SOME EXPLICIT FORMULAS FOR EULER-GENOCCHI POLYNOMIALS

H. Belbachir
Department of Mathematics, USTHB, RECITS Laboratory, Algiers, Algeria
hbelbachir@usthb.dz

S. Hadj-Brahim
Department of Mathematics, USTHB, RECITS Laboratory, Algiers, Algeria
bhadj@usthb.com

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Abstract

As an extension to both Euler and Genocchi polynomials, we study the Euler-Genocchi family of polynomials introduced recently via its generating function. We give some combinatorial properties and expressions in terms of alternate power sums and related sums for the Stirling numbers of the second kind. Moreover, we give a formula for the Jacobi-Stirling numbers in terms of the Euler-Genocchi family of polynomials using the determinantal approach.

1. Introduction

The falling factorial is defined as

$$(n)_r := \begin{cases} 1, & r = 0; \\ n(n-1)\cdots(n-r+1), & r \in \mathbb{N}. \end{cases}$$

The Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$ admit, respectively, the following generating functions (see [17]):

$$\begin{aligned} \frac{t}{e^t-1}e^{xt} &= \sum_{n\geq 0} B_n(x)\frac{t^n}{n!} \qquad (|t| < 2\pi), \\ \frac{2}{e^t+1}e^{xt} &= \sum_{n\geq 0} E_n(x)\frac{t^n}{n!} \qquad (|t| < \pi), \\ \frac{2t}{e^t+1}e^{xt} &= \sum_{n\geq 0} G_n(x)\frac{t^n}{n!} \qquad (|t| < \pi). \end{aligned} \tag{1}$$

The Bernoulli numbers B_n, the Euler numbers E_n and the Genocchi numbers G_n are given by $B_n := B_n(0)$, $E_n := E_n(0)$ and $G_n := G_n(0)$, (see [9, 13]).
Several approaches have been developed to deal with Appell polynomials. Together with the classical generating function methods, we mention the umbral calculus approach (see, for instance, Roman [17], Tempesta [19], and the references therein), and the determinantal approach introduced by Costabile and Longo [8]. Recently, Merca [14] expressed a convolution alterning formula for the Jacobi-Stirling numbers of the first kind $J_\gamma^s(n,k)$ and the second kind $J_\gamma^S(n,k)$ in terms of the Bernoulli polynomials. For other properties, see [2, 3, 11, 12].

In Section 2, we study some new properties of the Euler-Genocchi polynomials introduced in [6], such as the expression of the power of a variable, the Raabe-like formula, the linear recurrence and the difference equations. In Section 3, we evaluate the alternate power sums in terms of the Euler-Genocchi family of polynomials. In Section 4, related sums of the Stirling numbers of second kind with the Euler-Genocchi family of polynomials are given. In Section 5, we present a determinantal approach for the Euler-Genocchi polynomials. Finally, an extension in a determinantal representation for the Jacobi-Stirling numbers in terms of both Bernoulli and Euler-Genocchi polynomials is studied.

2. The Euler-Genocchi Polynomials

For $r \in \mathbb{N} \cup \{0\}$, Belbachir et al. [6] introduced the class of Euler-Genocchi polynomials $\{A_n^{(r)}(x)\}_{n\geq 0}$ defined by the generating function

$$\frac{2t^r}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} A_n^{(r)}(x)\frac{t^n}{n!} \qquad (|t|<\pi), \tag{2}$$

with $A_j^{(r)}(x) = 0$ for $j < r$. We denote by $A_n^{(r)} := A_n^{(r)}(0)$, the Euler-Genocchi numbers of order r. The Euler and Genocchi polynomials are $E_n(x) = A_n^{(0)}(x)$ and $G_n(x) = A_n^{(1)}(x)$ respectively.
In [5], the author established the ordinary generating function for this class of polynomials. From (2), it follows that

$$\sum_{n=0}^{\infty} A_n^{(r)}(x+1)\frac{t^n}{n!} = \frac{2t^r}{e^t+1}e^{xt}e^t = \left(\sum_{n=0}^{\infty} A_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}\right),$$

and using Cauchy's rule, we get the formula

$$A_n^{(r)}(x+1) = \sum_{k=0}^{n}\binom{n}{k}A_k^{(r)}(x). \tag{3}$$

On the other hand, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(A_n^{(r)}(x+1)+A_n^{(r)}(x)\right)\frac{t^n}{n!} &= \frac{2t^r}{e^t+1}e^{(x+1)t}+\frac{2t^r}{e^t+1}e^{xt}\\ &= 2t^r e^{xt}\\ &= \sum_{n=0}^{\infty} 2x^n\frac{t^{n+r}}{n!}\\ &= \sum_{n=0}^{\infty} 2(n)_r x^{n-r}\frac{t^n}{n!}.\end{aligned}$$

Equating the two series and using (3), we obtain the following theorem.

