TRIANGULAR RAMSEY NUMBERS

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Abstract

The purpose of this paper is to introduce the idea of triangular Ramsey numbers and provide values as well as upper and lower bounds for them. To do this, the combinatorial game Mines is introduced after some necessary theorems about triangular sets are proven. This game is easy enough that young children are able to play. The most basic variations of this game are analyzed, and theorems about winning strategies and the existence of draws are proven. The game of Mines is then used to define triangular Ramsey numbers. Lower bounds are found for these triangular Ramsey numbers using the probabilistic method and the theorems about triangular sets.

1. Combinatorial Games

Combinatorial games make it possible to easily explain the underlying workings of combinatorial problems, which in turn help build a deeper understanding of combinatorics. In fact, some combinatorial games are so simple that they where invented as tools to be used in grade school classrooms to introduce students to logical reasoning and mathematical concepts. For example, the game of Tri was introduced by Haggard and Schonberger in [9] with the goal of "developing logical skills of evaluating alternatives and their consequences." The game had the unintended learning outcome of developing the skill of visual disembedding, i.e., the skill of picking out simple figures from a more complex image. Haggard and Schonberger point out that this ability has been linked to success in solving mathematical problems. In this paper we introduce a new game in the spirit of Haggard and Schonberger. In theory, these games are simple enough that they can be used to help develop mathematical problem solving skills in primary school students.

This paper focuses on a new combinatorial game called Mines which we use to introduce the notion of a triangular Ramsey number. The game is called Mines because our original game boards were in the shape of a Reuleaux triangle, the logo of Colorado School of Mines (see Figure 1). Our main results concern theorems about this game, such as the existence of a winner and the possibility of a winning strategy. The existence of triangular Ramsey numbers follows from the work of Dobrinen and Todorcevic. The primary purpose of introducing the game of Mines is to provide a simplified presentation of the finite-dimensional Ramsey theory of the infinite-dimensional topological Ramsey space $\mathcal{R}_1$ introduced and studied [3]. The game provides a simplified method for defining triangular Ramsey numbers which are the direct analogue of the Ramsey numbers for the finite-dimensional Ramsey theory of $\mathcal{R}_1$.

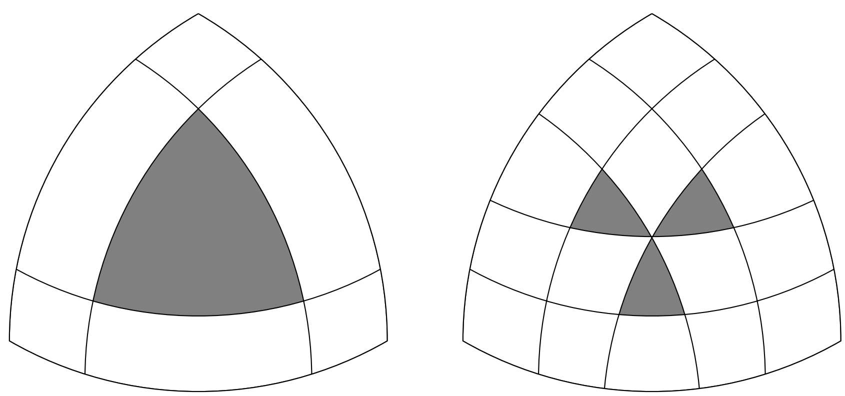

Figure 1: Two different sized Mines game boards

In Section 2, we introduce the games of Tri and Sim. Then, we describe how the games can be used to define Ramsey numbers. Near the end of the section we provide a short survey of some known Ramsey numbers and some bounds on unknown Ramsey numbers. The section concludes with a lemma about Tri needed later in the paper.

In Section 3, in order to help precisely describe the gameboards, we introduce the concept of a triangular set. The remainder of Section 3 is devoted to proving combinatorial results related to counting triangular sets.

In Section 4, we introduce the game of Mines and prove some theorems about its game play. For example, certain variations of Mines have the property that they can never end in a draw.

In Section 5, we use the game to define the notion of a triangular Ramsey number. Our main results in this section involve finding exact values of some triangular Ramsey numbers and bounds for other triangular Ramsey numbers. Section 5 ends by applying the combinatorial results from Section 3 and the probabilistic method as pioneered by Erdős to find lower bounds for triangular Ramsey numbers.

In Section 6, we collect together the main results of the paper in Table 11. We then discuss the connection between the triangular Ramsey numbers and the topological Ramsey space $\mathcal{R}_1$ introduced by Dobrinen and Todorcevic in [3]. We conclude with some open questions and problems related to Mines and triangular Ramsey numbers.

2. Tri, Sim, and Ramsey Numbers

Two games that have attracted attention in the literature are Sim and Tri. The game Sim_m was introduced by Simmons in 1969 [17], and Tri_m was introduced by Haggard and Schonberger in 1977 [9]. Both Sim_m and Tri_m are two player games played on a game board of $m > 2$ vertices with $\binom{m}{2}$ possible edges. Each player chooses a color; players alternate turns coloring uncolored edges using their color. Both games end when a monochromatic triangle is constructed (three vertices all of whose edges have the same color) or all edges have been colored. In Tri_m the winner is the player that constructs a monochromatic triangle. In Sim_m a player wins if they can force the other player to construct a monochromatic triangle. In either game, if no monochromatic triangle is constructed, we say the game ends in a draw. The finite Ramsey theorem for pairs implies that there exists a natural number m such that neither Tri_m nor Sim_m ever ends in a draw.

Problem 1. Find the smallest natural number $m > 2$ such that neither Tri_m nor Sim_m ever ends in a draw.

The solution to the problem is $m = 6$. For $n < m$, the two games have natural generalizations to $\mathrm{Tri}_m(n)$ and $\mathrm{Sim}_m(n)$. The only difference being that these versions end when a monochromatic complete graph with n vertices is constructed. In this notation Tri_m and Sim_m correspond to $\mathrm{Tri}_m(3)$ and $\mathrm{Sim}_m(3)$. The Finite Ramsey Theorem for pairs implies that for all natural numbers n there exists a natural number m such that neither $\mathrm{Tri}_m(n)$ nor $\mathrm{Sim}_m(n)$ ever ends in a draw.

Problem 2. Let n be a natural number greater than 2. Find the smallest natural number $m \geq n$ such that neither $\mathrm{Tri}_m(n)$ nor $\mathrm{Sim}_m(n)$ ever ends in a draw.

The solution to the second problem for the natural number m is called the *Ramsey number for n*, denoted by $R(n)$. It is known that $R(3) = 6$ and $R(4) = 18$; however, $R(5)$ still remains unknown. Figure 2 gives upper and lower bounds for some small Ramsey numbers. For example, from the table we have $43 \leq R(5) \leq 49$. In other words, there is a game of $\text{Tri}_{43}(5)$ that ends in a draw but no game of $\text{Tri}_{49}(5)$ can end in a draw. The lower bound in the second to last row of the table is the lower bound obtained by Erdős using the probabilistic method in [6]. The last row gives the best known upper and lower bounds.

n	Lower Bound	$R(n)$	Upper Bound	References
3	-	6	-	[8]
4	-	18	-	[8]
5	43	?	49	[7] [14]
6	102	?	165	[10] [13]
7	205	?	540	[16] [13]
n	$2^{n/2}$	?	4^{n-1}	[6][5]
n	$n2^{n/2}[\sqrt{2}/e + o(1)]$	?	$n^{-C\frac{\log n}{\log\log n}}4^n$	[18] [2]

Figure 2: Table of upper and lower bounds of some Ramsey numbers and their references.

The next lemma about Tri will be used later to obtain an upper bound for a small triangular Ramsey number. For natural numbers k the notation $R^k(n)$ denotes the Ramsey number $\underbrace{R(R(\cdots(R}_{k-times}(n))\cdots))$.

