INVARIANT MEASURES FOR NEW CLASSES OF PIECEWISE FRACTIONAL LINEAR MAPS

Fritz Schweiger
Department of Mathematics, University of Salzburg, Salzburg, Austria
fritz.schweiger@sbg.ac.at

Received: 4/16/18, Revised: 12/4/18, Accepted: 4/25/19, Published: 6/3/19

Abstract

This paper continues the investigations of invariant measures for the so-called Moebius maps T. Various modifications of these maps are considered. The left-most and the right-most branch do not map their domain onto the interval $[0,1]$, but since $T0 = T1$, the union of their images is $[0,1]$.

1. Introduction

In recent years the ergodic theory of non-invertible maps has been of interest. A possible frame work for these maps is the concept of *fibred systems* [8]. A map $T : B \to B$ is called a *fibred system* if there is a partition $\{B(k) : k \in I\}$ of B with the index set I that is finite or countable, such that the restriction of T to $B(k)$ is injective. This restriction is called a *branch* of T. The inverse map of a branch is denoted by V_k and its Jacobian by ω_k. A short account of the historical development of the related f-expansions can be found in [5].

A lot of work has already done on the ergodicity and the existence of invariant measures for such maps. Note that the existence of a finite invariant measure can be used to calculate probabilities by the use of ergodic theorems (see [8]). In many cases the existence of an invariant measure can be proved by the verification of some sufficient conditions. Much less is known about the shape of an invariant density. Lebesgue measure is invariant for g-adic expansions. Due to an old conjecture of Gauss, the invariant density for continued fractions is known, namely $h(x) = \frac{1}{1+x}$. One extension of continued fractions to higher dimensions is the Jacobi-Perron algorithm. For $n = 2$ the following map

$$T(x,y) = \big(\frac{y}{x} - a, \frac{1}{x} - b\big),\ a = a(x) = \lfloor \frac{y}{x} \rfloor,\ b = b(x) = \lfloor \frac{1}{x} \rfloor$$

is ergodic and admits a finite invariant measure, but the shape of the density is not known.

Another tricky example is the Bolyai algorithm which is related to the map

$$T : [1, 2] \to [1, 2],\, Tx = x^2, 1 \le x < \sqrt{2};$$
$$Tx = x^2 - 1, \sqrt{2} \le x < \sqrt{3};\, Tx = x^2 - 2,\, \sqrt{3} \le x < 2.$$

The shape of its invariant density is not known [2, 6]. Furthermore, we mention a/b-expansions [1, 3].

In this note modified Moebius maps are considered (in [7], the name Moebius maps has been proposed for fractional linear maps $T : [0, 1] \to [0, 1]$ with three branches). The left-most and the right-most branch do not map their domain of definition onto $[0, 1]$, but since $T0 = T1$ the union of their images build $[0, 1]$. The graphs of these maps have suggested the name *butterfly maps*.
Important tools are the dual Moebius map and the jump transformation (see [8]). We denote the *(natural) dual map* (or dual algorithm with respect to the kernel $K(x, y) = (1 + xy)^{-2}$) of T by T^*. This is a Moebius map, the branches of which are formed by the transposed matrices. The dual map T^* is called *differentiably isomorphic* to T if there is a bijective map $\psi(t) = \frac{b+dt}{a+bt}$ such that $\psi \circ T = T^* \circ \psi$. Otherwise we call T^* an *exceptional* dual. Since T can be written piecewise by 2×2 -matrices, one uses the matrix equations

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} T = T^* \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

to calculate a map ψ. If the dual map is defined on a set B^*, then the invariant density for T can be written down as

$$h(x) = \int_{B^*} \frac{dy}{(1 + xy)^2}.$$

It can happen that a constant map $\psi(t) = K$ satisfies $\psi \circ T = T^* \circ \psi$. Then $B^* = \{K\}$ and

$$h(x) = \frac{1}{(1 + Kx)^2}.$$

The case $K = 0$ means that Lebesgue measure is invariant.

2. Maps With a Fixed Partition

A point ζ is called a *central fixed point* of the butterfly map if $T\zeta = \zeta$ and $T0 = T1 = \zeta$.

