+1$, the edge v_1v_n would result in a common factor of 2 between the labels $x_1 = 2$ and x_n since x_n is even. Hence, we reassign $x_n = 4k+1$, which is the lowest available odd label. This label will now be relatively prime with its adjacent vertices whose labels are 1, 2, $4k-1$, and $4k-3$, making $\{x_1, \ldots, x_n\}$ a minimum coprime labeling.

Similarly, when $n = 3k+2$, the edge v_1v_{n-1} leads to adjacent vertices with even labels. We correct this while keeping the labeling minimum by reassigning

$x_{n-1} = 4k+1$ and $x_n = 4k+3$, which we note are the smallest available odd labels. The newly assigned x_{n-1} and x_n are coprime with any adjacent vertices' labels since their neighbors are only labeled by 1, 2, or an odd number of distance 2 or 4 from x_{n-1} or x_n. In each case, we obtain a minimum coprime labeling using the maximum amount of even labels and the smallest possible odd labels with the largest label being $4k-1$, $4k+1$, and $4k+3$ in the respective cases. □

Now we consider taking the third power of the path graph, P_n^3, which has additional edges from v_i to v_{i+3} for $i = 1, \ldots, n-3$. Since this graph contains P_n^2 as a spanning subgraph, it is also not prime when $n = 6$ and $n \geq 8$ by Observation 1. It is known that the independence number for the cubed path is $\left\lceil \frac{n}{4} \right\rceil$, which will determine how many even labels can be placed on its vertices. This fact results in P_n^3 not being prime for the additional cases of $n = 4$ and $n = 7$, but note that the path cubed on 4 vertices is simply the complete graph. It is, however, prime when $n = 5$, in which the path can be labeled using $2, 1, 3, 5, 4$.

Theorem 14. *Let $n \geq 6$. The minimum coprime number of P_n^3 is given by*

$$\mathfrak{pr}(P_n^3) = \begin{cases} 6k-1 & \text{if } n = 4k \text{ or } 4k+1 \\ 6k+1 & \text{if } n = 4k+2 \\ 6k+3 & \text{if } n = 4k+3. \end{cases}$$

Similar to our construction for the coprime labeling of P_n^2, we define a sequence of finite length, with the subsequent terms determined by shifting the initial sequence. For P_n^3, our initial sequence consists of 140 entries, and the shift is by 210 from y_i to y_{i+140}. Note that the sequence below is a repetition of even, odd, odd, and odd entries with the smallest possible odd numbers included and all even multiples of 3, 5, and 7 excluded. Additionally, some other even numbers were removed to maintain the inequality in Lemma 5 regarding the distance between adjacent labels. We define the labeling sequence as follows, where vertex v_n of P_n^3 is labeled by y_n:

$$\begin{aligned}\{y_1, \ldots, y_{140}\} = \{&2, 1, 3, 5, 4, 7, 9, 11, 8, 13, 15, 17, 16, 19, 21, 23, 22, 25, 27, 29, 26, 31,\\ &33, 35, 32, 37, 39, 41, 38, 43, 45, 47, 44, 49, 51, 53, 52, 55, 57, 59, 58,\\ &61, 63, 65, 62, 67, 69, 71, 68, 73, 75, 77, 74, 79, 81, 83, 82, 85, 87, 89,\\ &86, 91, 93, 95, 92, 97, 99, 101, 104, 103, 105, 107, 106, 109, 111, 113,\\ &116, 115, 117, 119, 118, 121, 123, 125, 122, 127, 129, 131, 128, 133,\\ &135, 137, 134, 139, 141, 143, 142, 145, 147, 149, 146, 151, 153, 155,\\ &152, 157, 159, 161, 158, 163, 165, 167, 164, 169, 171, 173, 172, 175,\\ &177, 179, 176, 181, 183, 185, 184, 187, 189, 191, 188, 193, 195, 197,\\ &194, 199, 201, 203, 202, 205, 207, 209\}\end{aligned}$$

and $y_i = y_{i-140} + 210$ for $i > 140$.