Lemma 1. *Let k be a natural number. If k games of $\text{Tri}_{R^k(3)}(3)$ are played on the same game board, then there exists a complete graph with three vertices that is monochromatic for each of the k games.*

Proof. By the definition of Ramsey number there is a complete graph with $R^{k-1}(3)$ vertices that is monochromatic for the first game. Then, restrict the second game to this complete subgraph of the game board. Again by the definition of Ramsey number there is a complete subgraph of this graph with $R^{k-2}(3)$ vertices that is monochromatic for the first and second games. Continuing this way for k steps we obtain a complete graph with three vertices that is monochromatic for all n games. □

3. Combinatorics of Triangular Sets

A *triangular number* is a number that can be represented by a triangular arrangement of equally spaced points. For example, the number 15 can be arranged into a triangle with five levels (see Figure 3).

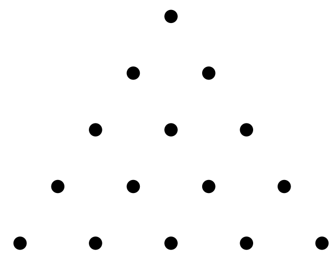

Figure 3: Triangular arrangement of 15 points.

For this reason, 15 is called a triangular number. The first four triangular numbers are 1, 3, 6, and 10 whose arrangements are given in Figure 4. If T_n denotes the n^{th} triangular number, then by construction $T_{n+1} = T_n + n + 1$ with $T_1 = 1$. It is well known that these numbers can be represented as follows:

$$T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

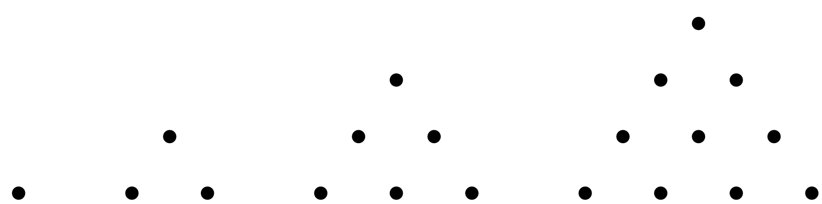

Figure 4: Triangular arrangement of 1, 3, 6, and 10 points

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of natural numbers. A subset X of $\mathbb{N}$ is *triangular* if $|X|$ is a triangular number. Let $\{x_1, x_2, x_3, \dots, x_{T_n}\}$ be an increasing enumeration of X. The numbers in X can be naturally arranged into a triangle with n levels as shown in Figure 5.

Let $\triangle$ denote the collection of all triangular subsets of $\mathbb{N}$. For $k \in \mathbb{N}$, let $\triangle_k$ denote the triangular sets with k levels, i.e., those subsets of $\mathbb{N}$ such that $|X| = T_k$.

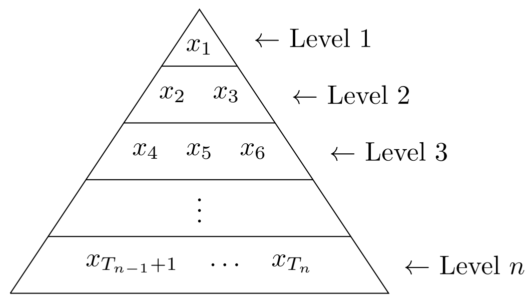

Figure 5: Triangular arrangement of $\{x_1, x_2, x_3, \ldots, x_{T_n}\}$

The next partial order is an adaptation of the order on $\mathcal{R}_1$ considered by Dobrinen and Todorcevic in [3] to our current setting. It can be seen as a restriction (to triangular sets) of the partial order used by Laflamme, which inspired the work in [3], to study complete combinatorics in [12].

Definition 1. For $X, Y \in \triangle$, $X \leqslant Y$ means that $X \subseteq Y$ and every level of X is contained in a single distinct level of Y.

For example, if we let $W = \{1, 2, 3, 4, 5, 6\}$, $X = \{2, 4, 5\}$, $Y = \{3, 5, 6\}$, and $Z = \{5\}$, then $W \in \triangle_3$, $X, Y \in \triangle_2$, and $Z \in \triangle_1$ such that $Z \subseteq X, Y$ and $X, Y \subseteq W$. Each level of X and Y are contained in a distinct single level of W and Z, being only one element, is contained in X and Y. Figure 6 displays this configuration and the associated Hasse diagram in the partial order $(\triangle, \leq)$.

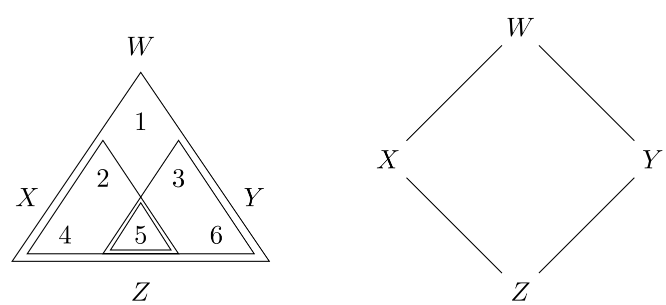

Figure 6: Hasse diagram and possible configuration for W, X, Y and Z

Definition 2. For $k \in \mathbb{N}$ and $X \in \triangle$, let $\triangle_k(X) = \{Y \in \triangle_k : Y \leq X\}$.

Suppose $X \in \triangle_3$ and let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be an increasing enumeration of X. Then $\triangle_2(X)$ contains the elements $\{x_1, x_2, x_3\}$, $\{x_1, x_4, x_5\}$, $\{x_1, x_4, x_6\}$,

$\{x_1, x_5, x_6\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_4, x_6\}$, $\{x_2, x_5, x_6\}$, $\{x_3, x_4, x_5\}$, $\{x_3, x_4, x_6\}$, and $\{x_3, x_5, x_6\}$. Thus for any $X \in \triangle_3$, $|\triangle_2(X)| = 10$.

Note that if $Y \in \triangle_4$, then $|\triangle_3(Y)| = 41$. To see this, note that there are $\binom{4}{3} = 4$ ways to choose three elements from the last row of Y. Each one of these possibilities can be added onto any element of $\triangle_2(Y')$ where Y' is the element of $\triangle_3$ obtained by removing the last level of Y to obtain a distinct element of $\triangle_3(Y)$. In particular, there are $\binom{4}{3} \cdot |\triangle_2(X)| = 40$ elements of $\triangle_3(Y)$ obtained this way. The only other element of $\triangle_3(Y)$ is Y'. So $|\triangle_3(Y)| = \binom{4}{3} \cdot |\triangle_2(Y')| + 1 = 41$. By a similar argument, one can show that for all $k \in \mathbb{N}$ and for all $Y \in \triangle_k$ if $Y' \in \triangle_{k-1}$, then

$$|\triangle_{k-1}(Y)| = 1 + k|\triangle_{k-2}(Y')|.$$

To better express these types of combinatorial relationships we introduce a variant of the binomial coefficient $\binom{n}{m}$. The next definition should be contrasted with the recursive definition of the binomial coefficients using Pascal's Triangle.

Definition 3.

$$\begin{cases} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \binom{n}{k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ \\ \begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \ \begin{bmatrix} n \\ n \end{bmatrix} = 1 \end{cases}$$

With this definition, note that for all natural numbers k

$$\begin{bmatrix} k \\ k-1 \end{bmatrix} = \begin{bmatrix} k-1 \\ k-1 \end{bmatrix} + \binom{k}{k-1} \begin{bmatrix} k-1 \\ k-2 \end{bmatrix} = 1 + k \begin{bmatrix} k-1 \\ k-2 \end{bmatrix}.$$

Since $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1$, the argument in the previous paragraph implies that for all natural numbers k and for all $Y \in \triangle_k$, $|\triangle_{k-1}(Y)| = \left[\begin{smallmatrix} k \\ k-1 \end{smallmatrix}\right]$ as they both satisfy the same recursive formula. The next theorem generalizes this result.