Theorem 2.1. *Let n and r be two non-negative integers such that $r \leq n$. The following formula holds:*

$$x^n = \frac{1}{2(n+r)_r}\left[A_{n+r}^{(r)}(x)+\sum_{k=0}^{n}\binom{n+r}{k+r}A_{k+r}^{(r)}(x)\right].$$

Corollary 2.2. [15, 18] *We have,*

$$x^n = \frac{1}{2}\left[E_n(x)+\sum_{k=0}^{n}\binom{n}{k}E_k(x)\right] \tag{4}$$

and

$$x^n = \frac{1}{2(n+1)}\left[G_{n+1}(x)+\sum_{k=0}^{n}\binom{n+1}{k+1}G_{k+1}(x)\right]. \tag{5}$$

Remark 2.3. Formulas (4) and (5) has been proved by Pintér and Srivastava [18, 15]. The corresponding formula for Bernoulli polynomials is (see for instance [18, Eq (27)])

$$x^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x).$$

Classically, a multiplication theorem for both Euler and Bernoulli polynomials has been proved by Raabe [16] (see also Bayad and Komatsu [4]). For odd m, we have

$$\sum_{k=0}^{m-1} B_n\left(\frac{x+k}{m}\right) = \frac{1}{m^{n-1}}B_n(x)$$

and

$$\sum_{k=0}^{m-1}(-1)^k E_n\left(\frac{x+k}{m}\right)=\frac{1}{m^n}E_n(x).$$

The corresponding formula for Euler-Genocchi polynomials is given by the following theorem.

Theorem 2.4. *Let r and m be non-negative integers with m odd, and $\{A_n^{(r)}(x)\}_{n\geq 0}$ be the class of polynomials given by (2). Then*

$$\sum_{k=0}^{m-1}(-1)^k A_n^{(r)}\left(\frac{x+k}{m}\right)=\frac{1}{m^{n-r}}A_n^{(r)}(x).$$

Proof. From (2), it follows that

$$\begin{aligned}
\sum_{n\geq 0}\sum_{k=0}^{m-1}(-1)^k A_n^{(r)}\left(\frac{x+k}{m}\right)\frac{t^n}{n!} &= \sum_{k=0}^{m-1}(-1)^k\sum_{n\geq 0}A_n^{(r)}\left(\frac{x+k}{m}\right)\frac{t^n}{n!}\\
&= \sum_{k=0}^{m-1}(-1)^k\frac{2t^r}{e^t+1}\exp\left((x+k)t/m\right)\\
&= \frac{2t^r}{exp(t)+1}\exp(xt/m)\frac{[1-(-\exp(t/m))^m]}{1+\exp(t/m)}\\
&= \frac{2t^r}{\exp(t/m)+1}\exp(xt/m)\\
&= m^r\sum_{n\geq 0}A_n^{(r)}(x)\frac{(t/m)^n}{n!}\\
&= \sum_{n\geq 0}\frac{1}{m^{n-r}}A_n^{(r)}(x)\frac{t^n}{n!}.
\end{aligned}$$

Equating the coefficients of t^n on both sides, we get the result. □

Theorem 2.5. *The Euler-Genocchi polynomials satisfy the following linear homogeneous recurrence relation*

$$\left(1-\frac{r}{n+1}\right)A_{n+1}^{(r)}(x)+(2-x)A_n^{(r)}(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{A_{k+r}^{(r)}}{(k+r)_r}A_{n-k}^{(r)}(x). \tag{6}$$