We prove this theorem as was done for squared paths by using a sequence of lemmas.

Lemma 5. *Given adjacent vertices v_i and v_j in P_n^3 for some $1 \leq i, j \leq n$, the labels in the sequence $\{y_1, \ldots, y_n\}$ satisfy $|y_i - y_j| \leq 9$.*

Proof. The graph P_n^3 has a neighborhood for each vertex v_i consisting of $\{v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3}\}$ with $i = 4, \ldots, n-3$. Through careful inspection of the 140 initial labels, the maximum distance between labels of a vertex v_i with $i \leq 137$ and any of its neighbors is 9, which is attained for example by vertices v_9 and v_{12} having labels 8 and 17, respectively.

We next examine adjacent vertices with labels with one from the initial sequence $\{y_1, \ldots, y_{140}\}$ and one from the shifted sequence $\{y_{141}, \ldots, y_{280}\}$. Since $y_{138} = 205$, $y_{139} = 207$, $y_{140} = 209$, $y_{141} = 212$, $y_{142} = 211$, and $y_{143} = 213$, the adjacent labels in this case with greatest distance apart are $|y_{138} - y_{141}| = 7$, which is within the desired distance.

As in Lemma 1, adjacent vertices with indices larger than 140 maintain the same distance for their labels as their corresponding vertices with indices between 1 and 143, making the inequality hold for all adjacent vertices. □

Lemma 6. *The labels $\{y_1, \ldots, y_n\}$ are a coprime labeling of P_n^3.*

Proof. By Lemma 5, the distance between adjacent labels is at most 9; thus, no adjacent labels have a common factor of 10 or higher. The construction of the sequence results in even labels being four indices apart and hence not adjacent, so no labels share a common factor of 2. The even labels were chosen to not contain a factor of 3, 5, or 7, and any pair of odd labels that both contain a multiple of 3, 5, or 7 have indices that are at least 4 apart, so they are not adjacent. Thus, each pair of adjacent labels are relatively prime. □

Lemma 7. *If $\max(y_1, \ldots, y_n)$ is odd, then the labels $\{y_1, \ldots, y_n\}$ are a minimum coprime labeling of P_n^3.*

Proof. Since the independence number of P_n^3 is $\left\lceil \dfrac{n}{4} \right\rceil$, our sequence of labels uses the maximum number of even labels. Therefore, the fact that we have used every odd number up to $\max(y_1, \ldots, y_n)$ implies that we have achieved a minimum coprime labeling. □

As with P_n^2, we define an altered labeling sequence for the case of $\max(y_1, \ldots, y_n)$ being even to switch out this largest even label to ensure the maximum is odd. We create our new sequence $\{y_i^*\}_{i=1}^n$ by defining $y_1^* = 14$, $y_m^* = 2$, and $y_i^* = y_i$ for all $i \in \{2, \ldots, n\} \setminus \{m\}$, where y_m was the maximum label of $\{y_1, \ldots, y_n\}$. It can be observed from the initial sequence $\{y_1, \ldots, y_{140}\}$ that $m = n$ or $n - 1$.

Lemma 8. *If* $\max(y_1, \ldots, y_n)$ *is even, then the labels* $\{y_1^*, \ldots, y_n^*\}$ *are a minimum coprime labeling of* P_n^3*.*

Proof. Through examination of the sequence $\{y_1, \ldots, y_n\}$, the first n value such that an even integer is the largest value in the sequence is $n = 69$ and $n = 70$ with $y_{69} = 104$. Replacing the maximum even label with 2 and the first label with 14 will result in there now being an odd label as the maximum. This allows us to apply Lemma 7 if our labeling remains a coprime labeling. Reassigning the label of v_1 to be 14 maintains our coprime property since it is only adjacent to the labels $1, 3$, and 5. The vertex v_m, whose maximum label was reassigned as the label 2, is only adjacent to vertices with odd labels; hence, the adjacent labels remain relatively prime. □