Theorem 1. *If $k < n$ and $X \in \triangle_n$, then $\begin{bmatrix} n \\ k \end{bmatrix} = |\triangle_k(X)|$. That is, $\begin{bmatrix} n \\ k \end{bmatrix}$ counts the number of $\triangle_k$'s in a given $\triangle_n$.*

Proof. Let $k < n$ and $X \in \triangle_n$. We show that $|\triangle_k(X)|$ satisfies the same recursive definition as $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$. It is clear that $|\triangle_n(X)| = 1$. If we consider $\emptyset$ to be the only element of $\triangle_0$ then $|\triangle_0(X)| = 1$. Thus, the base cases of the recursions are the same. We complete the proof by verifying that $|\triangle_k(X)| = |\triangle_k(X')| + \binom{n}{k}|\triangle_{k-1}(X')|$ where X' is the triangular set in $\triangle_{n-1}(X)$ obtained by removing the last level of X.

Note that $|\triangle_k(X)|$ can be thought of as the number of $\triangle_k$'s in a $\triangle_n$. It should be clear that $|\triangle_k(X)| = |\triangle_k(X')| + c$, where c is the amount of new $\triangle_k's$ formed when the last level of X is added back to X'. Each $\triangle_k$ contributing to c must have its last level in the last level of X. There are $\binom{n}{k}$ possibilities for those k points in the final level of X. For each collection of k points in the last level of X there are $|\triangle_{k-1}(X')|$ possibilities for triangular sets in $\triangle_k(X)$ whose last level is the given k points. Therefore, $c = \binom{n}{k}|\triangle_{k-1}(X')|$ and $|\triangle_k(X)| = |\triangle_k(X')| + \binom{n}{k}|\triangle_{k-1}(X')|$. □

Later in the paper we use the next corollary to obtain estimates for upper and lower bounds on $\begin{bmatrix} n \\ k \end{bmatrix}$. These estimates are needed to apply the probabilistic method to our combinatorial game and obtain lower bounds on triangular Ramsey numbers.

Corollary 1. *For* $0 < k < n$, $\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{0<i_1<\cdots<i_k\le n} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right)$.

Proof. We show that $\sum_{0<i_1<\cdots<i_k\le n} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right)$ satisfies the same recursive definition as $\begin{bmatrix} n \\ k \end{bmatrix}$. By the previous theorem, $\sum_{0<i_1\le n} \binom{i_1}{1} = T_n = \begin{bmatrix} n \\ 1 \end{bmatrix}$. Further,

$$\sum_{0<i_1<\cdots<i_n\le n} \left(\prod_{j=1}^{n} \binom{i_j}{j} \right) = \binom{1}{1}\binom{2}{2}\cdots\binom{n}{n} = 1 = \begin{bmatrix} n \\ n \end{bmatrix}.$$

Thus, the base case of the two recursions are the same. Note that

$$\begin{aligned} &\sum_{0<i_1<\cdots<i_k\le n-1} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right) + \binom{n}{k} \cdot \sum_{0<i_1<\cdots<i_{k-1}\le n-1} \left(\prod_{j=1}^{k-1} \binom{i_j}{j} \right) \\ &= \sum_{0<i_1<\cdots<i_k\le n-1} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right) + \sum_{0<i_1<\cdots<i_{k-1}\le n-1, i_k=n} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right). \end{aligned}$$

The right hand side of the previous equation is just the sum $\sum_{0<i_1<\ldots<i_k\le n} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right)$ broken into the parts where $i_k = n$ in the second sum and where it isn't in the first sum, so the equality holds. In particular, the formula satisfies the same recursion formula as $\begin{bmatrix} n \\ k \end{bmatrix}$. □

4. The Game of Mines

The most basic variant of the game of Mines, denoted by Mines_3, is played on a game board of size $\triangle_3$ with two players each assigned one of two markings and/or colors. Two example game boards are given in Figure 7 where the numbers represent the positions that can be marked. The players have alternating turns in which they may choose to mark one position on the game board or none at all. The game ends when all positions have been played.

Definition 4. Let X denote the set of moves made by Player 1 and Y the set of moves made by Player 2. Let x_m denote a move by Player 1 on position m and y_m a move by Player 2 on position m.

At any point in the game we have $X \cap Y = \emptyset$. If both players always choose to mark a position, then given that Player 1 goes first, we also have $|X| = |Y| - 1$ or $|X| = |Y|$, and $|X| + |Y| = T_3$ when all positions have been played.

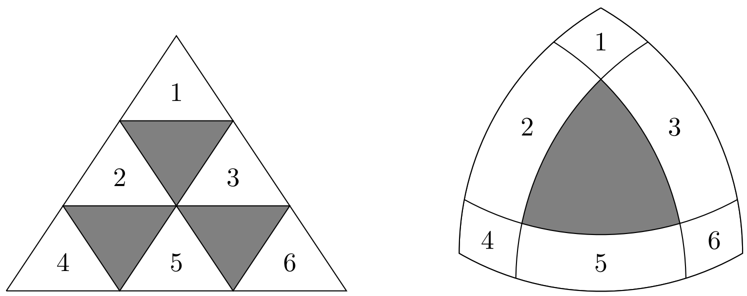

Figure 7: Two alternative gameboards for Mines$_3$.

Player 1 *wins* if $\triangle_2(X) \neq \emptyset$. Player 2 *wins* if $\triangle_2(Y) \neq \emptyset$. A *draw* occurs if all positions have been played but neither player has won the game. That is, $\triangle_2(X) = \triangle_2(Y) = \emptyset$ and $|X| + |Y| = T_3$.

Figure 8 describes a possible game such that $X = \{x_2, x_3, x_4\}$ and $Y = \{y_1, y_5, y_6\}$. Here $\triangle_2(Y) \neq \emptyset$ since $\{y_1, y_5, y_6\} \in \triangle_2$ indicating that Player 2 has won the game.

Theorem 2. *Both players cannot construct a* $\triangle_2$*. That is, it is impossible for both* $\triangle_2(X) \neq \emptyset$ *and* $\triangle_2(Y) \neq \emptyset$*.*

Proof. Assume this is not the case. In other words, there exists a situation such that $\triangle_2(X) \neq \emptyset$ and $\triangle_2(Y) \neq \emptyset$. Without loss of generality, we may assume that all positions have been played.

The largest row of the game board has 3 positions, and the largest level of a $\triangle_2$ has 2 positions. Therefore, both X and Y cannot construct their $\triangle_2$'s largest level in the same row of the game board. By the pigeon hole principle there exists $a, b \in \{4, 5, 6\}$ such that $x_a, x_b \in X$ or $y_a, y_b \in Y$. If $x_a, x_b \in X$, then $y_2, y_3 \in Y$. In order for $\triangle_2(X) \neq \emptyset$ we must have $x_1 \in X$ and we find $\triangle_2(Y) = \emptyset$, a contradiction. If instead we had $y_a, y_b \in Y$, then $x_2, x_3 \in X$. Once again, in order for $\triangle_2(X) \neq \emptyset$

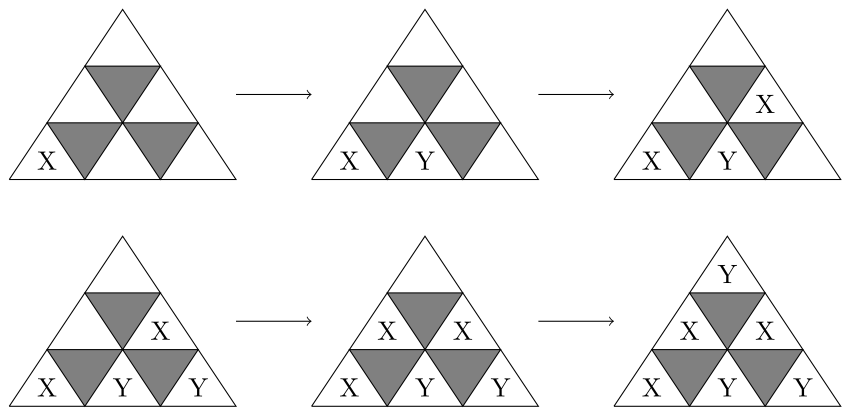

Figure 8: An example game of Mines_3.