Theorem 1. *The invariant density can be written down in an explicit form for butterfly maps with three branches on a fixed partition and a central fixed point if* *there is polynomial $P(X,Y)$ such that the parameters μ and ν of the map satisfy $P(\mu,\nu)=0$.*

Proof. (1) We first consider specific maps with three decreasing branches and the partition 0, $\frac{1}{4}$, $\frac{1}{2}$, 1. The choice of a fixed partition reduces the number of variables and is sufficient to show the method of proof.

We list the three branches of T

$$Tx = \frac{1-4x}{3+\delta x},\ 0 \le x < \frac{1}{4}$$
$$Tx = \frac{1-2x}{-1+6x},\ \frac{1}{4} \le x < \frac{1}{2}$$
$$Tx = \frac{5+\eta-4x}{3-\eta+4\eta x},\ \frac{1}{2} \le x \le 1$$

and their inverse maps

$$V_0x = \frac{1-3x}{4+\delta x},\ V_1x = \frac{1+x}{2+6x},\ V_2x = \frac{5+\eta+(\eta-3)x}{4+4\eta x}.$$

The essential conditions are

$$T(\frac{1}{3}) = \frac{1}{3}$$

and

$$T0 = T1 = \frac{1}{3}.$$

This means that the central fixed point is $\zeta = \frac{1}{3}$. To avoid poles in the intervals of definition, we require $\delta > -3$ and $\eta > -1$. Note that the condition that $\frac{1}{3}$ is a fixed point implies that the map V_1 is uniquely determined.

We are looking for a map $\psi(t) = \frac{b+dt}{a+bt}$ such that $\psi \circ T = T^* \circ \psi$ holds. This leads to three linear equations in a, b, and d. One finds a non-trivial solution if the condition

$$84 + \delta + 12\eta = \delta\eta$$

is satisfied. The map ψ can be calculated as

$$\psi(t) = \frac{-6+\delta+(42-\delta)t}{6+(\delta-6)t}.$$

If we look at $216+6\delta-\delta^2$, we see that ψ is increasing for $-1 < \delta \le 18$ and decreasing for $\delta > 18$.

The map T^* essentially is of the same type on the interval with endpoints $\psi(0) = \frac{\delta-6}{6}$ and $\psi(1) = \frac{36}{\delta}$. The crucial fixed point is $\psi(\frac{1}{3}) = 2$.

If $\delta \neq 18$ the invariant density has two parts h_I and h_{II}. The function h_I is the density on $[0, \frac{1}{3}]$ and h_{II} the density on $[\frac{1}{3}, 1]$. They satisfy the following conditions:

$$h_I(x) = h_I(V_0x)\omega_0(x) + h_{II}(V_1x)\omega_1(x),\ 0 \leq x < \frac{1}{3},$$

$$h_{II}(x) = h_I(V_1x)\omega_1(x) + h_{II}(V_2x)\omega_2(x),\ \frac{1}{3} \leq x < 1.$$

The map T^* on the interval with endpoints $\psi(0)$ and $\psi(1)$ is a dual algorithm. Let D_I be the interval with endpoints 2 and $\psi(0)$, and D_{II} be the interval with endpoints 2 and $\psi(1)$. Then we see that

$$((T^*)^{-1}D_I \cap B^*(0)) \cup ((T^*)^{-1}D_{II} \cap B^*(1)) = D_I$$

and

$$((T^*)^{-1}D_I \cap B^*(1)) \cup ((T^*)^{-1}D_{II} \cap B^*(2)) = D_{II}.$$

Therefore, the device of the dual algorithm gives the densities as

$$h_I(x) = \int_{D_I} \frac{dy}{(1+xy)^2} \text{ and } h_{II}(x) = \int_{D_{II}} \frac{dy}{(1+xy)^2}.$$

For $\delta = 18$ the set B^* shrinks to the point 2. Therefore the invariant density is given by $h(x) = \frac{1}{(1+2x)^2}$. Note that the second root of $216 + 6\delta - \delta^2 = 0$, namely $\delta = -12$, does not give an allowed value for the map V_0.

