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CYCLES IN THE COPRIME HYPERGRAPH OF INTEGERS

Jan-Hendrik de Wiljes
Institute of Mathematics and Applied Computer Science, University of Hildesheim, Hildesheim, Germany
wiljes@uni-hildesheim.de

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Abstract

In this paper we investigate the existence of different kinds of cycles in the coprime hypergraph of integers CHI_k and its induced subgraphs. We obtain conditions for the existence of cycles as well as examples which show that the results cannot be improved. We will also briefly discuss different forms of acyclicity of coprime hypergraphs. Our results generalize similar results of Erdős and Sárközy on cycles in the coprime graph of integers CHI_1, although not in the sense that they imply their findings (since we require $k \geq 2$).

1. Introduction

1.1. Definitions and Notations

We start by explaining the necessary notation. Let $[n]$ denote the set of the first n positive integers; similarly we define $[m,n] := [n] \setminus [m-1]$ for positive integers m,n with $n > m$. The number of integers from $[n]$, which have a divisor among the first j primes, is denoted by $f(n,j)$. For $A \subseteq \mathbb{N}$ and non-negative integers s,t, we define $A_{(s,t)} := \{a \in A : a \equiv t \mod s\}$. Further, $(a_1, a_2, \ldots, a_j)$ denotes the *greatest common divisor* of $a_1, a_2, \ldots, a_j \in \mathbb{Z}$. A set of size l is called an l-*set*, and $\mathcal{P}(A)$ denotes the *power set* of the set A. Finally, A^l denotes the set and $\binom{|A|}{l}$ the number of l-subsets of the finite set A.

A *hypergraph* H is a pair (V,E) consisting of a set $V \neq \emptyset$ of *vertices* and a set E of subsets of V which are called *(hyper)edges*. If all the elements of E have the same cardinality k, the hypergraph is called k-*uniform*. In this case we also call H a k-*graph*, in particular a 2-graph is just a *graph*. We can associate a graph $G(H)$ to H, its so called *primal graph*, which has vertex set V and contains the edge $\{u,v\}$ if and only if there exists some edge in E that contains both, u and v.

A *clique* in a hypergraph $H = (V,E)$ is a subset W of V such that, for any two elements u,v of W, there exists some edge in E that contains both, u and v. We call H *chordal* if it is *conformal*, i. e., every clique of H is contained in an edge of

H, and if $G(H)$ is *chordal*, i. e., $G(H)$ does not contain induced (normal graph) cycles of length greater than 3 (see [1, 2]).

Different types of cycles for hypergraphs have been introduced in the past. We will restrict ourselves to the following concepts (exemplary literature where those concepts can be found is given after every definition).

A *Berge cycle* of length l in a hypergraph G is a sequence $v_1, e_1, v_2, e_2, \dots, v_l, e_l$ of distinct vertices $v_1, v_2, \dots, v_l$ and distinct edges $e_1, e_2, \dots, e_l$ of G, where $v_1, v_l \in e_l$ and $v_i, v_{i+1} \in e_i$ for every $i \in [l-1]$ (see [2]).

A k-graph $C = (V, E)$ is a *t-cycle* if the elements of V can be ordered, say $v_1, v_2, \dots, v_n$, such that E consists of

$$\{v_1, v_2, \dots, v_k\}, \{v_{k-t+1}, v_{k-t+2}, \dots, v_{2k-t}\}, \dots,$$
$$\dots, \{v_{n-k+t+1}, v_{n-k+t+2}, \dots, v_n, v_1, v_2, \dots, v_t\},$$

i. e., consecutive edges intersect in precisely t vertices.[1] The number of edges of C is called its *length*. We say that a hypergraph contains a t-cycle of length r if it has a subhypergraph which is a t-cycle of length r (see [12, 13, 14]). Only the cases $t = 1$ and $t = k - 1$, which are also known as *loose* and *tight cycles* respectively, will be discussed in detail in this article.

The following cycle type is quite different from those considered above, but we find it interesting to study. It is motivated by a definition of a graph cycle using matroids (see [2]).