Proof of Theorem 14. The preceding lemmas have proven that P_n^3 has a mimimum coprime labeling using either $\{y_1, \ldots, y_n\}$ or $\{y_1^*, \ldots, y_n^*\}$, leaving us to verify that the correct minimum coprime number was attained. The labels consist of $\left\lceil \frac{n}{4} \right\rceil$ even labels, so we consider the cases of $n \pmod 4$. For the case of $n = 4k$, $\left\lceil \frac{n}{4} \right\rceil = k$ even labels were used. Therefore, $\mathfrak{pr}(P_n^3)$ is the $(n-k)^{\text{th}}$ odd number, which is $2(n-k) - 1 = 6k - 1$. The other three cases follow similarly to find their minimum coprime numbers. □

We next demonstrate a minimum coprime labeling of C_n^3, which we note has an independence number of $\left\lfloor \frac{n}{4} \right\rfloor$. Also observe that $C_n^3 = K_n$ for $n \leq 7$.

Theorem 15. *Let* $n \geq 8$*. The minimum coprime number of* C_n^3 *is given by*

$$\mathfrak{pr}(C_n^3) = \begin{cases} 6k-1 & \text{if } n = 4k \\ 6k+1 & \text{if } n = 4k+1 \\ 6k+5 & \text{if } n = 4k+2 \\ 6k+7 & \text{if } n = 4k+3. \end{cases}$$

Proof. We begin by considering the labeling sequence $\{y_1, \ldots, y_n\}$ that was used for P_n^3 on the vertices $v_1, \ldots, v_n$ of the cycle. Before considering each case, it is important to note that the vertex v_n is adjacent to v_3, which is labeled by $y_3 = 3$, in C_n^3 since their distance in the cycle graph is 3. Observe from our labeling sequence that y_n is a multiple of 3 if and only if $n = 4k + 3$.

For $n = 4k$, the labeling $\{y_1, \ldots, y_n\}$ is a minimum coprime labeling of C_n^3. The additional edges in C_n^3 that were not in P_n^3 have endpoints with relatively prime labels because $y_3 = 3$ with y_n not being a multiple of 3 as previously stated above, $y_2 = 1$, and $y_1 = 2$ with the three labels y_{n-2}, y_{n-1}, and y_n all being odd.

When $n = 4k+1$, the independence number being $\left\lfloor \frac{n}{4} \right\rfloor$ implies that the sequence $\{y_1, \ldots, y_n\}$ cannot be used for the labeling since it would include $k+1$ even labels.

Instead, we reassign $y_n = 6k+1$, which is the smallest unused odd label. This label is not a multiple of 3, so it is relatively prime with $y_3 = 3$. This final label $y_n = 6k+1$ is also clearly relatively prime with the labels 2 and 1 of the vertices v_1 and v_2, in addition to the odd labels of $y_{n-3} = 6k-5$, $y_{n-2} = 6k-3$, and $y_{n-1} = 6k-1$ since it is not a multiple of 3, resulting in a minimum coprime labeling.

Assuming $n = 4k+2$, as in the previous case, there are too many even labels which requires a reassignment of the last even label to be $y_{n-1} = 6k+1$. This label is again relatively prime with y_1 and y_2. The last label, y_n, which originally was also $6k+1$, cannot be reassigned to be the next smallest odd label of $6k+3$, which is a multiple of 3. This is because v_n is adjacent to v_3, which is labeled as $v_3 = 3$. Furthermore, labels that are multiples of 3 cannot be shifted in any way to accommodate $6k+3$ because of the independence number being $\left\lfloor \frac{n}{4} \right\rfloor = k$ and the fact that there are already k multiples of 3 in our labeling sequence. Thus, we set $y_n = 6k+5$, the smallest possible label that is not even or a multiple of 3. Since it is coprime with the odd labels of $y_{n-3} = 6k-3$, $y_{n-2} = 6k-1$, and $y_{n-1} = 6k+1$, as well as y_1 and y_2, we have a minimum coprime labeling.