(2) Now we look shortly at three increasing branches with the same partition. The essential conditions are again

$$T(\frac{1}{3}) = \frac{1}{3}$$

and

$$T(0) = T(1) = \frac{1}{3}.$$

We find

$$V_0x = \frac{1-3x}{-8-\delta+\delta x},\ V_1x = \frac{1+x}{4},\ V_2x = \frac{1+(3+\eta)x}{2+\eta x}.$$

If $2\eta = \delta$, the map ψ can be calculated as

$$\psi(t) = \frac{\delta - 3\delta x}{-8-\delta+\delta x}.$$

The invariant densities

$$h_I(x) = |\int_0^{\psi(1)} \frac{dy}{(1+xy)^2}|,\ h_{II}(x) = |\int_{\psi(0)}^0 \frac{dy}{(1+xy)^2}|$$

satisfy the invariance conditions

$$h_I(x) = h_I(V_1x)\omega_1(x) + h_{II}(V_2x)\omega_2(x),\ 0 \le x < \frac{1}{3}$$

$$h_{II}(x) = h_I(V_0x)\omega_0(x) + h_{II}(V_1x)\omega_1(x),\ \frac{1}{3} \le x < 1.$$

(3) The first mixture of (1) and (2) is not really interesting. Its inverse branches are given by

$$V_0x = \frac{1-3x}{4+\delta x},\ V_1x = \frac{1+x}{4},\ V_2x = \frac{5+\eta+(\eta-3)x}{4+4\eta x}.$$

One sees that this map has two ergodic components, namely the intervals $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$. The dual algorithm of the first map is given by the map ψ on $[0, \frac{1}{3}]$

$$\psi(t) = \frac{\delta - 3\delta t}{4+\delta t},$$

and the the dual algorithm of the second map is given by by the map ψ on $[\frac{1}{3}, 1]$

$$\psi(t) = \frac{\eta - 3\eta t}{-\eta - 2 + \eta t}.$$

If $4\eta + \eta\delta + 2\delta = 0$ then we obtain the same function ψ.

(4) The second mixture has a little different behavior. Its inverse branches are given by

$$V_0x = \frac{1-3x}{-8+\delta+\delta x},\ V_1x = \frac{1+x}{2+6x},\ V_2x = \frac{1+(3+\eta)x}{2+\eta x}.$$

The map T^2 has two ergodic components, namely the intervals $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$. Under the condition $6 + 2\delta = 3\eta$, the map

$$\psi(t) = \frac{6-\delta+(30+7\delta)t}{6+\delta+(6-\delta)t}$$

allows us to write down an invariant density for T.

We take a short look at T^2 on the interval $[0, \frac{1}{3}]$. It has four branches

$$V_{01}x = \frac{1-3x}{16+\delta+(48+5\delta)x},\ V_{02}x = \frac{1+(9+2\eta)x}{16+\delta+(-3\delta+8\eta)x},$$

$$V_{12}x = \frac{3+(3+2\eta)x}{10+(18+8\eta)x},\ V_{11}x = \frac{3+7x}{10+18x}$$

with the partition 0, $\frac{1}{16+\delta}$, $\frac{1}{4}$, $\frac{3}{10}$, and $\frac{1}{3}$. □

In the proof of theorem 1 we constructed invariant densities for case (1) and (2) with the help of an appropriate function ψ. The next theorem shows that no exceptional dual maps exist.

Theorem 2. *There is no exceptional dual in case (1) or (2) .*

Proof. (1) To construct a dual map on an interval B^*, we have to find points α and β as its endpoints and ζ as a central fixed point such that

$$\alpha = V_0^*\zeta,\ V_0^*\alpha = V_1^*\beta,\ V_1^*\zeta = \zeta,\ V_1^*\alpha = V_2^*\beta,\ V_2^*\zeta = \beta.$$

The map $y \mapsto \frac{6+y}{2+y}$ has two fixed points $\zeta = 2$ and $\zeta = -3$.

(1a) If $\zeta = 2$ then $\alpha = \frac{\delta-6}{6}$ and $\beta = \frac{-3+3\eta}{7+\eta}$. The equation $V_0^*\alpha = V_1^*\beta$ gives

$$\eta\delta = \delta + 12\eta + 84.$$

This is exactly the condition which was given for the existence of a differentiably isomorphic dual. The equation $V_1^*\alpha = V_2^*\beta$ leads to

$$\frac{30+\delta}{6+\delta} = \frac{7\eta^2+16\eta+9}{3\eta^2+16\eta+13} = \frac{(7\eta+9)(\eta+1)}{(3\eta+13)(\eta+1)} = \frac{7\eta+9}{3\eta+13}.$$

This equation gives the same condition. Note that $\eta = -1$ is not an admissible value for the parameter η. Therefore this dual is not exceptional.