we must have $x_1 \in X$ and we find $\triangle_3(Y) = \emptyset$, a contradiction. Therefore, it is impossible for both players to construct a $\triangle_2$. □

Theorem 3. *Mines_3 never ends in a draw.*

Proof. Assume a full game of Mines_3 has been played. By the pigeonhole principle, at least two of the bottom three elements must be of the same color. If any of the three elements above the bottom row are of the same color as the two on the bottom, then a $\triangle_2$ has been constructed in that color. For this not to happen, the three elements in the top two rows must all be in the opposite color. If this is the case, then a $\triangle_2$ has been constructed in the opposite color. Therefore, it is impossible for a game of Mines_3 to be played in which neither player constructs a $\triangle_2$, and no game can be played in which both players construct a $\triangle_2$. □

An interesting variation of the game play exploits the fact that the game board has 120° rotational symmetry about its center. There are three directions to the game board given by the perpendicular from any of the three edges to its adjacent vertex. We let D_n denote the orientation of the game board in the n direction (see Figure 9). We use the notation $\triangle_{k,n}(X)$ to denote all $\triangle_k$'s on the game board in the D_n direction. For example, $\{1,3,4\} \in \triangle_{2,2}(X)$ but $\{1,3,4\} \notin \triangle_{2,1}(X)$. Note that if $\triangle_{k,n}(X) \neq \emptyset$, then there exists a $\triangle_k$ in the direction of D_n contained in X. Likewise, if $\triangle_{k,n}(Y) \neq \emptyset$, then there exists a $\triangle_k$ in the direction of D_n contained in Y. The player that wins in two of the three directions wins this variation of the game.

By Theorem 2 and Theorem 3 we see that there can never be a draw in a single direction. In addition, given that there are three directions one player must win in at least two directions. Therefore this variation of Mines_3 can never end in a draw.

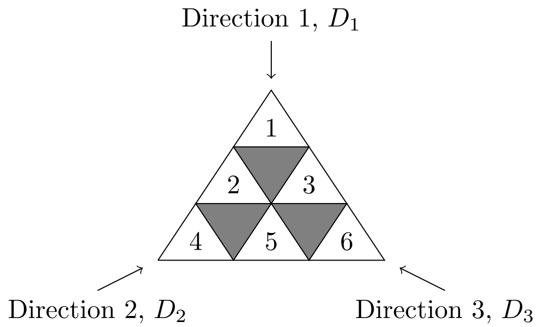

Figure 9: A game board for Mines_3 with directions D_1, D_2 and D_3 labeled.

The next theorem is true for both variations of Mines_3. We give the proof for the omnidirectional case as it is more informative.

Theorem 4. *Player 1 has a winning strategy for Mines_3.*

Proof. In the first three moves of the game we have $|X| = 2$ and $|Y| = 1$. Therefore, Player 1 can guarantee there exists $q, r \in \{1, 4, 6\}$ such that $x_q, x_r \in X$.

By rotating the game board 120° to the left or right, we can, without loss of generality, assume $x_4, x_6 \in X$. At this point in the game $\{y_1, y_2, y_3\} \not\subseteq Y$ and $\{x_1, x_2, x_3\} \not\subseteq X$ since $|X| + |Y| = 3$. On the next move Player 1 plays position 1, 2 or 3, whichever is available. Player 1 wins in direction D_1 since $\triangle_{3,1}(X) \neq \emptyset$. Thus, Player 1 only needs to win in one other direction to win the game. If $x_1 \in X$ then, $\{x_1, x_4, x_6\} = X$ and $\triangle_{3,2}(X) \neq \emptyset$. If $x_2 \in X$, then $\{x_2, x_4, x_6\} = X$ and $\triangle_{3,3}(X) \neq \emptyset$. If $x_3 \in X$, then $\{x_3, x_4, x_6\} = X$ and $\triangle_{3,2}(X) \neq \emptyset$. Thus, in any case, Player 1 wins in at least two out of the three directions. □

4.1. The Game of $\text{Mines}_m(p, q, k)$

Let p, q, m and k be positive integers with $k \leq p, q \leq m$. The game $\text{Mines}_m(p, q, k)$ is played on a game board of size $\triangle_m$ with two players each assigned one of two markings and/or colors. Players have alternating turns in which they may choose to mark one position, in this case positions are the $\triangle_k$'s on the game board, or not mark a position. If $p = q = n$ then we use the notation $\text{Mines}_m(n, k)$. If $p = q = n$ and $k = 1$ then we use the notation $\text{Mines}_m(n)$. In this notation, $\text{Mines}_3 = \text{Mines}_3(2) = \text{Mines}_3(2, 1) = \text{Mines}_3(2, 2, 1)$ since the positions played by both players in Mines_3 are the $\triangle_1$'s on the game board.

We again let $X \subseteq \triangle_k$ denote the set of moves made by Player 1 and $Y \subseteq \triangle_k$ the set of moves made by Player 2. At any point in the game we have $X \cap Y = \emptyset$. If both players always choose to mark a position then given that Player 1 goes first we also have $|X| = |Y| - 1$ or $|X| = |Y|$, and $|X| + |Y| = \begin{bmatrix} m \\ k \end{bmatrix}$ when all positions have been played. Player 1 *wins* if they construct a $Z \in \triangle_p$ such that $\triangle_k(Z) \subseteq X$ before the second player is able to construct a $Z \in \triangle_q$ such that $\triangle_k(Z) \subseteq Y$. Player 2 *wins* if they construct a $Z \in \triangle_q$ such that $\triangle_k(Z) \subseteq Y$ before the first player is able to construct a $Z \in \triangle_p$ such that $\triangle_k(Z) \subseteq X$. A *draw* occurs if all positions have been played but neither player has won the game. That is, $|X| + |Y| = \begin{bmatrix} m \\ k \end{bmatrix}$ and for all $Z \in \triangle_n$, $\triangle_k(Z) \not\subseteq X$ and $\triangle_k(Z) \not\subseteq Y$. The game ends when all positions have been played or one of the players wins.

For some small game boards it is unnecessary to keep track of which player first constructs the winning triangular set. Instead the players can simply fill out the game board completely, then check to see who wins. For example, $\text{Mines}_5(3) = \text{Mines}_5(3, 3, 1)$ has this property. Figure 10 gives two examples of game boards for $\text{Mines}_5(n)$.

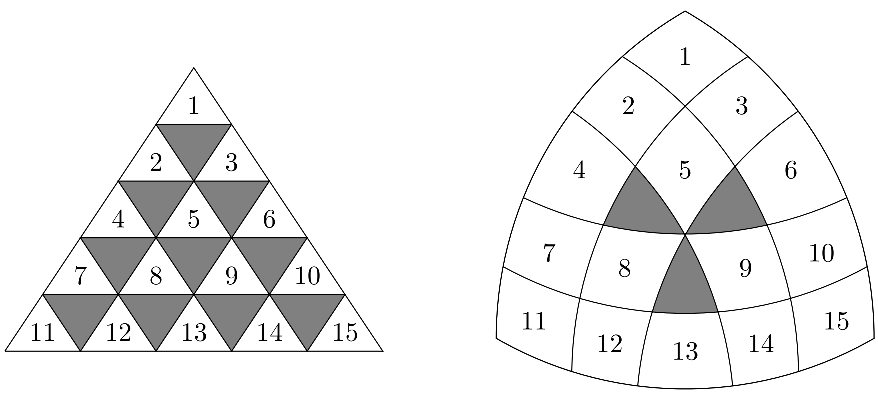

Figure 10: Two example game boards of size $\triangle_5$

Theorem 5. *In the game of Mines*$_5(3)$*, both players cannot construct a* $\triangle_3$*. That is, it is impossible for both* $\triangle_3(X) \neq \emptyset$ *and* $\triangle_3(Y) \neq \emptyset$*.*