We use the same reasoning for the $n = 4k+3$ to reassign the labels $y_{n-2} = 6k+1$, $y_{n-1} = 6k+5$ (to avoid the multiple of 3), and $y_n = 6k+7$. The labeling is again minimum because the independence number limits the number of even integers and multiples of 3 that can be used. □

The next logical step would be to generalize our constructions for P_n^k and C_n^k or at least continue with finding a minimum coprime labeling of P_n^4. However, to keep even labels spaced out enough, a sequence for P_n^4 would require repetition of the pattern of even, odd, odd, odd, and odd. Using the smallest possible odd numbers fails quickly though as it would begin $2, 1, 3, 5, 7, 4, 9$, resulting in the labels 3 and 9 being adjacent due to their vertices being distance 4 apart. Having to frequently skip odd numbers within our label sequence greatly increases the difficulty of finding minimum coprime labeling of P_n^k for $k \geq 4$, so we leave this as an open problem in Section 6.

5. The Join of Paths and Cycles

In this section we will establish the minimum coprime number for the join of two paths, two cycles, or a cycle and a path. As we did in Section 4, we will use the results on paths to find solutions for the join of cycles. In [12, 14], it was shown that $P_m + K_1 = P_m + P_1$ is prime, $P_m + \overline{K}_2$ is prime if and only if $m \geq 3$ is odd, and $P_m + \overline{K}_n$ is not prime for $n \geq 3$. We will consider similar join graphs in which edges are added to $\overline{K}_n$ to create a path P_n.

We will exploit the size of the gap between prime numbers in order to ensure we can form a minimum coprime labeling. We define the gap between two primes as $g(p_n) = p_{n+1} - p_n$. By the prime number theorem, for all $\varepsilon > 0$, there exists an integer N such that for all $n > N$, $g(p_n) < \varepsilon p_n$. More specifically, particular values for ε and N are mentioned in [8].

Theorem 16. [6, 9, 11] *Let $\varepsilon > 0$. For any positive integer $n > N$, $g(p_n) < \varepsilon p_n$ where $(N, \varepsilon) \in \{(9, 1/5), (118, 1/13), (2010760, 1/16597)\}$.*

We first investigate a minimum coprime labeling of $P_m + P_2$. We know from [14] that $P_m + \overline{K_2}$ is prime if and only if m is odd or $m = 2$. By Observation 1, adding an edge to create the second path results in $P_m + P_2$ not being prime when m is even and $m \geq 4$. Note that $P_2 + P_2$ would also not be prime, but this graph is simply K_4.

Theorem 17. *If $m \geq 4$ is even, then $\mathfrak{pr}(P_m + P_2) = m + 3$.*

Proof. For the case of $m = 4$, Figure 9 shows an example of a minimum coprime labeling of $P_4 + P_2$. Since $P_m + P_2$ is known to not be prime and the maximum label is $|V(P_4 + P_2)| + 1$, we see that $\mathfrak{pr}(P_4 + P_2) = 7$.

For $m = 6$, label the vertices of $V(P_6) = \{v_1, \dots, v_6\}$ using the sequence $2, 9, 4, 5, 8, 3$, and label P_2 using 1 and 7. Then it is clear the result follows. Suppose that $m \geq 8$ is even. We label the vertices of $V(P_m) = \{v_1, \dots, v_m\}$ using the sequence

$$2, m+3, 4, 3, 10, 9, 8, 5, 6, p_4, 12, p_5, 14, 15, 16, p_6, 18, \dots.$$

Notice that p_i is the label of $v_{p_{i+1}-1}$ for $i \geq 4$. By Theorem 16, m is large enough so that $p_{i+1} < 1.2p_i$. Since the smallest number that is divisible by p_i that is not p_i is $2p_i$, we know p_i is relatively prime with the labels $p_{i+1} - 1$ and $p_{i+1} + 1$ that are adjacent to $v_{p_{i+1}-1}$ for all $i \geq 4$. Let p' be the largest prime that is less than or equal to $m + 1$, which we note is not a label on P_m based on our shift of the prime numbers within our labeling sequence. Then we label the vertices of P_2 using the labels 1 and p'.