A nonempty set $S \subseteq \mathcal{P}(B^k)$, where B is a finite set, is called *dependent* if every set $A \subseteq B^{k-1}$ is a subset of an even number of elements from S, and it is called *reduced* if no proper nonempty subset of S is dependent. For a k-graph H, we call a nonempty l-set C of edges, which is dependent and reduced, a *matroid cycle* of H of length l (see [8]).

1.2. The Coprime Hypergraph of Integers

The object of interest, as introduced in [4], is the $(k+1)$-uniform *coprime hypergraph of integers CHI_k*, which has vertex set $\mathbb{Z}$ and $k+1$ elements of $\mathbb{Z}$ form a hyperedge if their greatest common divisor equals 1. Let A be a set of positive integers. The coprime hypergraph of A, denoted by $CHI_k(A)$, is the induced subgraph in CHI_k by A. The set of edges of $CHI_k(A)$ will be denoted by $E_k(A)$.

In [4, 5, 6] the maximal size of complete subgraphs of $CHI_k([n])$ was investigated. We will consider another natural question, namely the existence of cycles in CHI_k and its induced subgraphs. For $CHI_1(A)$, the existence of cycles depends on the size of $A \subseteq [n]$ compared to $f(n, 1)$ for even cycles and to $f(n, 2)$ for odd cycles,

[1] To avoid unnecessarily long notation we sometimes say that $v_1, v_2, \dots, v_n$ yields or induces a t-cycle.

see [10]. It turns out that for $k \geq 2$ the existence of (certain types of) cycles in $CHI_k(A)$ only depends on $f(n,1)$.

It should be mentioned that our results generalize the ones from [10] to coprime hypergraphs, but not in the sense that they imply the results from [10] (since they only hold for $k \geq 2$).

The following lemma from [9] will be used extensively.

Lemma ESS. (Erdős, Sárközy & Szemerédi 1980). *There exists some constant n_0 such that if $n \geq n_0$, $A \subseteq [n]$ with $|A| > f(n,1)$, and $r = \min\{|A_{(2,1)}|, \lfloor \frac{1}{10} \log\log n \rfloor\}$, then $CHI_1(A)$ contains a complete bipartite graph on $2r$ vertices. In particular, for every $l \in [r]$ we find elements $v_1, v_2, \ldots, v_{2l}$ in A with $(v_i, v_{i+1}) = (v_1, v_{2l}) = 1$ for all $i \in [2l-1]$.*

2. Different Types of Cycles in Coprime Hypergraphs

2.1. Berge Cycles

By filling up edges of $CHI_1([n])$ with arbitrary elements we get $(k+1)$-sets, which are edges of $CHI_k([n])$. Thus, we can certainly use the results from [10] to find Berge cycles. More precisely, Lemma ESS gives us Berge cycles of length l in $CHI_k([n])$ for every $l \in [3, 2r]$ and $n > k \geq 2$. Although certainly there are examples with longer Berge cycles for $k = 2$ (hence improved results might be possible), we could not find a general way to do better in this case. For $k \geq 3$ on the other hand a much stronger result is possible.

Theorem 1. *Let k, n be positive integers with $f(n,1) \geq k \geq 3$ and $A \subseteq [n]$. If $|A| > \max\{f(n,1), k+1\}$ then $CHI_k(A)$ contains a Berge cycle of length l for every $l \in \{2, 3, \ldots, |A|-2\}$.*

Proof. Using the pigeonhole principle, we find two elements $a, b \in A$ with $(a,b) = 1$. If $l = 2$, we pick two distinct $(k-1)$-subsets S_1, S_2 of $A \setminus \{a,b\}$. Then

$$a, S_1 \cup \{a,b\}, b, S_2 \cup \{a,b\}$$

is the desired Berge cycle of length 2.

So, from now on we require $l \geq 3$. We have to distinguish two cases. If $l \leq k$, we pick k elements $v_1, v_2, \ldots, v_k$ from $A \setminus \{a,b\}$ and define $e_i := (\{a,b\} \cup \{v_j : j \in [k]\}) \setminus \{v_i\}$ for every $i \in [l]$. Then

$$v_1, e_3, v_2, e_4, \ldots, v_{l-2}, e_l, v_{l-1}, e_1, v_l, e_2$$

is the desired Berge cycle of length l.