Proof. Differentiating both sides of (2) with respect to t yields

$$\begin{aligned}\frac{d}{dt}\sum_{n\geq 0} A_n^{(r)}(x)\frac{t^n}{n!} &= \frac{2xt^r}{e^t+1}e^{xt} + \frac{2rt^{r-1}(e^t+1)-2t^re^t}{(e^t+1)^2}e^{xt}\\ &= r\sum_{n\geq 0} A_n^{(r)}(x)\frac{t^{n-1}}{n!} + (x-2)\sum_{n\geq 0} A_n^{(r)}(x)\frac{t^n}{n!} + \frac{1}{e^t+1}\sum_{n\geq 0} A_n^{(r)}(x)\frac{t^n}{n!}\\ &= r\sum_{n=0}^{\infty}\frac{1}{n+1}A_{n+1}^{(r)}(x)\frac{t^n}{n!} + (x-2)\sum_{n=0}^{\infty} A_n^{(r)}(x)\frac{t^n}{n!}\\ &+ \left(\sum_{n\geq 0} A_n^{(r)}\frac{t^{n-r}}{n!}\right)\left(\sum_{n\geq 0} A_n^{(r)}(x)\frac{t^n}{n!}\right),\end{aligned}$$

$$\begin{aligned}\sum_{n\geq 0} A_{n+1}^{(r)}(x)\frac{t^n}{n!} &= r\sum_{n\geq 0}\frac{1}{n+1}A_{n+1}^{(r)}(x)\frac{t^n}{n!} + (x-2)\sum_{n\geq 0} A_n^{(r)}(x)\frac{t^n}{n!}\\ &+ \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+r)_r}A_{k+r}^{(r)}A_{n-k}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Then, using simple manipulations and equating the coefficients of t^k on both sides leads to the result. □

As $A_n^{(r)}(x)$ belongs to the family of Appell polynomials, we have

$$D_x^k A_n^{(r)}(x) = (n)_k A_{n-k}^{(r)}(x), \quad \text{where} \quad D_x^k := \frac{d^k}{dx^k}.$$

The recurrence relation (6) can be written as

$$A_{n+1}^{(r)}(x) = \frac{n+1}{n-r+1}\left[(x-2) + \sum_{k=0}^{n}\frac{A_{k+r}^{(r)}}{k!(k+r)_r}D_x^k\right]A_n^{(r)}(x).$$

Introducing the shift operator

$$E := \frac{n+1}{n-r+1}\left[(x-2) + \sum_{k=0}^{n}\frac{A_{k+r}^{(r)}}{k!(k+r)_r}D_x^k\right]$$

and the derivative operator

$$\widehat{E} := \frac{1}{n+1}D_x$$

and applying both operators $\widehat{E}$ and E to $A_n^{(r)}(x)$, we obtain

$$\widehat{E}EA_n^{(r)}(x) = A_n^{(r)}(x). \tag{7}$$

From (6) and (7), we get the following theorem:

Theorem 2.6. *The Euler-Genocchi polynomials* $A_n^{(r)}(x)$ *satisfy the differential equation*

$$\frac{A_{r+n-1}^{(r)}}{(n-1)!(r+n-1)_r}y^{(n)}+\cdots+\frac{A_{r+1}^{(r)}}{(r+1)_r}y^{(2)}+\left(\frac{A_r^{(r)}}{(r)_r}+(x-2)\right)y'+(r-n)y=0.$$

3. The Power Sum and the Alternate Power Sum

The power sum $S_k(n)$ and the alternate power sum $T_k(n)$ are defined respectively by:

$$S_k(n):=\sum_{i=0}^{n}i^k \qquad \text{and} \qquad T_k(n):=\sum_{i=0}^{n}(-1)^i i^k.$$

The exponential generating function of $(S_k(n))_k$ and $(T_k(n))_k$ are given by

$$\sum_{k=0}^{\infty}S_k(n)\frac{t^k}{k!}=\frac{1-e^{(n+1)t}}{1-e^t} \qquad \text{and} \qquad \sum_{k=0}^{\infty}T_k(n)\frac{t^k}{k!}=\frac{1-(-1)^{n+1}e^{(n+1)t}}{1+e^t}. \tag{8}$$

The power sum $S_k(n)$ and the alternate power sum $T_k(n)$ are related to Bernoulli and Euler polynomials in integer variable, (see [1, Eq. (23.1.4)]) via:

$$S_k(n)=\frac{B_{k+1}(n+1)-B_{k+1}}{k+1} \qquad \text{and} \qquad T_k(n)=\frac{(-1)^nE_k(n+1)+E_k}{2}. \tag{9}$$

For the alternate sums, we have an extension to the Euler-Genocchi polynomials in integer variables.