Proof. Assume this is not the case. There exists a situation such that $\triangle_3(X) \neq \emptyset$ and $\triangle_3(Y) \neq \emptyset$. The largest level of a $\triangle_5$ has 5 positions, and the largest level of a $\triangle_3$ has 3 positions. Therefore, both X and Y cannot construct their $\triangle_3$'s largest level in the same row of the game board. By the pigeonhole principle there exist $a, b, c \in \{11, 12, 13, 14, 15\}$ such that $x_a, x_b, x_c \in X$ or $y_a, y_b, y_c \in Y$. Suppose that $x_a, x_b, x_c \in X$. We have the following two cases:

Case 1: There exist $d, e, f \in \{7, 8, 9, 10\}$ such that $y_d, y_e, y_f \in Y$. Since $\triangle_3(Y) \neq \emptyset$, either there exist $g, h \in \{4, 5, 6\}$ such that $y_g, y_h \in Y$ or $y_2, y_3 \in Y$. If $g, h \in \{4, 5, 6\}$, then $x_2, x_3 \in X$. If $y_2, y_3 \in Y$, then there exist $i, j \in \{4, 5, 6\}$ such that

$x_i, x_j \in X$. In order for $\triangle_3(X) \neq \emptyset$ we must have $x_1 \in X$ which causes $\triangle_3(Y) = \emptyset$, a contradiction.

Case 2: There exist $d, e, f \in \{4, 5, 6\}$ such that $y_d, y_e, y_f \in Y$. Then either there exist $g, h \in \{7, 8, 9, 10\}$ such that $x_g, x_h \in X$ or $x_2, x_3 \in X$. If $x_2, x_3 \in X$, then clearly $\triangle_3(Y) \neq \emptyset$. If there exist $g, h \in \{7, 8, 9, 10\}$ such that $x_g, x_h \in X$, then we must have $y_2, y_3 \in Y$. In order for $\triangle_3(X) \neq \emptyset$ we must have $x_1 \in X$ which causes $\triangle_3(Y) = \emptyset$, a contradiction.

If instead $y_a, y_b, y_c \in Y$, then a similar argument gives a contradiction. So it is impossible for both $\triangle_3(X) \neq \emptyset$ and $\triangle_3(Y) \neq \emptyset$. □

The previous theorem can be extended, in the game of $\text{Mines}_{2n-1}(n)$ both players cannot construct a $\triangle_n$. That is, it is impossible for both $\triangle_n(X) \neq \emptyset$ and $\triangle_n(Y) \neq \emptyset$. We leave the proof to the interested reader.

5. Triangular Ramsey Numbers

The Finite Ramsey Theorem for $\mathcal{R}_1$, which follows from the work of Dobrinen and Todorcevic in [3], can be used to show that for all $p, q, k \in \mathbb{N}$ with $k \leq p, q$ there is a game board of size $\triangle_m$ with $m \geq p, q$ such that $\text{Mines}_m(p, q, k)$ never ends in a draw.

Problem 3. Let p, q and k be natural numbers such that $k \leq p, q$. Find the smallest natural number $m \geq p, q$ such that $\text{Mines}_m(p, q, k)$ never ends in a draw.

The solution to the problem when $p, q = 2$ and $k = 1$ is $m = 3$. The solution to the problem for the natural numbers p, q and k is called the *triangular Ramsey number for p, q and k* and denoted by $\mathcal{R}_1(p, q, k)$. If $p = q = n$ then we denote $\mathcal{R}_1(p, q, k)$ by $\mathcal{R}_1(n, k)$.

Lemma 2. *For all natural numbers p and q, $\mathcal{R}_1(p, q, 1) > p + q - 2$.*

Proof. Suppose a game of $\text{Mines}_{p+q-2}(p, q, 1)$ is to be played. It is possible for the bottom row, of size $p + q - 2$, to contain $p - 1$ elements in X and the remaining $q - 1$ elements in Y. Note that neither player has constructed the bottom row of a winning triangle in the bottom row. Since the rows decrease in size as players move up the triangle, it is possible for both players to fail to construct the bottom row of their winning triangle anywhere on the board. Therefore, $\mathcal{R}_1(p, q, 1) > p + q - 2$. □

Theorem 6. *For all natural numbers p and q, $\mathcal{R}_1(p, q, 1) = p + q - 1$. In particular, for all numbers n, $\mathcal{R}_1(n, 1) = 2n - 1$.*

Proof. By the previous lemma, $\mathcal{R}_1(p, q, 1) > p + q - 2$. The result follows by showing via induction on $p + q$ that $\mathcal{R}_1(p, q, 1) \leq p + q - 1$. The base case occurs when $p = q = 1$, i.e., $p + q = 2$. The base case is trivial; the first person to color an element creates a complete $\triangle_1$ and wins. In other words, $\mathcal{R}_1(1, 1, 1) = 1$.

Now, suppose $p + q = n + 1$ and the results holds when $p + q = n$. Note that $(p - 1) + q = n$ and $p + (q - 1) = n$. Thus by the inductive hypothesis, $\mathcal{R}_1(p - 1, q) = \mathcal{R}_1(p, q - 1) \leq p + q - 2$. We can now prove by contradiction that no game of $\text{Mines}_{p+q-1}(p, q, 1)$ ends in a draw. Towards a contradiction, suppose a full game of $\text{Mines}_{p+q-1}(p, q, 1)$ has been played on a board of height n and ends in a draw, i.e., $\triangle_p(X) = \emptyset$ and $\triangle_q(Y) = \emptyset$. Since $p + q - 1 = n$ and the bottom level of the game board contains n positions, the pigeon hole principle mandates that the bottom row of the game board must contain either p elements in X, considered *Case 1*, or q elements in Y, considered *Case 2*.

Case 1: The bottom row of the game board contains Z', a set of p elements in X. By the equation above, $\mathcal{R}_1(p - 1, q) = p + q - 1$, and the inductive hypothesis, we can see that there exists either $Z \in \triangle_{p-1}(X) \neq \emptyset$ or $\zeta \in \triangle_q(Y) \neq \emptyset$. If the first is the case, then $Z \cup Z' \in \triangle_p(X)$ and the game is won by Player 1. If the second is the case, then the game is won by Player 2. In either, we have a contradiction.

Case 2: The bottom row of the game board contains q elements in Y. By a similar method to that used above, we can show that either $Z \in \triangle_{q-1}(Y) \neq \emptyset$ or $\zeta \in \triangle_p(X)$ exists. In either situation, a fully colored triangle is made, and we have a contradiction.

In either case, we obtain a contradiction. Therefore, $\mathcal{R}_1(p, q, 1) \leq p + q - 1$ when $p + q = n + 1$. □

Next, we define a sequence $(M_{n,k})$ which we use to establish upper bounds for triangular Ramsey numbers. We let $R(n, k)$ denote the smallest number of vertices of a complete graph such that, for any coloring of its complete subgraphs with k vertices with two colors, there exists a complete subgraph with n vertices where the coloring is monochromatic. The existence of these Ramsey numbers also follows from Ramsey's theorem and could also be introduced by generalizing the game of Tri. Here we let $R^l(n, k)$ denote $\underbrace{R(R(\cdots(R}_{l-times}(n, k), k)\cdots), k)$.

Theorem 7. *Let $(M_{n,k})$ be the sequence recursively defined by*

$$\begin{cases} M_{n,k} = n & \text{if } k = 1, \\ M_{n,k} = n & \text{if } n = k, \\ M_{n+1,k} = R^{\left\lceil \frac{\mathcal{R}_1(M_{n,k},k-1)}{k-1} \right\rceil}(n+1, k) & \text{if } n > k > 1. \end{cases}$$

If $n \geq k$, then $\mathcal{R}_1(n, k) \leq M_{2n-1,k}$.

