PARTITIONS RELATED TO POSITIVE DEFINITE BINARY QUADRATIC FORMS

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Abstract

The purpose of this paper is to present a collection of interesting generating functions for partitions which have connections to positive definite binary quadratic forms. In establishing our results we obtain some new Bailey pairs.

1. Introduction and Main Theorems

In a study on lacunary partition functions [7], Lovejoy offered a collection of interesting partition functions which satisfy an estimate given by P. Bernays. Therein, Theorem 1 is constructed using a special Bailey pair which connects q-series to positive definite binary quadratic forms. There appears to be few studies in the literature developing connections between positive definite quadratic forms and partitions through these types of Bailey pairs. By positive definite, we take the usual definition where we write $Q(x, y) = ax^2 + bxy + cy^2$, with $a > 0$, $b^2 - 4ac < 0$, for $a, b, c \in \mathbb{Z}$.

It is interesting to mention that the generating function for representations by binary quadratic forms gives a modular form of weight 1 (see [11]). As a consequence, the right–hand sides of Theorems 1.1–1.3 are all weight one modular forms on some congruence subgroup, and thus so is the q-series on the left side in the same way. Therefore, we provide further examples of modular q-series which are of independent interest. Additionally, we offer a collection of partitions that we believe to be new and interesting, and follow from new Bailey pairs that are similar to the one offered in [7].

We will be applying q-series notation that is widely used throughout the literature [6]. We shall put $(z; q)_n = (z)_n := (1 - z)(1 - zq) \cdots (1 - zq^{n-1})$. It is taken that $q \in \mathbb{C}$, and all of our series converge in the unit circle, $0 < |q| < 1$.

In our first example, we consider a partition function that is related to the $f_1(q)$ studied in [4], but with a different weight function.

Theorem 1. *Let $P_{m,j}(n)$ denote the number of partitions of n into m distinct parts and one part $2m+1$ that may repeat any number of times or not appear at all. Here j is the largest distinct part. Then*

$$\sum_{m,n,j\geq 0}(-1)^j P_{m,j}(n)q^n = \frac{1}{2}\sum_{n\geq 0} q^{n^2+n/2}(1+q^{n+1/2})\sum_{|j|\leq n} q^{j^2/2}$$
$$+\frac{1}{2}\sum_{n\geq 0}(-1)^n q^{n^2+n/2}(1-q^{n+1/2})\sum_{|j|\leq n}(-1)^j q^{j^2/2}.$$

We recall that an overpartition is a partition of n where the first occurrence of a number may be overlined [9].

Theorem 2. *Let $Q_{m,j}(n)$ denote the number of overpartition pairs (μ,λ) of n where μ is an overpartition into even parts $\leq 2m$ with j equal to the number of parts, and λ is a partition into odd parts $\leq 2m+1$ where (i) all odd numbers $< 2m+1$ appear as a part and an even number of times (ii) the part $2m+1$ may repeat any number of times or not appear at all. Then*

$$\sum_{m,n,j\geq 0}(-1)^j Q_{m,j}(n)q^n = \frac{1}{2}\sum_{n\geq 0} q^{3n^2/2}(1+q^{2n+1})(1+q^{n+1/2})\sum_{|j|\leq n} q^{j^2/2}$$
$$+\frac{1}{2}\sum_{n\geq 0}(-1)^n q^{3n^2/2}(1+q^{2n+1})(1-q^{n+1/2})\sum_{|j|\leq n}(-1)^j q^{j^2/2}.$$

Theorem 3. *Let $R_{m,j}(n)$ denote the number of overpartition pairs (μ',π) of n where μ' is an overpartition into even parts $\leq 2m$ with j equal to the number of parts, and π is a partition into odd parts $\leq 2m+1$ where (i) all odd numbers $< 2m+1$ appear as a part and at least once (ii) the part $2m+1$ may repeat any number of times or not appear at all. Then*

$$\sum_{m,j,n\geq 0}(-1)^j R_{m,j}(n)q^n = \frac{1}{2}\sum_{n\geq 0} q^{n^2/2}(1+q^{2n+1})(1+q^{n+1/2})\sum_{|j|\leq n} q^{j^2/2}$$
$$+\frac{1}{2}\sum_{n\geq 0}(-1)^n q^{n^2/2}(1+q^{2n+1})(1-q^{n+1/2})\sum_{|j|\leq n}(-1)^j q^{j^2/2}.$$