It is clear that this is a coprime labeling based on our discussion of p_i being relatively prime to its adjacent labels in the sequence. It is a minimum coprime labeling with $\mathfrak{pr}(P_m + P_2) = m + 3$ since the graph is not prime and the maximum label is $|V(P_m + P_2)| + 1$. □

See Figure 9 for an example of the labeling in the following theorem when $m = 12$.

Theorem 18. *If $m \geq 3$, then the minimum coprime number of $P_m + P_3$ is*

$$\mathfrak{pr}(P_m + P_3) = \begin{cases} m + 4 & \textit{if } m \textit{ is odd} \\ m + 5 & \textit{if } m \textit{ is even.} \end{cases}$$

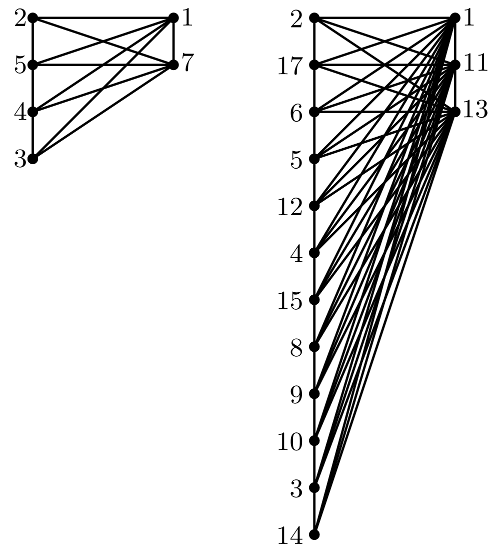

Figure 9: Minimum coprime labeling of $P_4 + P_2$ (left), $P_{12} + P_3$ (right)

Proof. Based on the independence number of $P_m + P_3$ being $\lceil \frac{m}{2} \rceil$, a minimum coprime labeling can at best use the smallest $\lfloor \frac{m}{2} \rfloor + 3$ odd integers. This would correspond to the largest label being $m+4$ if m is odd or $m+5$ if m is even. Thus, constructing labelings using these as the maximum label would prove our result.

Let $V(P_m) = \{v_1, \ldots, v_m\}$. For the initial cases of $m = 3, 4, 5$, and 6, by using the sequence $2, 3, 4, 9, 8, 11$ as the labels for $v_1, \ldots, v_6$, respectively, and labeling the vertices in P_3 with $1, 5, 7$, we see that $\mathfrak{pr}(P_3+P_3) = 7$, $\mathfrak{pr}(P_4+P_3) = \mathfrak{pr}(P_5+P_3) = 9$, and $\mathfrak{pr}(P_6 + P_3) = 11$. By using the sequence $6, 5, 4, 3, 2, 9, 8, 13, 10$ as the labels for $v_1, \ldots, v_9$, respectively, and labeling the vertices in P_3 with $1, 7, 11$, we have $\mathfrak{pr}(P_7 + P_3) = 11$ and $\mathfrak{pr}(P_8 + P_3) = \mathfrak{pr}(P_9 + P_3) = 13$. By using the sequence $6, 5, 4, 3, 2, 9, 10, 7, 8, 15, 14$ as the labels for $v_1, \ldots, v_{11}$, respectively, and labeling the vertices in P_3 with $1, 11, 13$, we see that $\mathfrak{pr}(P_{10} + P_3) = \mathfrak{pr}(P_{11} + P_3) = 15$.

Suppose that $m \geq 12$. We label the vertices of $V(P_m) = \{v_1, \ldots, v_m\}$ using the first m entries in the sequence

$$2, a, 6, 5, 12, 7, 4, b, 8, 9, 10, 3, 14, 15, 16, p_5, 18, p_6, 20, 21, 22, p_7, \ldots$$

where a and b are the values of m' and $m'+2$, with m' being the smallest odd number larger than $m + 1$. If m' is not a multiple of 3, we set $a = m'$ and $b = m' + 2$, but if $3 \mid m'$, we set $b = m'$ and $a = m' + 2$ to avoid a multiple of 3 being adjacent to the label 6. We know that m' will either be $m + 2$ or $m + 3$ depending on whether $m + 1$ is even or odd, implying the largest label is $m' + 2 = m + 4$ or $m + 5$, as described in our claim for $\mathfrak{pr}(P_m + P_3)$.