Proof. We begin by establishing a simpler result by induction on n.

Claim 1. *Suppose that a game of Mines*$_{M_{n,k}}(n,k)$ *is played to completion. There exists a* $Z \in \triangle_n$ *on the game board such that given any level* i *of* Z*, either all* $\triangle_k$*'s contained in* Z *whose last level is contained in the* i^{th} *level of* Z *are played by Player 1 or all* $\triangle_k$*'s contained in* Z *whose last level is contained in the* i^{th} *level of* Z *are played by Player 2.*

Proof of Claim 1. First note that by the previous theorem $\mathcal{R}_1(n,1) = 2n-1 = M_{2n-1,1}$. Thus, the claim holds when $k=1$. Next, fix $k \geq 2$. Consider the base case when $n=k$ and $M_{n,k}=n$. If a game of Mines$_n(n,k)$ has been played to completion, then since there is only one playable position on the game board, the base case holds trivially.

Assume that the claim holds for $M_{n,k}$. Suppose that a game of Mines$_{M_{n+1,k}}(n+1,k)$ has been played to completion. As usual, let X denote the moves made by Player 1 and Y denote those made by Player 2. For each element $Z \in \triangle_{k-1}$ in the first $\mathcal{R}_1(M_{n,k},k-1)$ levels of the game board, we play a game of Tri$_{M_{n+1,k}}(n+1,k)$ on the final level of the game board as follows: Player 1 plays the k-element set $\{i_1, i_2, \ldots i_k\}$ if $Z \cup \{i_1, i_2, \ldots, i_k\} \in X$ and Player 2 plays the k-element set $\{i_1, i_2, \ldots i_k\}$ if $Z \cup \{i_1, i_2, \ldots, i_k\} \in Y$. By an argument similar to the proof of Lemma 1, there exists an $(n+1)$-element set in the last level of the game board $\{z_1, z_2, \ldots z_{n+1}\}$ such that for all $Z \in \triangle_{k-1}$ in the first $\mathcal{R}_1(M_{n,k},k-1)$ levels of the game board either (†) for all $i_1, i_2, \ldots i_k \in \{z_1, z_2, \ldots z_{n+1}\}$, $Z \cup \{i_1, i_2, \ldots i_k\} \in X$ or (‡) for all $i_1, i_2, \ldots i_k \in \{z_1, z_2, \ldots z_{n+1}\}$, $Z \cup \{i_1, i_2, \ldots i_k\} \in Y$.

Next, consider the following hypothetical game of Mines$_{\mathcal{R}_1(M_{n,k},k-1)}(M_{n,k},k-1)$ played on the first $\mathcal{R}_1(M_{n,k},k-1)$ levels of the our original game board. Let $\bar{X}$ denote the moves made by Player 1 and $\bar{Y}$ denote those made by Player 2. In this game, Player 1 plays position $Z \in \triangle_{k-1}$ if (†) holds, and Player 2 plays position $Z \in \triangle_{k-1}$ if (‡) holds. By definition, this game does not end in a draw. If Player 1 wins this game, then there exists $W \in \triangle_{M_{n,k}}$ such that all $\triangle_k$'s whose first $k-1$ levels are in W and whose last level is contained in $\{z_1, z_2, \ldots z_{n+1}\}$ are played by Player 1. If Player 2 wins this game, then there exists $W \in \triangle_{M_{n,k}}$ such that all $\triangle_k$'s whose first $k-1$ levels are in W and whose last level is contained in $\{z_1, z_2, \ldots z_{n+1}\}$ are played by Player 2.

By the induction hypothesis there exists a $Z' \in \triangle_n(W)$ such that given any level i of Z', either all $\triangle_k$'s contained in Z' whose last level is contained in the i^{th} level of Z' are played by Player 1, or all $\triangle_k$'s contained in Z' whose last level is contained in the i^{th} level of Z' are played by Player 2.

Let $Z'' = Z' \cup \{z_1, z_2, \ldots z_{n+1}\} \in \triangle_{n+1}$. Then, either all $\triangle_2$'s contained in Z'' whose last level is contained in the i^{th} level of Z'' are played by Player 1, or all $\triangle_2$'s contained in Z'' whose last level is contained in the i^{th} level of Z'' are played by Player 2. Therefore, the claim holds by induction. □

To prove the inequality, assume toward a contradiction that a game of Mines$_{M_{2n-1,k}}$

ends in a draw. By the previous claim, there exists a $Z \in \triangle_{2n-1}$ on the game board such that given any level i of Z, either all $\triangle_k$'s contained in Z whose last level is contained in the i^{th} level of Z are played by Player 1 or all $\triangle_k$'s contained in Z whose last level is contained in the i^{th} level of Z are played by Player 2. By the pigeonhole principle there are either at least n levels where Player 1 plays all $\triangle_k$'s or at least n levels where Player 2 plays all $\triangle_k$'s. If there are at least n levels where Player 1 wins, then any $W \in \triangle_n$ whose levels come from these n levels witnesses a win for Player 1, a contradiction. Similarly, if there are at least n levels where Player 2 wins, then any $W \in \triangle_n$ whose levels come from these n levels witnesses a win for Player 2, a contradiction. Therefore, this game could not have ended in a draw. □

When $n = k+1$ the previous proof can be simplified and we obtain smaller upper bounds. In fact, in this special case, induction on n is unnecessary.

Theorem 8. *Suppose that $k \geq 2$. Then*

$$\mathcal{R}_1(k+1,k) \leq R^{\left[\mathcal{R}_1(k+1,k-1) \atop k-1\right]}(k+1,k).$$

Proof. Let $M = R^{\left[\mathcal{R}_1(k+1,k-1) \atop k-1\right]}(k+1,k)$. Toward a contradiction, suppose that a game of $\text{Mines}_M(k+1,k)$ ends in a draw. As usual, let X denote the moves made by Player 1 and Y denote those made by Player 2. For each element $Z \in \triangle_{k-1}$ in the first $\mathcal{R}_1(k+1,k-1)$ levels of the game board, we play a game of $\text{Tri}_M(k+1,k)$ on the final level of the game board as follows: Player 1 plays the k-element set $\{i_1, i_2, \ldots i_k\}$ if $Z \cup \{i_1, i_2, \ldots, i_k\} \in X$ and Player 2 plays the k-element set $\{i_1, i_2, \ldots i_k\}$ if $Z \cup \{i_1, i_2, \ldots, i_k\} \in Y$. By Lemma 1, there exists a $(k+1)$-element set in the last level of the game board $\{z_1, z_2, \ldots z_{k+1}\}$ such that for all $Z \in \triangle_{k-1}$ in the first $\mathcal{R}_1(k+1,k-1)$ levels of the game board, either (†) for all $i_1, i_2, \ldots i_k \in \{z_1, z_2, \ldots z_{k+1}\}$, $Z \cup \{i_1, i_2, \ldots i_k\} \in X$ or (‡) for all $i_1, i_2, \ldots i_k \in \{z_1, z_2, \ldots z_{k+1}\}$, $Z \cup \{i_1, i_2, \ldots i_k\} \in Y$.

Next, consider the following hypothetical game of $\text{Mines}_{\mathcal{R}_1(k+1,k-1)}(k+1,k-1))$. Let $\bar{X}$ denote the moves made by Player 1 and $\bar{Y}$ denote those made by Player 2. In this game, Player 1 plays position Z if (†) holds and Player 2 plays position Z if (‡) holds. In other words, $\bar{X} = \{Z \in \triangle_{k-1} : Z \cup \{i_1, i_2, \ldots i_k\} \in X\}$ and $\bar{Y} = \{Z \in \triangle_{k-1} : Z \cup \{i_1, i_2, \ldots i_k\} \in Y\}$. By definition, this game does not end in a draw.