Theorem 3.1. *Let* n *and* k *be nonnegative integers. The following formula holds:*

$$T_k(n)=\frac{(-1)^nA_{k+r}^{(r)}(n+1)+A_{k+r}^{(r)}}{2(k+r)_r}. \tag{10}$$

Proof. From (8), we have

$$\begin{aligned}\sum_{k=0}^{\infty}T_k(n)\frac{t^k}{k!}&=\frac{1-(-1)^{n+1}e^{(n+1)t}}{e^t+1}\\&=\frac{2}{2(e^t+1)}-\frac{2(-1)^{n+1}e^{(n+1)t}}{2(e^t+1)}.\end{aligned}$$

Multiplying both sides by t^r, we get

$$\sum_{k=0}^{\infty}T_k(n)\frac{t^{k+r}}{k!}=\frac{2t^r}{2(e^t+1)}-\frac{2t^r(-1)^{n+1}e^{(n+1)t}}{2(e^t+1)}.$$

Then, using (2) and a direct computation gives

$$\sum_{k=0}^{\infty}(k)_r T_{k-r}(n)\frac{t^k}{k!} = \frac{1}{2}\sum_{k=0}^{\infty}A_k^{(r)}\frac{t^k}{k!} - \frac{(-1)^{n+1}}{2}\sum_{k=0}^{\infty}A_k^{(r)}(n+1)\frac{t^k}{k!}.$$

Then equating the coefficients of t^k on both sides leads to the result. □

Remark 3.2. For $r = 0$, Expression (10) gives the alternate sum in terms of Euler polynomials (9). For $r = 1$, we express the alternate sum in terms of Genocchi polynomials:

$$T_k(n) = \frac{(-1)^n G_{k+1}(n+1) + G_{k+1}}{2(k+1)}.$$

4. Related sums with Stirling numbers of the second kind

Given nonnegative integers n and k ($k \leq n$), it is known that the Stirling numbers of the second kind $S(n,k)$ satisfy [7, p. 206]

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

In the following theorem, we give a link between the Euler-Genocchi numbers of order r, Bernoulli numbers, and Stirling numbers.

Theorem 4.1. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}A_{n-k}^{(r)}B_k = 2^{n-r}(n)_r\sum_{k=0}^{n-r}\frac{(-1)^k}{k+1}k!S(n-r,k). \tag{11}$$

Proof. Let t be a real number with $|t| < \pi$, and set $\mathbb{T}(t) := \dfrac{2t^r}{e^t+1} \times \dfrac{t}{e^t-1}$. Taking into account the right-hand side of (1) and (2) for $x = 0$, a direct computation gives

$$\mathbb{T}(t) = \sum_{n\geq 0}\left\{\sum_{k=0}^{n}\binom{n}{k}A_{n-k}^{(r)}B_k\right\}\frac{t^n}{n!}.$$

On the other hand, we have

$$\begin{aligned}
\mathbb{T}(t) = \frac{2t^{r+1}}{e^{2t}-1} &= \frac{t^r \ln\left((e^{2t}-1)+1\right)}{e^{2t}-1} \\
&= t^r \sum_{k\geq 0} \frac{(-1)^k}{k+1}(e^{2t}-1)^k \\
&= t^r \sum_{k\geq 0} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e^{2jt} \\
&= t^r \sum_{k\geq 0} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n\geq 0} 2^n j^n \frac{t^n}{n!} \\
&= \sum_{n\geq 0} 2^n \frac{t^{n+r}}{n!} \sum_{k\geq 0} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \\
&= \sum_{n\geq 0} 2^n \frac{t^{n+r}}{n!} \sum_{k\geq 0} \frac{(-1)^k}{k+1} k! S(n,k) \\
&= \sum_{n\geq 0} 2^{n-r} (n)_r \frac{t^n}{n!} \sum_{k\geq 0} \frac{(-1)^k}{k+1} k! S(n-r,k).
\end{aligned}$$