If $l > k$, we pick l elements $v_1, v_2, \ldots, v_l$ from $A \setminus \{a,b\}$ and define

$$e_i := \{v_i, v_{i+1} \ldots, v_{k+i-2}\} \cup \{a,b\}$$

for $i \in [l]$, where $v_{l+j} = v_j$ for every $j \in \mathbb{N}$. Then $v_1, e_1, v_2, e_2, \ldots, v_l, e_l$ is the desired Berge cycle of length l. $\square$

It is not possible to replace $f(n, 1)$ in the statement of the theorem by a smaller number, since $CHI_k([n]_{(2,0)})$ contains no Berge cycles.

Remark. Different notions for acyclicity of hypergraphs, apart from not containing a Berge cycle, have been introduced in the past (see for example [11]), the weakest[2] one being *α-acyclicity*, which is equivalent to being chordal [1]. One might ask if there exists a set $A \subseteq [n]$ with $|A| > f(n, 1)$ such that $CHI_k(A)$ is α-acyclic. This is not the case if $|A| > k + 1 \geq 4$, since the vertices of the Berge cycle of length $|A| - 2$ given in Theorem 1, together with a and b, obviously form a clique of size at least $k + 2$. Therefore, $CHI_k(A)$ is not conformal.

2.2. Loose Cycles

We can easily see that given $A = [n]_{(2,0)} \cup ([2(r-1)+1]_{(2,1)} \setminus \{1\})$ (a similar set was used in [10]), the vertex set of a loose cycle in $CHI_k(A)$ of length $2r - 1$ has to contain at least r odd integers (one for every two edges), which cannot be the case. This gives us the lower bound $f(n, 1) + r$ for the cardinality of A which guarantees a loose cycle of given length $2r - 1$ (as well as $2r$) in $CHI_k(A)$.

Using results from [9] and [10], one can show that $|A| > f(n, 2)$ guarantees the existence of loose cycles of almost every length in $CHI_k(A)$ for $k \geq 2$ (just fill up the missing vertices in the hyperedges by arbitrarily picked numbers, as will be done in the proof of Theorem 2).

For $k \geq 2$ we can even prove a better bound which matches the lower bound from above (therefore in that sense the following theorem cannot be improved), more precisely we have:

Theorem 2. *Let $k \geq 2$ be an integer. There exists some constant n_k such that if $n \geq n_k$, $A \subseteq [n]$, and $|A| \geq f(n, 1) + r$, where $r = \min\{|A_{(2,1)}|, \lfloor \frac{1}{10} \log \log n \rfloor\}$, then $CHI_k(A)$ contains a loose cycle of length $2l$ and a loose cycle of length $2l - 1$ for every l with $2 \leq l \leq r$.*

Proof. We choose $n_k \geq n_0$ (with n_0 from Lemma ESS) such that $f(n, 1) \geq (2k - 1)\lfloor \frac{1}{10} \log \log n \rfloor$ for every $n \geq n_k$. Then, for every l with $2 \leq l \leq r$, we find $v_1, v_2, \ldots, v_{2l} \in A$ with $(v_i, v_{i+1}) = (v_1, v_{2l}) = 1$ for all $i \in [2l - 1]$ using Lemma ESS. Further, since the choices of A and n_k guarantee

$$|A| \geq f(n, 1) + r \geq (2k - 1) \left\lfloor \frac{1}{10} \log \log n \right\rfloor + r \geq 2rk \geq 2lk,$$

[2]Weakest in the sense that it is implied by Berge acyclicity.