Suppose that Player 1 wins and let W be some element of $\triangle_{k+1}$ witnessing a win for Player 1 (in our hypothetical game of $\text{Mines}_{\mathcal{R}_1(k+1,k-1)}(k+1,k-1))$. Note that not all $\triangle_k's$ in W are played by Player 2 (in our original game) because otherwise the game would not have ended in a draw. So, without loss of generality, we can assume that there is at least one $W' \in \triangle_k(W)$ played by Player 1 (in the original game). However, this is a contradiction because $W' \cup \{z_1, z_2, \ldots z_{k+1}\}$ then witnesses a win for Player 1 (in our original game). If instead Player 2 wins, we can let $\bar{Z}$ be some element of $\triangle_3(\bar{Y})$ and we obtain a similar contradiction. □

5.1. The Probabilistic Method

The work in this section follows closely from the probabilistic method described by Erdös [6]. In our case, we apply it to a randomly played game of Mines.

Theorem 9. *Let p, q, k and m be natural numbers such that $k \leq p \leq q \leq m$. If $\begin{bmatrix} m \\ p \end{bmatrix} \cdot 2^{-\begin{bmatrix} p \\ k \end{bmatrix}} + \begin{bmatrix} m \\ q \end{bmatrix} \cdot 2^{-\begin{bmatrix} q \\ k \end{bmatrix}} < 1$, then $\mathcal{R}_1(p, q, k) > m$.*

Proof. Let p, q, k and m be given. To use the probabilistic method, we consider two players playing a game of $\text{Mines}_m(p, q, k)$ with a fair coin. The players pick a position and flip the coin to see if they will play that triangular set or skip their turn. The game is then played to completion using these random moves.

Consider the random variables X and Y where X counts the number of winning $\triangle_p$'s colored by Player 1 and Y the number of winning $\triangle_q$'s colored by Player 2. By Theorem 1, the probability that a randomly chosen $\triangle_p$ from the game board witnesses a win for Player 1 is $2^{-\begin{bmatrix} p \\ k \end{bmatrix}}$ and for a randomly chosen $\triangle_q$ the probability that it witnesses a win for Player 2 is $2^{-\begin{bmatrix} q \\ k \end{bmatrix}}$. By Theorem 1, there are exactly $\begin{bmatrix} m \\ p \end{bmatrix}$ possible $\triangle_p$'s on the game board and $\begin{bmatrix} m \\ q \end{bmatrix}$ possible $\triangle_q$'s. Thus, $\begin{bmatrix} m \\ p \end{bmatrix} \cdot 2^{1-\begin{bmatrix} p \\ k \end{bmatrix}} = E[X]$ and $\begin{bmatrix} m \\ q \end{bmatrix} \cdot 2^{1-\begin{bmatrix} q \\ k \end{bmatrix}} = E[Y]$. Hence, $E[X+Y] = E[X]+E[Y] = \begin{bmatrix} m \\ p \end{bmatrix} \cdot 2^{-\begin{bmatrix} p \\ k \end{bmatrix}} + \begin{bmatrix} m \\ q \end{bmatrix} \cdot 2^{-\begin{bmatrix} q \\ k \end{bmatrix}} < 1$. Since the expected value of $X + Y$ is less than one there exists some game of $\text{Mines}_m(p, q, k)$ that ends in a draw. Therefore, $m < \mathcal{R}_1(p, q, k)$. □

The previous theorem can be used to find lower bounds for small values of p, q and k by searching for the largest value of m that satisfies the inequality. A summary of these values for small p, q and k can be found in Figure 11. The next theorem provides asymptotic estimates for large values of p, q and k with $p = q = n$ which don't require the computation of $\begin{bmatrix} n \\ k \end{bmatrix}$ nor $\begin{bmatrix} m \\ n \end{bmatrix}$ for any m.

Theorem 10. *For all natural numbers n and k with $n \leq k$,*

$$\mathcal{R}_1(n, k) \geq (2\pi T_n)^{\frac{1}{4T_n}} \cdot \sqrt{\frac{2T_n}{e}} \cdot 2^{\frac{n^k - k^k}{2k^k T_n}} - 1.$$

Proof. Let $m = \mathcal{R}_1(n, k)$. By the previous theorem, we must have $\begin{bmatrix} m \\ n \end{bmatrix} 2^{-\begin{bmatrix} n \\ k \end{bmatrix}+1} \geq 1$. Now we also know that

$$\begin{bmatrix} m \\ n \end{bmatrix} \leq \binom{T_m}{T_n}.$$

This is because if we are counting $\triangle_n$'s (which has T_n points) in a $\triangle_m$ (which has T_m points) we cannot exceed the number of ways of choosing T_n points arbitrarily from T_m points. Therefore, we are counting some T_n sized subset of T_m points with certain properties. We know that $\binom{m}{n} \leq \frac{m^n}{n!}$. Substituting in these inequalities gives,

$$\frac{T_m^{T_n}}{T_n!} \cdot 2^{-\left[{n \atop k}\right]+1} \geq 1$$

$$\frac{(m(m+1))^{T_n}}{T_n!} \cdot 2^{-\left[{n \atop k}\right]+1-T_n} \geq 1.$$

Then by Sterling's formula since $n! > \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$ we have

$$\frac{(m(m+1))^{T_n}}{\sqrt{2\pi T_n} \cdot T_n^{T_n} \cdot e^{-T_n}} \cdot 2^{-\left[{n \atop k}\right]+1-T_n} \geq 1$$

$$\frac{(m+1)^{2T_n}}{\sqrt{2\pi T_n} \cdot T_n^{T_n} \cdot e^{-T_n}} \cdot 2^{-\left[{n \atop k}\right]+1-T_n} \geq 1$$

$$(m+1)^{2T_n} \geq \sqrt{2\pi T_n} \cdot \left(\frac{T_n}{e}\right)^{T_n} \cdot 2^{\left[{n \atop k}\right]+T_n-1}.$$

From Corollary 1, $\left[{n \atop k}\right] = \sum_{0<i_1<\ldots<i_k\leq n} \left(\prod_{j=1}^{k} \binom{i_j}{j} \right)$. The smallest product in this summation is $\binom{1}{1}\binom{2}{2}\cdots\binom{n}{n} = 1$ since there are $\binom{n}{k}$ terms in the sum $\left[{n \atop k}\right] \geq \binom{n}{k} \geq \left(\frac{n}{k}\right)^k$. So

$$(m+1)^{2T_n} \geq \sqrt{2\pi T_n} \cdot \left(\frac{T_n}{e}\right)^{T_n} \cdot 2^{\left(\frac{n}{k}\right)^k+T_n-1}.$$

Isolating m in the previous inequality and simplifying gives the result. □

6. Conclusion

Recall that the existence of triangular Ramsey numbers follows from the work of Dobrinen and Todorcevic. Our primary purpose for introducing the game of Mines was to provide a simplified presentation of the finite-dimensional Ramsey theory of the infinite-dimensional topological Ramsey space $\mathcal{R}_1$ introduced and studied by Dobrinen and Todorcevic. The next table summarizes the main results for small values of p, q and k. Note that $\mathcal{R}_1(p, q, k) = \mathcal{R}_1(q, p, k)$ so we only give values for $p \leq q$.