Also, $S(n-r,k) = 0$ when $k > n-r$. Hence

$$\mathbb{T}(t) = \sum_{n=0}^{\infty} 2^{n-r} (n)_r \sum_{k=0}^{n-r} \frac{(-1)^k}{k+1} k! S(n-r,k) \frac{t^n}{n!}.$$

Comparing the preceding two expansions of $\mathbb{T}(t)$, we get Equation (11). □

In particular, we get the following corollary

Corollary 4.2. *For $r = 0$ and $r = 1$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} E_{n-k} B_k = 2^n \sum_{k=0}^{n} \frac{(-1)^k}{k+1} k! S(n,k)$$

and

$$\sum_{k=0}^{n} \binom{n}{k} G_{n-k} B_k = 2^{n-1} n \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} k! S(n-1,k).$$

5. A Determinantal Approach for Euler-Genocchi Polynomials

Belbachir et al. [6] proposed a determinantal form to express x^n in terms of both Euler-Genocchi and Bernoulli polynomials. We give an alternate version to Theorem 3.1. in [6].

Theorem 5.1. *For every integer $n \geq r$ $(r \geq 1)$, the following formula holds:*

$$x^n = \sum_{k=0}^{n+r} \Lambda_{n,k}^{(r)} \times \nabla_{n-(k-r),k}^{(r-1)}(x+1, x), \tag{12}$$

where $\nabla_{n,k}^{(r)}(x,y) = \begin{vmatrix} B_n(x) & A_k^{(r)}(y) \\ B_n(y) & A_k^{(r)}(x) \end{vmatrix}$ *and* $\Lambda_{n,k}^{(r)} = \frac{1}{2^{n+1}(n+r)_r}\binom{n+r}{k}$.

Proposition 5.2. *Let n and k be nonnegative integers and $r \geq 1$. Then the following identity holds:*

$$\sum_{s=0}^{m-1} \nabla_{n,k}^{(r)}(s+1, s) = \nabla_{n,k}^{(r)}(m, 0).$$

Proof. We have

$$\begin{aligned} \sum_{s=0}^{m-1} \nabla_{n,k}^{(r)}(s+1,s) &= \sum_{s=0}^{m-1} \begin{vmatrix} B_n(s+1) & A_k^{(r)}(s) \\ B_n(s) & A_k^{(r)}(s+1) \end{vmatrix} \\ &= \sum_{s=0}^{m-1} \left[B_n(s+1)A_k^{(r)}(s+1) - B_n(s)A_k^{(r)}(s) \right] \\ &= \left[B_n(m)A_k^{(r)}(m) - B_n(0)A_k^{(r)}(0) \right] \\ &= \begin{vmatrix} B_n(m) & A_k^{(r)}(0) \\ B_n(0) & A_k^{(r)}(m) \end{vmatrix}. \end{aligned}$$

□

Theorem 5.3. *For all integers n, m and $r \geq 0$, we have*

$$S_n(m) = \sum_{k=0}^{n+r} \Lambda_{n,k}^{(r)} \times \nabla_{n-(k-r),k}^{(r-1)}(m+1, 0). \tag{13}$$

Proof. Letting $x = s$ in (12), we get:

$$s^n = \sum_{k=0}^{n+r} \Lambda_{n,k}^{(r)} \times \nabla_{n-(k-r),k}^{(r-1)}(s+1, s).$$

We conclude by summing up the s^n and using Proposition 5.2. □

6. Jacobi-Stirling Numbers in a Determinantal Form Representation

For n and k nonnegative integers and γ a positive real number, the Jacobi-Stirling numbers of the first kind $J_{\gamma}^{s}(n,k)$ and the second kind $J_{\gamma}^{S}(n,k)$ are given respectively by

$$E(t) := \prod_{k=1}^{n}(1+k(k+2\gamma-1)t) = \sum_{k=0}^{\infty} J_{\gamma}^{s}(n+1,n+1-k)\,t^{k} \tag{14}$$

and

$$H(t) := \prod_{k=1}^{n}(1-k(k+2\gamma-1)t)^{-1} = \sum_{k=0}^{\infty} J_{\gamma}^{S}(n+k,n)\,t^{k}. \tag{15}$$