we can pick arbitrary $2l(k-1)$ elements from $A \setminus \{v_1, v_2, \dots, v_{2l}\}$ (and put them in place of the stars below) to obtain

$$v_1, \underbrace{*, \dots, *}_{k-1 \text{ elements}}, v_2, \underbrace{*, \dots, *}_{k-1 \text{ elements}}, v_3, *, \dots, \dots, *, v_{2l-1}, \underbrace{*, \dots, *}_{k-1 \text{ elements}}, v_{2l}, \underbrace{*, \dots, *}_{k-1 \text{ elements}}$$

and

$$v_1, \underbrace{*, \dots, *}_{k-1 \text{ elements}}, v_2, \underbrace{*, \dots, *}_{k-1 \text{ elements}}, v_3, *, \dots,$$
$$\dots, *, v_{2l-2}, \underbrace{*, \dots, *}_{k-1 \text{ elements}}, v_{2l-1}, \underbrace{*, \dots, *}_{k-2 \text{ elements}}, v_{2l},$$

while using only $2(l-1)(k-1)-1$ of the arbitrarily picked elements in the second case. These sequences yield loose cycles in $CHI_k(A)$ of the desired lengths. □

We can do better for $l = 2$:

Proposition 1. *Let k, n be positive integers with $f(n,1) \geq k \geq 2$ and $A \subseteq [n]$ with $|A| \geq \max\{2k, f(n,1)+1\}$. Then $CHI_k(A)$ contains a loose cycle of length 2.*

Proof. As in the proof of Theorem 1, we find $a, b \in A$ with $(a,b) = 1$. Pick $2k-2$ elements $v_1, v_2, \dots, v_{2k-2}$ from $A \setminus \{a, b\}$. Then

$$a, v_1, v_2, \dots, v_{k-1}, b, v_k, v_{k+1}, \dots, v_{2k-2}$$

induces a loose cycle of length 2 in $CHI_k(A)$. □

2.3. Tight Cycles

Since $\{2,3,4,8,9\}$ is the vertex set of a tight cycle of length 5 in $CHI_2([n])$ for $n \geq 9$, the set $[n]_{(2,0)} \cup [n]_{(3,0)}$ from [10] does not pose problems for the existence of odd cycles if $k \geq 2$. But using the set $A = [n]_{(2,0)} \cup ([2\lceil \frac{r}{k+1} \rceil]_{(2,1)} \setminus \{1\})$ instead, we can see that, since a tight cycle of length l in $CHI_k(A)$ has to contain at least $\lceil \frac{l}{k+1} \rceil$ odd integers, $CHI_k(A)$ cannot contain any tight cycle of length l for $l \geq r$. This gives us the lower bound $f(n,1) + \lceil \frac{r}{k+1} \rceil$ for the cardinality of A which guarantees tight cycles of given length r in $CHI_k(A)$.

Any (normal graph) cycle of length l in $CHI_1(A)$ certainly induces a tight cycle of length l in $CHI_k(A)$ for every $k < l$. Applying the results from [10] would therefore guarantee the existence of tight cycles of almost any length in $CHI_k(A)$ for a set $A \subseteq [n]$ with $|A| > f(n,2)$ (for sufficiently large n).

As in the loose cycle case, the lower bound from above is sufficient to guarantee the existence of tight cycles (under certain additional conditions). A similar argument as in the proof of Theorem 2 yields:

Theorem 3. *Let $k \geq 2$ be an integer. There exists some constant n_k such that if $n \geq n_k \geq k+3$, $A \subseteq [n]$, $|A| \geq f(n,1) + \lceil \frac{r}{k+1} \rceil$, where*

$$\lceil \frac{r}{k+1} \rceil = \min\{|A_{(2,1)}|, \lfloor \frac{1}{10} \log\log n \rfloor\},$$

then $CHI_k(A)$ contains a tight cycle of length l for every integer l with $k+1 < l \leq r$.