These upper and lower bounds have applications to characterizing the Dedekind cuts in nonstandard models of arithmetic that arise from ultrafilter mapping that are associated to the space $\mathcal{R}_1$ introduced by Dobrinen and Todorcevic in [3]. In [1], Blass uses upper and lower bounds for Ramsey numbers to characterize, under the continuum hypothesis, the Dedekind cuts that can be associated to ultrafilter mappings from Ramsey and weakly-Ramsey ultrafilters. Trujillo in [19], characterizes

p	q	k	Lower Bound	$\mathcal{R}_1(p,q,k)$	Upper Bound	Result
1	2	1	–	2	–	Thm. 6
2	2	1	–	3	–	Thm. 6
2	3	1	–	4	–	Thm. 6
3	3	1	–	5	–	Thm. 6
2	2	2	–	2	–	trivial
2	3	2	–	3	–	trivial
3	3	2	6	?	$R^{15}(3)$	Thm. 8
3	4	2	6	?	$M_{7,2}$	Thm. 7&Thm. 9
4	4	2	25	?	$M_{7,2}$	Thm. 7&Thm. 9
3	3	3	–	3	–	trivial
3	4	3	–	4	–	trivial
4	4	3	20	?	$R^{\lceil \frac{M_{7,2}}{2} \rceil}(4,3)$	Thm. 8&Thm. 9
4	5	3	20	?	$M_{9,3}$	Thm. 7&Thm. 9
5	5	3	9.39×10^7	?	$M_{9,3}$	Thm. 7&Thm. 9
4	4	4	–	4	–	trivial
4	5	4	–	5	–	trivial
5	5	4	3425	?	$R^{\lceil \frac{M_{9,3}}{3} \rceil}(5,4)$	Thm. 8&Thm. 9
30	30	20	221	?	$M_{59,20}$	Thm. 7&Thm. 10
35	35	20	4.70^{18}	?	$M_{69,20}$	Thm. 7&Thm. 10
40	40	20	7.29×10^{193}	?	$M_{79,20}$	Thm. 7&Thm. 10

Figure 11: Table of some triangular Ramsey numbers.

the Dedekind cuts that can be associated to ultrafilter mappings among ultrafilters within the Tukey-type of a Ramsey for $\mathcal{R}_1$ ultrafilter. This motivates the following open problems:

Problem 4. Use upper and lower bounds for triangular Ramsey numbers to characterize the Dedekind cuts that can be associated to ultrafilter mappings from Ramsey for $\mathcal{R}_1$ ultrafilters.

In a followup paper, Dobrinen and Todorcevic [4] introduce a hierarchy of spaces $\mathcal{R}_\alpha$ for $\alpha < \omega$ that extend the space $\mathcal{R}_1$.

Problem 5. Introduce a combinatorial game, similar to Mines, that provides a simplified presentation of the finite-dimensional Ramsey theory of the infinite-dimensional topological Ramsey spaces $\mathcal{R}_\alpha$ for $\alpha < \omega_1$ defined and studied by Dobrinen and Todorcevic in [4]. Then find upper and lower bounds for Ramsey numbers based on these games.

Upper and lower bounds for the Ramsey numbers associated to the spaces $\mathcal{R}_\alpha$, $\alpha > 1$, also have similar applications to characterizing the Dedekind cuts that can be associated to ultrafilter mappings from Ramsey for $\mathcal{R}_\alpha$ ultrafilters. In addition to these open problems, there are problems still open related to playing the game of Mines.

Theorem 4 provides an explicit description of a winning strategy for Mines_3. The proof of Zermelo's theorem in [11] can be adapted to show that either Player 1 or Player 2 must have a winning strategy for any game of $\text{Mines}_m(n, k)$ where $m \geq \mathcal{R}_1(n, k)$. A standard strategy stealing argument can then be used to show that Player 1 must have a winning strategy for the game. However, the proof of Zermelo's theorem does not provide for an explicit description of how Player 1 should play to win the game. In the game of $\text{Mines}_5(3, 1)$, Player 1 has the opening move allowing them to guarantee two of the three corners (assuming this is still the optimal strategy with larger game boards) which could give them the win. However, Player 2 has the ability to react to Player 1's moves and possibly prevent them from forming a $\triangle_3$. In addition, some game boards will have an odd number of positions giving Player 1 an additional position over Player 2; perhaps this gives them an even bigger advantage. Regardless, giving an explicit description of the winning strategy for a game board of size $\triangle_m$ seems like a difficult problem.

Problem 6. Find an explicit description of the winning strategy for Player 1 in a game of $\text{Mines}_m(n, k)$ where $m \geq \mathcal{R}_1(n, k)$. In particular, describe the winning strategy for Player 1 in a game of $\text{Mines}_5(3, 1)$.

The complexity of Mines increases as the number of players is increased from 2 players to k players. In this variation, we have a different set of Ramsey numbers. It is evident that as the number of players increases, the size of the associated triangular Ramsey numbers also increase.

Problem 7. How does the game of Mines change when adding more players? In particular, find upper and lower bounds for triangular Ramsey numbers for variations of Mines with more than two players.

In the off-diagonal case of the game $\text{Mines}_m(p, q, 1)$, if $p < q$, then Player 1 has an advantage as they are now constructing a smaller triangular set. Player 2 can be given an advantage in this game by allowing them to play more than one $\triangle_1$ on each turn. The question then becomes how many $\triangle_1$s should Player 2 be allowed to color per turn such that the game is fair? Note that, if $m = p + q - 1$ and we let Player 2 play q $\triangle_1$s per turn, then they clearly have a winning strategy provided that $p > 1$.

Problem 8. Suppose p and q are given with $1 < p < q$. What is the smallest number of $\triangle_1$s Player 2 can be allowed to color per turn such that Player 2 has the winning strategy?

In addition to simplifying the finite Ramsey theory of the Ramsey space $\mathcal{R}_1$, a secondary goal was to introduce a game that was simple enough to played by young children. In this way, the game can be played with the intention of developing logical skill. In [9], the authors give evidence that the game of Tri can be used to develop logical thinking skills in young children. Our final problem will be of interest mainly to researchers in math education.

Problem 9. Give concrete evidence that the game Mines can be played by young children and can be used to develop their visual disembedding skills.

References

[1] A. Blass, Ultrafilter mappings and their Dedekind cuts, *Trans. Amer. Math. Soc.*, 327-340, (1974), 327-340.

[2] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.*, (2009), 941-960.

[3] N. Dobrinen and S. Todorcevic, A new class of Ramsey-classification theorems and their application in the Tukey theory of ultrafilters, Part 1, *Trans. Amer. Math. Soc.*, **7** (2015), 4627–4659.

[4] N. Dobrinen and S. Todorcevic, A new class of Ramsey-classification theorems and their application in the Tukey theory of ultrafilters, Part 2, *Trans. Amer. Math. Soc.*, **367(7)** (2015), 4627-4659.

[5] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compo. Math.*, **2** (1935), 463-470.

[6] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.*, **53(4)** (1947), 292-294.

[7] G. Exoo, A lower bound for $R(5, 5)$, *J. Graph Theory*, **13(1)** (1989), 97-8.

[8] R. Greenwood and A. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. of Math.*, **7(1)** (1955), 7.

[9] G. Haggard and A. Schonberger, The game of Tri, *Arithmetic Teacher*, (1977), 318-320.

[10] J. Kalbfleisch, Construction of special edge-chromatic graphs, *Canad. Math. Bull.*, **8** (1965), 575-584.

[11] Y. Khomskii, *Infinite Games*, Summer course at the University of Sofia, Bulgaria, 2010.

[12] C. Laflamme, Forcing with filters and complete combinatorics, *Ann. of Pure Appl. Logic*, **42(2)** (1989), 125-163.

[13] J. Mackey, *Combinatorial Remedies*, Ph.D. thesis, Department of Mathematics, University of Hawaii, 1994.

[14] B. McKay and S. Radziszowski, Subgraph counting identities and Ramsey numbers, *J. Comb. Theory Ser. B*, **69(2)** (1997), 193-209.

[15] F. Ramsey, On a problem of formal logic, *Proc. Lond. Math. Soc. (2)* , **1** (1930), 264-286.

[16] J. Shearer, Lower bounds for small diagonal Ramsey numbers, *J. Comb. Theory Ser. A*, **42(2)** (1986), 302-304.

[17] G. Simmons, The game of SIM, *Mathematical Solitaires & Games*, (1980), 50.

[18] J. Spencer, Ramsey's theorem - a new lower bound, *J. Combin. Theory Ser. A*, **18(1)** (1975), 108-115.

[19] T. Trujillo, Ramsey for R1 ultrafilter mappings and their Dedekind cuts, *MLQ Math. Log. Q.*, **61(4-5)** (2015), 263-273.