Merca [14] expresses a convolution alternating formula with Jacobi-Stirling numbers of both kinds in terms of Bernoulli polynomials in an integer variable as follows:

Theorem 6.1. *We have*

$$\sum_{j=1}^{k}(-1)^{j-1} j J_{\gamma}^{s}(n+1,n+1-j)\,J_{\gamma}^{S}(n+k-j,n) = \sum_{j=0}^{k}\frac{B_{k+j+1}(n+1)-B_{k+j+1}}{k+j+1} \times \binom{k}{j}(2\gamma-1)^{k-j}.$$

In the following Theorem, we propose another version to express the convolution alternating formula with Jacobi-Stirling numbers of both kinds in determinantal form.

Theorem 6.2. *We have*

$$\sum_{j=1}^{k}(-1)^{j-1} j J_{\gamma}^{s}(n+1,n+1-j)\,J_{\gamma}^{S}(n+k-j,n) = \sum_{j=0}^{k}\binom{k}{j}(2\gamma-1)^{k-j}\sum_{t=0}^{k+j+r}\Lambda_{k+j,t}^{(r)} \times \nabla_{k+j-(t-r),t}^{(r-1)}(n,0). \tag{16}$$

Proof. It is easy to verify that

$$\frac{d}{dt}\ln H(t) = E^{'}(-t)H(t).$$

We replace t by $-t$ in the power series (14) and we calculate the derivative. We get:

$$E^{'}(-t) = \sum_{k=0}^{\infty} kJ_{\gamma}^{s}(n+1,n+1-k)\,(-t)^{k-1}.$$

On the one hand, we have

$$\frac{d}{dt}\ln H(t) = \sum_{k=1}^{n}\frac{k(k+2\gamma-1)}{1-k(k+2\gamma-1)t} = \sum_{k=1}^{n}\sum_{j=1}^{\infty}k^{j}(k+2\gamma-1)^{j}t^{j-1}.$$

Interchanging the order of the resulting double sums, we find

$$\sum_{k=1}^{\infty}\sum_{j=1}^{n} j^k (j+2\gamma-1)^k t^{k-1} = \left(\sum_{k=1}^{\infty} kJ_\gamma^s\left(n+1,n+1-k\right)(-t)^{k-1}\right)\left(\sum_{k=0}^{\infty} J_\gamma^S\left(n+k,n\right)t^k\right).$$

Taking into account the series product and equating coefficients of t^{k-1} on both sides, we find the following identity:

$$\sum_{j=1}^{n} j^k (j+2\gamma-1)^k = \sum_{j=1}^{k}(-1)^{j-1} j J_\gamma^s\left(n+1,n+1-j\right) J_\gamma^S\left(n+k-j,n\right). \tag{17}$$

On the other hand, taking into account the left hand side of (17), we get

$$\begin{aligned}\sum_{j=1}^{n} j^k (j+2\gamma-1)^k &= \sum_{j=1}^{n} j^k \sum_{i=0}^{k}\binom{k}{i} j^i (2\gamma-1)^{k-i}\\ &= \sum_{j=0}^{n-1}\sum_{i=0}^{k}\binom{k}{i} j^{k+i}(2\gamma-1)^{k-i}\\ &= \sum_{j=0}^{k}\binom{k}{j}(2\gamma-1)^{k-j}\sum_{i=0}^{n-1} i^{k+j}.\end{aligned}$$

Now, upon replacing the factor i^{k+j} in Formula (13) by its expression, we obtain

$$\sum_{j=1}^{n} j^k (j+2\gamma-1)^k = \sum_{j=0}^{k}\binom{k}{j}(2\gamma-1)^{k-j}\sum_{t=0}^{k+j+r}\Lambda_{k+j,t}^{(r)}\times\nabla_{k+j-(t-r),t}^{(r-1)}(n,0). \tag{18}$$

Comparing the two expansions (17) and (18), we get the result. □

For $\gamma = 1$, (16) gives Legendre-Stirling numbers of both kinds introduced by Everitt et al. [10] in 2002. Some of their properties may be found in [14, 2, 3].
We have the following property.