2. Proof of Theorems

Here we give the proofs of our theorems, which will require some lemmas from the literature and also some new Bailey pairs. First we note that a pair $(\alpha_n(a;q),\beta_n(a,q))$ is said to be a *Bailey pair* [2, 13] with respect to (a,q) if

$$\beta_n(a,q) = \sum_{0\leq j\leq n}\frac{\alpha_j(a,q)}{(q;q)_{n-j}(aq;q)_{n+j}}. \tag{1}$$

It is known [13] that

$$\sum_{n\geq 0}(X_1)_n(X_2)_n(aq/X_1X_2)^n\beta_n(a,q) \tag{2}$$

$$= \frac{(aq/X_1)_\infty(aq/X_2)_\infty}{(aq)_\infty(aq/X_1X_2)_\infty}\sum_{n\geq 0}\frac{(X_1)_n(X_2)_n(aq/X_1X_2)^n\alpha_n(a,q)}{(aq/X_1)_n(aq/X_2)_n}.$$

We need to mention a result that was established by Lovejoy [8, eq.(2.4)–(2.5)]

Lemma 1 ([8]). *If $(\alpha_n(a;q),\beta_n(a,q))$ is a Bailey pair, then so is $(\alpha_n^*(aq,b,q)$, $\beta_n^*(aq,b,q))$, where*

$$\alpha_n^*(aq,b,q)$$
$$= \frac{(1-aq^{2n+1})(aq/b;q)_n(-b)^nq^{n(n-1)/2}}{(1-aq)(bq)_n}\sum_{n\geq j\geq 0}\frac{(b)_j}{(aq/b)_j}(-b)^{-j}q^{-j(j-1)/2}\alpha_j(a,q), \tag{3}$$

$$\beta_n^*(aq,b,q) = \frac{(1-b)}{1-bq^n}\beta_n(a,q). \tag{4}$$

If we let $a=1$ in Lemma 2.1, divide both sides by $(1-b)$, and add the resulting Bailey pair to itself after replacing b by $-b$, we obtain the next Bailey pair.

Lemma 2. *If $(\alpha_n(1,q),\beta_n(1,q))$ is a Bailey pair, then $(L_{1(n)}(q,b,q),L_{2(n)}(q,b,q))$ is a Bailey pair relative to (q,q), where*

$$L_{1(n)}(q,b,q) = \frac{1}{2(1-b)}\alpha_n^*(q,b,q)+\frac{1}{2(1+b)}\alpha_n^*(q,-b,q), \tag{5}$$

$$L_{2(n)}(q,b,q) = \frac{\beta_n(1,q)}{1-b^2q^{2n}}. \tag{6}$$

We need a result that appeared in [10].

Lemma 3. *If $(\alpha(a,q),\beta(a,q))$ forms a Bailey pair with respect to (a,q), then $(\alpha'_n(a^2,q^2),\beta'_n(a^2,q^2))$ forms a Bailey pair with respect to (a^2,q^2), where*

$$\alpha'_n(a^2,q^2) = \frac{(1+aq^{2n})}{(1+a)q^n}\alpha_n(a,q), \tag{7}$$

$$\beta'_n(a^2,q^2) = \frac{q^{-n}}{(-a;q)_{2n}}\sum_{n\geq j\geq 0}\frac{(-1)^{n-j}q^{(n-j)^2-(n-j)}}{(q^2;q^2)_{n-j}}\beta_j(a,q). \tag{8}$$

We take the E(1) Bailey pair relative to $a=1$ from Slater's list [11], $\alpha_0(1,q)=1$,

$$\alpha_n(1,q) = 2(-1)^nq^{n^2}, \tag{9}$$

when $n > 0$, and

$$\beta_n(1,q) = \frac{1}{(q^2;q^2)_n}, \tag{10}$$

and insert it into Lemma 2.2. We then insert the resulting pair into Lemma 2.3 to obtain our main Bailey pair. To obtain the following lemma, we require use of an identity found in Fine's text [5, pg. 26, eq. (20.41)] (with $a = b/q$, $c = 0$ therein);

$$(t)_\infty \sum_{n\geq 0} \frac{t^n}{(q)_n(1-bq^n)} = \sum_{n\geq 0} \frac{(-t)^n b^n q^{n(n-1)/2}}{(b)_{n+1}}.$$

Lemma 4. *Define*

$$U_n(q,b,q) := \frac{(1-q^{4n+2})(q/b;q)_n(-b)^n q^{n(n-1)/2}}{q^n(1-b)(1-q^2)(bq)_n}\left(1+2\sum_{n\geq j>0}\frac{(b)_j}{(q/b)_j}(b)^{-j}q^{j(j+1)/2}\right).$$