Proof. We choose $n_k \geq n_0$ (with n_0 from Lemma ESS) such that $f(n,1) \geq k\lfloor \frac{1}{10} \log\log n \rfloor$ for every $n \geq n_k$. Then we define $l' := \lceil \frac{l}{k+1} \rceil$ and $x := l'(k+1) - l \in [0,k]$. We find $v_1, v_2, \ldots, v_{2l'} \in A$ with $(v_i, v_{i+1}) = (v_1, v_{2l'}) = 1$ for all $i \in [2l'-1]$ using Lemma ESS. Further, since the choices of A and n_k guarantee

$$|A| \geq f(n,1) + \left\lceil \frac{r}{k+1} \right\rceil \geq k \left\lfloor \frac{1}{10} \log\log n \right\rfloor + \left\lceil \frac{r}{k+1} \right\rceil \geq \left\lceil \frac{r}{k+1} \right\rceil (k+1) \geq r \geq l,$$

we can pick arbitrary $l - 2l' = (k-1)l' - x$ elements from $A \setminus \{v_1, v_2, \ldots, v_{2l'}\}$ (and put them in place of the stars below) to obtain

$$v_1, v_2, \underbrace{*, \ldots, *}_{k-1 \text{ elements}}, v_3, v_4, \underbrace{*, \ldots, *}_{k-1 \text{ elements}}, v_5, v_6, *, \ldots, \ldots, *, v_{2l'-1}, v_{2l'}, \underbrace{*, \ldots, *}_{(k-1)-x \text{ elements}}.$$

This sequence yields a tight cycle of length l in $CHI_k(A)$ if $x \in [0, k-1]$. For $x = k$ we need to consider

$$v_1, v_2, \underbrace{*, \ldots, *}_{k-1 \text{ elements}}, v_3, v_4, \underbrace{*, \ldots, *}_{k-1 \text{ elements}}, v_5, v_6, *, \ldots,$$
$$\ldots, *, v_{2l'-3}, v_{2l'-2}, \underbrace{*, \ldots, *}_{k-2 \text{ elements}}, v_{2l'-1}, v_{2l'}.$$

This sequence induces a tight cycle of length l in $CHI_k(A)$. $\square$

Remark. It is possible to extend Theorems 2 and 3 to the existence of t-cycles with $1 < t < k$ of length r in $CHI_k(A)$ depending on the size of A. Since the proof of an analogue statement is rather messy, we will just sketch the idea for finding a cycle for fixed t and r. The vertex set of a t-cycle of length r in $CHI_k(A)$ contains precisely $r(k+1-t)$ elements. Therefore, we require $|A| \geq r(k+1-t)$. Further, we need r coprime elements (Lemma ESS). Then, using the idea from Theorem 2, the sequence

$$v_1, \underbrace{*, \ldots, *}_{k-t \text{ elements}}, v_2, \underbrace{*, \ldots, *}_{k-t \text{ elements}}, v_3, *, \ldots, \ldots, *, v_{r-1}, \underbrace{*, \ldots, *}_{k-t \text{ elements}}, v_r, \underbrace{*, \ldots, *}_{k-t \text{ elements}}$$

induces a t-cycle of length r if $1 < t \leq \frac{k+1}{2}$. If $k > t > \frac{k+1}{2}$ this idea would use unnecessarily many elements v_i per edge, therefore we use the idea of Theorem 3 and just push v_{2i} and v_{2i+1} apart. More precisely, the sequence

$$v_1, \underbrace{*,\ldots,*}_{t \text{ elements}}, v_2, \underbrace{*,\ldots,*}_{k-t \text{ elements}}, v_3, \underbrace{*,\ldots,*}_{t \text{ elements}}, v_4, \underbrace{*,\ldots,*}_{k-t \text{ elements}}, v_5, *, \ldots,$$
$$\ldots, *, v_{2j-1}, \underbrace{*,\ldots,*}_{t \text{ elements}}, v_{2j}, \underbrace{*,\ldots,*}_{k-t \text{ elements}}$$

induces a t-cycle of length $j \cdot \lceil \frac{k+1}{k+1-t} \rceil$.[3] It should be mentioned that depending on k, r and t and for a given number of v_i (analogous to Theorems 2 and 3), the number of elements between v_{2j-1} and v_{2j} might be different (but still in $[t]$) to obtain a cycle (this is the messy part).

2.4. Matroid Cycles

We start with the construction of matroid cycles for the graph $CHI_k([n])$.

Proposition 2. *Let n, k be positive integers. Then $CHI_k([n])$ does not contain a matroid cycle of length l in the following cases:*

(i) $n \leq k+1$,

(ii) $n > k+1, l \leq k+1$,

(iii) $n > k+1, 2 \nmid (k+1)l$.