Notice that p_i is the label of $v_{p_{i+2}-1}$ for $i \geq 5$. By Theorem 16, m is large enough so that $p_{i+2} < 1.44p_i$, thus p_i is relatively prime with the labels $p_{i+2}-1$ and $p_{i+2}+1$ on either side of v_{p_i-1} for all $i \geq 5$. Let p' and p'' be the two largest primes in the sequence $2, 3, 4, \ldots, m+1$. Then we label the vertices in P_3 using the sequence $1, p', p''$. It is clear that this sequence is a coprime labeling, and since we achieved the goal of using a maximum label of $m+4$ and $m+5$ for the odd and even cases, respectively, it is also a minimum coprime labeling. □

Next, we consider the join of a path P_m with P_4, where we note the cases of $m < 4$ have already been covered by previous results.

Theorem 19. *If $m \geq 4$, then the minimum coprime number of $P_m + P_4$ is given by*

$$\mathfrak{pr}(P_m + P_4) = \begin{cases} m+6 & \text{if } m \text{ is odd} \\ m+7 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Observe that the independence number of $P_m + P_4$ is once again $\lceil \frac{m}{2} \rceil$, as it was for $P_m + P_3$. We now have one additional odd label that must be used in our labeling, meaning $\mathfrak{pr}(P_m + P_4) \geq m+6$ for m odd and $m+7$ for m even. We now construct labelings that use this as the maximum label.

Let $V(P_m) = \{v_1, \ldots, v_m\}$. By using the sequence $2, 3, 4, 9, 8, 13, 6$ as the labels for $v_1, \ldots, v_7$, respectively, and labeling the vertices of P_4 with $1, 5, 7, 11$, we have $\mathfrak{pr}(P_m + P_4) = 11, 11, 13, 13$ for the cases of $m = 4, 5, 6, 7$, respectively. By using the sequence $6, 5, 4, 3, 10, 9, 8, 15, 2, 17, 16, 19, 18$ as the labels for $v_1, \ldots, v_{13}$, respectively, and labeling the vertices of P_4 with $1, 7, 11, 13$, we see that $\mathfrak{pr}(P_m + P_4) = m+6$ for each $m = 9, 11, 13$ and $m+7$ for $m = 8, 10, 12$.

Suppose that $m \geq 14$. We label the vertices of $V(P_m) = \{v_1, \ldots, v_m\}$ using the first m values of the sequence

$$2, a, 6, b, 10, 3, 4, c, 8, 9, 14, 5, 12, 15, 16, p_4, 18, p_5, 20, 21, \ldots$$

where $\{a, b, c\} = \{m', m'+2, m'+4\}$, with m' being the smallest odd number larger than $m+1$. Note that at most two of m', $m'+2$, and $m'+4$ can be divisible by 3 or 5, or at most one can be divisible by both 3 and 5. In the case of one of the three being a multiple of three and a different value being a multiple of 5, set a to equal the multiple of 5, c to be the multiple of 3, and b to be the other value. If one of m', $m'+2$, or $m'+4$ is a multiple of 3 and 5, set c to be that value, and set a and b to be the other two values. Again, whether $m+1$ is even or odd will determine if m' is $m+2$ or $m+3$, resulting in the largest label being $m'+4 = m+6$ or $m+7$.