Corollary 6.3. *Let $J_1^s(n,k)$ and $J_1^S(n,k)$ be the Legendre-Stirling numbers of the first and second kind respectively, we have*

$$\sum_{j=1}^{k}(-1)^{j-1} j J_1^s\left(n+1,n+1-j\right) J_1^S\left(n+k-j,n\right) = \sum_{j=0}^{k}\binom{k}{j}\sum_{t=0}^{k+j+r}\Lambda_{k+j,t}^{(r)}\nabla_{k+j-(t-r),t}^{(r-1)}(n,0).$$

Letting $\gamma = 1/2$ in Formula (17), and using the same approach as in the proof of Theorem 6.2, allows us to express the Chebyshev-Stirling numbers (see [12]) in terms of a determinantal representation in the following corollary.

Corollary 6.4. *Let $J^{s}_{1/2}(n,k)$ and $J^{S}_{1/2}(n,k)$ be Chebyshev-Stirling numbers of the first and second kind respectively. We have*

$$\sum_{j=1}^{k}(-1)^{j-1}jJ^{s}_{1/2}(n+1,n+1-j)\,J^{S}_{1/2}(n+k-j,n)=\sum_{t=0}^{2k+r}\Lambda^{(r)}_{2k,t}\times\nabla^{(r-1)}_{2k-(t-r),t}(n+1,1).$$

For $r=1$ and $r=2$, we have the determinantal representation for Jacobi-Stirling numbers in terms of Bernoulli-Euler numbers and Bernoulli-Genocchi numbers respectively. Hence, we get the following corollary.

Corollary 6.5. *We have*

$$\sum_{j=1}^{k}(-1)^{j-1}jJ^{s}_{\gamma}(n+1,n+1-j)\,J^{S}_{\gamma}(n+k-j,n)=\sum_{j=0}^{k}\binom{k}{j}(2\gamma-1)^{k-j}$$
$$\times\sum_{t=0}^{k+j+1}\frac{1}{2^{k+j+1}(k+j+1)}\binom{k+j+1}{t}\begin{vmatrix}B_{k+j-(t-1)}(n) & E_t\\ B_t & E_{k+j-(t-1)}(n)\end{vmatrix}\tag{19}$$

and

$$\sum_{j=1}^{k}(-1)^{j-1}jJ^{s}_{\gamma}(n+1,n+1-j)\,J^{S}_{\gamma}(n+k-j,n)=\sum_{j=0}^{k}\binom{k}{j}(2\gamma-1)^{k-j}$$
$$\times\sum_{t=0}^{k+j+2}\frac{1}{2^{k+j+1}(k+j+1)(k+j+2)}\binom{k+j+2}{t}\begin{vmatrix}B_{k+j-(t-2)}(n) & G_t\\ B_t & G_{k+j-(t-2)}(n)\end{vmatrix}.$$

The determinantal representation (19) of Jacobi-Stirling numbers involves the Bernoulli and Euler polynomials. However, these formulas can be expressed in terms of the Bernoulli polynomials, using some well known closed relations between Bernoulli and Euler polynomials. Indeed, it is well-known that (see [9, 18]):

$$E_n(x)=\frac{2}{n+1}\left[B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right].\tag{20}$$

Combining (20) and (19), we obtain

$$\nabla^{(0)}_{m,k}(n,0)=\frac{2}{k+1}\begin{vmatrix}B_m(n) & \left(1-2^{k+1}\right)B_{k+1}\\ B_m & B_{k+1}(n)-2^{k+1}B_{k+1}(\frac{n}{2})\end{vmatrix}.$$

For all integers s, k and a, we let

$$\mathbb{B}_{s,k}(a):=\begin{vmatrix}B_s(a) & \left(1-2^{k}\right)B_k\\ B_s & B_k(a)-2^{k}B_k(\frac{a}{2})\end{vmatrix}.$$

Expression (19) may then be written in terms of Bernoulli polynomials:

$$\sum_{j=1}^{k}(-1)^{j-1}jJ_{\gamma}^{s}\left(n+1,n+1-j\right)J_{\gamma}^{S}\left(n+k-j,n\right)$$
$$=\sum_{j=0}^{k}\binom{k}{j}(2\gamma-1)^{k-j}\sum_{t=0}^{k+j+1}\frac{2}{t+2}\Lambda_{k+j,t}^{(0)}\times\mathbb{B}_{k+j-(t-1),t+1}(n).$$

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