Proof. Let C be a subset of $E_k([n])$ of size l. Since for $n \leq k+1$ we have $l \leq 1$, there cannot exist a matroid cycle, therefore (i) is true.

For instance, from [3] we know that circuits of $(k+1)$-simplicial matroids (which correspond to matroid cycles) have lengths greater than $k+1$, implying (ii).

If C is a matroid cycle, every k-subset of some $c \in C$ has to occur an even number of times as a subset of the elements of C. This cannot be the case if in total there are an odd number, namely $(k+1)l$, of subsets (counted with multiplicities). Therefore, there cannot exist any odd matroid cycle in $CHI_k(A)$ if $2 \nmid (k+1)l$, which proves (iii). □

Theorem 4. *Let n, k be positive integers with $n > k+1 \geq 3$. Then $CHI_k([n])$ contains a matroid cycle of length l if $l \in [k+2, (n-k-1)k+2]_{(k,2)}$.*

Proof. The set $C_1 = [k+2]^{k+1}$ has exactly $k+2$ elements and every element of $[k+2]^k$ is a subset of an even number of elements of C_1. Therefore, the elements of

[3] The set of edges can be partitioned by the property of containing an element v_{2i-1} and for each such vertex there exist $\lceil \frac{k+1}{k+1-t} \rceil$ corresponding edges.

C_1, which are all edges of $CHI_k([n])$ since each contains two consecutive integers, form a matroid cycle of length $k+2$. Now we need to verify that the set

$$C_i = \underbrace{(C_{i-1} \setminus [i, k+i])}_{A} \cup \underbrace{\left\{D \cup \{k+i+1\} : D \in [i, k+i]^k\right\}}_{B}$$

of size $ik+2$ is a matroid cycle in $CHI_k([n])$ if C_{i-1} is one. First of all, since each element of B contains two consecutive integers, we have $C_i \subseteq E_k([n])$.

Let $S \subseteq [k+i+1]^k$. We consider the following four cases. If $S \subseteq [i, k+i]$, then S lies in an odd number of elements of A since C_{i-1} is dependent and exactly in one element of B, namely $S \cup \{k+i+1\}$. If $[i-1] \cap S \neq \emptyset$ and $k+i+1 \in S$, then S is not a subset of any of the elements of C_i. If $[i-1] \cap S \neq \emptyset$ and $k+i+1 \notin S$, then $S \cap B' = \emptyset$ for every $B' \in B$, and S lies in an even number of elements from A since C_{i-1} is dependent. Lastly, if $[i-1] \cap S = \emptyset$ and $k+i+1 \in S$, we have $S \cap A' = \emptyset$ for every $A' \in A$, and S lies in exactly two elements of B (since $|[i, k+i+1] \setminus S| = 2$). Therefore, C_i is dependent.

Suppose, C_i was not reduced and S would be a proper subset of C_i, which is dependent. Note that $S \subseteq B$ is impossible. If $B \not\subseteq S$, then $B \cap S = \emptyset$, since the subsets of the form $D \cup \{k+i+1\}$ with $D \in [i, k+i]^{k-1}$ occur only in elements of B (each exactly twice). But then $S \subsetneq C_{i-1}$ contradicts the fact that C_{i-1} is reduced. Therefore, we must have $B \subseteq S$ and $A \not\subseteq S$. But then $(S \setminus B) \cup [i, k+i]$ is a dependent proper subset of C_{i-1} in contradiction of C_{i-1} being a matroid cycle. $\square$

Remark. Certainly any other element of the existing cycle might be substituted in the given way (using the subsets of this element) to get a new cycle, provided these subsets together with the new element $k+i$ form an edge of $CHI_k([n])$. Therefore, if one finds a matroid cycle C of length l with $(i-1)k+2 < l < ik+2$, the method used in the proof of Theorem 4 can be applied to get a cycle of length $l+yk$, where y is at most the number of elements of $[n]$ which are not used in the elements of C.