Notice that p_i is the label for $v_{p_i+3} - 1$ for $i \geq 4$. By Theorem 16, m is large enough so that $p_{i+3} < 1.728p_i$ and so p_i is relatively prime with the numbers on either side of the position of p_{i+3} for all $i \geq 4$. Let q_1, q_2, q_3 be the three largest primes in the sequence $2, 3, 4, \ldots, m+1$. Then we label the vertices in P_4 using the sequence $1, q_1, q_2, q_3$. It is clear that this sequence is a coprime labeling. Since its maximum label matches the lower bound for the minimum coprime number, the result follows. □

We now consider the more general join of paths P_m and P_n with restrictions set on m and n, particularly a larger lower bound for m. It is likely the case that $\mathfrak{pr}(P_m + P_n)$ satisfies the equality below for $m \leq 118$, but the task of completing these cases is better left to a computer.

Theorem 20. *Let $m > 118$ and $2 \leq n \leq 10$ be positive integers. Then the minimum coprime number of $P_m + P_n$ is given by*

$$\mathfrak{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Since $m > n$, the independence number of the general $P_m + P_n$ is still $\lceil \frac{m}{2} \rceil$ as in the previous proofs. A coprime labeling could be minimized by using the smallest $\lfloor \frac{m}{2} \rfloor + n$ odd integers, resulting the smallest possible minimum coprime number being $m + 2n - 2$ if m is odd and $m + 2n - 1$ if m is even.

Theorems 17, 18, and 19 prove the smallest cases of n, so we let $5 \leq n \leq 10$. We label the vertices of $P_m = \{v_1, \ldots, v_m\}$ using the first n terms of the sequence

$$2, x_1, 4, x_2, 8, x_3, 16, x_4, 32, x_5, 64, x_6, 10, x_7, 20, x_8, 40, x_9, 50, 7, 6, 5, 12, 31, 18, 37, 24, 17, 30, 19, 36, 11, 42, 25, 48, 35, 54, 49, 60, 13, 66, 65, 72, 23, 78, 29, 70, 3, 14, 9, 28, 15, 56, 39, 22, 21, 26, 27, 34, 33, 38, 45, 44, 51, 46, 55, 52, 57, 58, 63, 62, 75, 68, 77, 74, 69, 76, p_{13}, 80, 81, 82, p_{12}, 84, 85, 86, 87, 88, p_{14}, 89, 90, 91, 92, 93, 94, 95, 96, p_{15}, 98, 99, 100, p_{16}, 102, p_{17}, 104, \ldots$$

where $x_1, x_2, \ldots, x_9$ are the 9 smallest odd integers larger than $m+1$. Observe that at most two of the integers in the sequence $x_1, \ldots, x_9$ are divisible by 5. Hence there are at least 4 integers not divisible by 5 that we will use to label the vertices $v_{12}, v_{14}, v_{16}, v_{18}$ to avoid adjacent pairs with a common factor of 5.

If $n < 10$ then the $10 - n$ largest of the labels in $\{x_1, x_2, \ldots, x_9\}$ will be reassigned to use $10 - n$ of the smallest primes in $\{p_{14}, p_{15}, \ldots, p_{22}\}$, and the placement of the primes will shift by $10 - n$ prime positions lower than they currently are in the sequence above. Whether $m + 1$ is even or odd will determine if $m' = \min(\{x_1, \ldots, x_9\})$ is $m + 2$ or $m + 3$. The largest label will be $m' + 2n - 4 = m + 2n - 2$ or $m + 2n - 1$ depending on the parity of m.

By Theorem 16, m is large enough so that $p_{i+n-10} < (14/13)^9 < p_i$ and hence p_i is relatively prime with the labels on either side of $v_{p_{i+n-10}-1}$ for all $i \geq 4$. Let $q_1, q_2, \ldots, q_{n-1}$ be the $n - 1$ largest primes in the sequence $2, 3, 4, \ldots, m + 1$. Then we label the vertices in $V(P_n)$ using the sequence $1, q_1, q_2, \ldots, q_{n-1}$. It is clear that this sequence is a coprime labeling using the desired minimum coprime label which is our claim for $\mathfrak{pr}(P_m + P_n)$. □

By applying similar methods as shown in the proofs of Theorem 19 and 20, the authors believe the following conjecture to be true.