Then $(U_{1(n)}(q^2,b,q^2), U_{2(n)}(q^2,b,q^2))$ *forms a Bailey pair relative to* (q^2,q^2), *where* $U_{1(n)}(q^2,b,q^2) = \frac{1}{2}U_n(q,b,q) + \frac{1}{2}U_n(q,-b,q)$ *and*

$$U_{2(n)}(q^2,b,q^2) = \frac{(-b^2)^n q^{n(n-2)}}{(-q)_{2n}(b^2;q^2)_{n+1}}.$$

Proof of Theorem 1. We take the Bailey pair that results from inserting (2.9)–(2.10) into Lemma 2.2 with $b = q^{1/2}$. Noting $\lim_{X_2\to\infty}(X_2)_n/X_2^n = (-1)^n q^{n(n-1)/2}$, we apply (2.2) with $X_1 = q$ and let $X_2 \to \infty$ to obtain

$$\sum_{n\geq 0}\frac{(-1)^n q^{n(n+1)/2}}{(-q)_n(1-q^{2n+1})} = \frac{1}{2}\sum_{n\geq 0} q^{n^2+n/2}(1+q^{n+1/2})\sum_{|j|\leq n} q^{j^2/2} \tag{11}$$

$$+\frac{1}{2}\sum_{n\geq 0}(-1)^n q^{n^2+n/2}(1-q^{n+1/2})\sum_{|j|\leq n}(-1)^j q^{j^2/2}.$$

It is clear to see that $(-1)^m q^{m(m+1)/2}/(-q)_m$ generates partitions of n into m distinct parts weighted by -1 raised to the largest part. The component $(1-q^{2m+1})^{-1}$ generates a partition of the part $2m+1$ which may repeat or not appear at all. Combining the two gives the generating function for the partition described in the theorem. □

Proof of Theorem 2. We take the Bailey pair in Lemma 2.4 with $b = q^{1/2}$ and then insert it into (2.2) with $X_1 = q^2$, $X_2 \to \infty$, to get

$$\sum_{n\geq 0}\frac{(q^2;q^2)_n q^{2n^2}}{(-q^2;q^2)_n(q^2;q^4)_n(1-q^{2n+1})} = \frac{1}{2}\sum_{n\geq 0} q^{3n^2/2}(1+q^{2n+1})(1+q^{n+1/2})\sum_{|j|\leq n} q^{j^2/2} \tag{12}$$

$$+\frac{1}{2}\sum_{n\geq 0}(-1)^n q^{3n^2/2}(1+q^{2n+1})(1-q^{n+1/2})\sum_{|j|\leq n}(-1)^j q^{j^2/2}.$$

Now $(q^2;q^2)_m/(-q^2;q^2)_m$ generates an overpartition of n into even parts $\leq 2m$ weighted by -1 raised to the number of parts. The function $q^{m^2}/(q^2;q^4)_m(1-q^{2m+1})$ generates the partition λ given in the theorem. To see this, we write

$$\frac{q^{2n^2}}{(q^2;q^4)_n(1-q^{2n+1})}=\frac{q^{1+1+3+3+\cdots(2n-1)+(2n-1)}}{(1-q^{1+1})(1-q^{3+3})\cdots(1-q^{2n-1+2n-1})(1-q^{2n+1})}.$$

□

Proof of Theorem 3. We take the Bailey pair in Lemma 2.4 with $b=q^{1/2}$ and then insert it into (2.2) with $X_1=q^2$, $X_2=-q$, and rewrite to get

$$\sum_{n\geq 0}\frac{(q^2;q^2)_n q^{n^2}}{(-q^2;q^2)_n(q;q^2)_{n+1}}=\frac{1}{2}\sum_{n\geq 0}q^{n^2/2}(1+q^{2n+1})(1+q^{n+1/2})\sum_{|j|\leq n}q^{j^2/2} \qquad (13)$$

$$+\frac{1}{2}\sum_{n\geq 0}(-1)^n q^{n^2/2}(1+q^{2n+1})(1-q^{n+1/2})\sum_{|j|\leq n}(-1)^j q^{j^2/2}.$$

The partition generating function on the left side is quite similar to our previous theorem, and so the remaining details are left to the reader. □

Here we can observe that more partition functions may be obtained by instead selecting different Bailey pairs from Slater's list [12] in conjunction with Lemma 2.1. This principal idea is of course aided with the application of the work [3] to obtain simple forms of $\beta_n(a,q)$ in the same way we have done here with Lemma 2.3.

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