In case we only want to know if, for a given l, there exists some n such that $CHI_k([n])$ contains a matroid cycle of length l, the problem becomes purely combinatorial since we can use arbitrarily many primes to solve the only obstacle of the sets being edges.

For arbitrary A, the construction of matroid cycles in $CHI_k(A)$ can be done similarly, although A has to contain suitable elements (such as $k+2$ consecutive integers). It does not seem to be easy to obtain the possible length of a matroid cycle given only the size of A (although certainly the ideas in the proof of Theorem 4 should be helpful). This should be part of future research. Therefore, we settle for sufficient conditions on $|A|$ and $|E_k(A)|$ which guarantee a matroid cycle.

Proposition 3. *Let n, k be positive integers and $A \subseteq [n]$.*

(i) If $|A| \geq \max\{f(n,1)+2, k+2\}$ then $CHI_k(A)$ contains a matroid cycle.

(ii) *The hypergraph* $CHI_k([n]_{(2,0)} \cup \{a\})$*, where* $a \in [n]_{(2,1)}$*, does not contain a matroid cycle. In particular this means that* $|A| > f(n,1)$ *does not imply the existence of matroid cycles in* $CHI_k(A)$*.*

(iii) *If* $|E_k(A)| > \binom{|A|}{k}$ *then* $CHI_k(A)$ *contains a matroid cycle.*

Proof. As in the proof of Theorem 1 we find four distinct integers a, b, c, d in A with $(a, b) = (c, d) = 1$. Let S be a $(k + 2)$-subset of A containing a, b, c and d. Then S^{k+1} is a matroid cycle of $CHI_k(A)$ (using the same argument as in the proof of Theorem 4), which settles (i).

All the edges of $CHI_k([n]_{(2,0)} \cup \{a\})$ necessarily contain a. Let e be such an edge. Since $e \setminus \{a\}$ cannot be a subset of any element from $E_k([n]_{(2,0)} \cup \{a\})$, the edge e cannot lie in a matroid cycle of $CHI_k([n]_{(2,0)} \cup \{a\})$. This settles (ii).

We fix an ordering $S_1, S_2, \ldots, S_t$, where $t = \binom{|A|}{k}$, of the k-subsets of A and let $f(e)$ be the characteristic vector in $\mathbb{F}_2^t$ of an edge e with respect to the given ordering, meaning it has entry 1 at coordinate i if $S_i \subseteq e$ and entry 0 everywhere else.[4] From linear algebra we know that $\{f(e) : e \in E_k(A)\}$ is linearly dependent over $\mathbb{F}_2$ since $|E_k(A)| > t$. This means that $E_k(A)$ is dependent and, therefore, contains a matroid cycle. □

3. Future Work

We constructed different types of cycles in $CHI_k(A)$ for different sets $A \subseteq [n]$. Other sets should be investigated and the question of the existence of longer cycles in the considered cases should be addressed. Further, it would be natural to try to count the number of cycles of given length.

Since two consecutive numbers are coprime, the graph $CHI_k([n])$ contains a Hamiltonian cycle (cycle containing every vertex of the graph) of every of the first three discussed types. Therefore, it is natural to ask similar extremal questions, namely what size of $A \subseteq [n]$ guarantees the existence of a Hamiltonian cycle in $CHI_k(A)$. Certainly $|A| > f(n, 1)$ is necessary for the existence of loose and tight Hamiltonian cycles, but for small k and large n that is not a sufficient condition: $CHI_2([n]_{(2,0)} \cup \{1\})$ neither contains a loose nor a tight Hamiltonian cycle if $n \geq 10$.

There are other types of cycles in hypergraphs whose existence in $CHI_k(A)$ might be investigated. Examples are: (i) k-cycles in [8], which are derived from a generalization of characterizing graph cycles via cut edges and 2-colourings; (ii) t-tight Berge cycles in [7], which are sequences of vertices and edges like Berge cycles but with the condition that $\{v_i, v_{i+1}, \ldots, v_{i+t-1}\} \subseteq e_i$ holds for $i \in [l]$ (indices mod l).

[4] Actually this way the underlying matroid is defined (see for example [8]).

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