Conjecture 2. For any positive integer N, there exists a positive integer M such that for all $m > M$ and $n \leq N$, the minimum coprime number of $P_m + P_n$ is

$$\mathfrak{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

Consequently, a stronger conjecture may be posed based solely on the size of $m + n$.

Conjecture 3. Let m and n be positive integers such that $m \geq n$. Then the minimum coprime numbers of $P_m + P_n$ is

$$\mathfrak{pr}(P_m + P_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

We now move our discussion to the join of two cycles. We will use our work on the join of two paths to find the following results with relative ease. Notice that by Observation 1, $C_m + C_n$, $C_m + P_n$ are not prime labelings when $m, n \geq 2$.

Theorem 21. *Let m and n be positive integers such that $m \geq n$, $n \leq 10$, and $m > 118$ when $n \geq 5$. Then the minimum coprime number of $C_m + C_n$ is given by*

$$\mathfrak{pr}(C_m + C_n) = \begin{cases} m + 2n & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Consider the vertices of C_m to be $\{v_1, \ldots, v_m\}$ and those of C_n to be $\{u_1, \ldots, u_n\}$. Assume m is even. From the join of paths, two edges $v_1 v_m$ and $u_1 u_n$ are added to form $C_m + C_n$. For the labelings developed in Theorems 17, 18, 19, and 20, the label on v_1 (it is 2 in each case) is relatively prime with the odd integer label on v_m, and the label 1 on u_1 is relatively prime with any integer, so the result follows.

Suppose that m is odd. The labels on C_n still satisfy the coprime property. The labeling sequences from the previous $P_m + P_n$ results all have $m + 1$ as the label of v_m in the case of m being odd, but this label is even and thus is not relatively prime with the label of v_1. We can reassign the label on v_m to the next available odd number which will only increase the coprime labeling number by 2 from $\mathfrak{pr}(P_m + P_n)$, which is necessary since the independence number of C_m is $\frac{m-1}{2}$. □

The following corollary is directly from combining Observation 1 with Theorem 21 and from our reasoning in the previous result about the edge u_1u_n not violating the coprime condition.

Corollary 1. *Let m and n be positive integers such that $m \geq n$, $n \leq 10$, and $m > 118$ when $n \geq 5$. Then we have*

$$\mathfrak{pr}(C_m + P_n) = \begin{cases} m + 2n & \text{if } m \text{ is odd, } m \neq n \\ m + 2n - 1 & \text{if } m \text{ is even,} \end{cases}$$

and

$$\mathfrak{pr}(P_m + C_n) = \begin{cases} m + 2n - 2 & \text{if } m \text{ is odd} \\ m + 2n - 1 & \text{if } m \text{ is even.} \end{cases}$$

6. Concluding Remarks

We conclude by posing several open questions regarding minimum coprime numbers.

Question 1. Can the minimum coprime number be determined for P_n^k and C_n^k for $k \geq 4$?

Question 2. Trees and grid graphs are conjectured to be prime, meaning their minimum coprime number would match their order. Can the bound shown by Salmasian in [10] that $\mathfrak{pr}(T) \leq 4n$ for a tree T of order n be improved, and can a similar upper bound be found for the grid graph $P_m \times P_n$?

Question 3. Berliner et al. [1] investigated the minimum coprime number of $K_{n,n}$, but were not able to determine this number for all n. Does there exist a formula for $\mathfrak{pr}(K_{n,n})$, and can one determine the minimum coprime number more generally for all complete bipartite graphs $K_{m,n}$ in the cases in which there is no prime labeling?

Question 4. Many graphs that are not always prime are left to study in terms of minimum coprime labelings, such as Möbius ladders and $K_{1,n} + K_2$. Can their minimum coprime numbers be determined?

Question 5. Is the following true: If $mn \leq p_{n-1} - n - 1$ then $\mathfrak{pr}(K_n \odot \overline{K}_m) = p_{n-1}$?

Question 6. Is the following true: If $mn > p_{n-1} - n - 1$ then $\mathfrak{pr}(K_n \odot \overline{K}_m) = mn + n + 1$?

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