(1b) If $\zeta = -3$, then $\alpha = \delta + 9$ and $\beta = \frac{9+\eta}{-11-3\eta}$. The equation $V_1^*\alpha = V_2^*\beta$ leads to

$$27\eta\delta + 356\eta + 83\delta + 1092 = 0.$$

The equation $V_1^*\alpha = V_2^*\beta$ gives the relation

$$\delta\eta + 8\eta = \delta + 24.$$

These equations represent hyperbolas with parallel asymptotic lines. The common points are $(\delta,\eta) = (-12,-3)$ and $(\delta,\eta) = (-\frac{144}{11}, -\frac{15}{7})$ which do not correspond to a suitable map T.

(2) We proceed in a similar way as before. The map $y \mapsto \frac{y}{4+y}$ has two fixed points $\zeta = 0$ and $\zeta = -3$.

(2a) If $\zeta = 0$ then $\alpha = \frac{\delta}{-8-\delta}$ and $\beta = \frac{\eta}{2}$. We first use the equation $V_0^*\beta = V_1^*\alpha$ which leads to the relation

$$2\eta\delta - \delta^2 - 12\delta + 24\eta = (\delta - 2\eta)(-\delta - 12).$$

Since $\delta = -12$ is not allowed we see that $\delta = 2\eta$. The condition $V_1^*\beta = V_2^*\alpha$ implies

$$\delta + 6\delta - 12\eta - 2\eta^2 = (-2\eta + \delta)(\eta + 6) = 0.$$

Therefore we find $\delta = 2\eta$ and this dual is not exceptional.

(2b) The value $\zeta = -3$ gives $\alpha = \frac{\delta+9}{-\delta+11}$ and $\beta = 9 + 2\eta$. $V_0^*\beta = V_1^*\alpha$ gives

$$\delta^2 - 8\eta\delta - 27\delta - 96\eta - 468 = 0,$$

and $V_1^*\beta = V_2^*\alpha$ implies

$$2\eta^2 - 4\eta\delta - 27\eta - 24\delta - 234 = 0.$$

Then

$$\begin{aligned} 0 &= \delta^2 - 8\eta\delta - 27\delta - 96\eta - 468 - 2(2\eta^2 - 4\eta\delta - 27\eta - 24\delta - 234) \\ &= \delta^2 - 4\eta^2 + 21\delta - 42\eta = (\delta - 2\eta)(\delta + 2\eta + 21). \end{aligned}$$

If we insert $\delta = 2\eta$ and $\delta = -2\eta - 21$ into one of the equations, we eventually find the four common points of these hyperbolas: $(-12, -6)$, $(-13, -\frac{13}{2})$, $(-9, -6)$, and $(-12, -\frac{9}{2})$ which do not represent admissible parameters. □

3. Maps with a Variable Partition

In this section we consider a fractional linear map $g(x)$ such that $Tx = g(x) \bmod 1$ is a butterfly map with three branches. The partition of the unit interval then depends on the given map $g(x)$.

Theorem 3. *The invariant density can be written down in an explicit form for butterfly maps with three branches, given by a fractional linear map and a central fixed point.*

Proof. (1) We start with a decreasing map

$$g(x) = \frac{A + Bx}{C + Dx}$$

and the conditions

$$g(1) = \beta,\ g(0) = \beta + 2, 0 < \beta < 1$$

and

$$\frac{A + B\beta}{C + D\beta} - 1 = \beta.$$

Then we choose $C = 1$ and the parameter $\delta = D$. We calculate $A = \beta + 2$, $B = -\frac{\delta+4}{\delta+2}$, and $\beta = \frac{1}{2+\delta}$, which implies $-1 < \delta$. The map $Tx = g(x) \bmod 1$ then is given by the three branches

$$Tx = \frac{1 - (2\delta^2 + 5\delta + 4)x}{2 + \delta + (2\delta + \delta^2)x},\ 0 \le x < \frac{1}{4 + 5\delta + 2\delta^2},$$

$$Tx = \frac{3+\delta-(\delta^2+3\delta+4)x}{2+\delta+(2\delta+\delta^2)x}, \ \frac{1}{4+5\delta+2\delta^2} \le x < \frac{3+\delta}{4+3\delta+\delta^2},$$

$$Tx = \frac{5+2\delta-(\delta+4)x}{2+\delta+(2\delta+\delta^2)x}, \ \frac{3+\delta}{4+3\delta+\delta^2} \le x < 1.$$

In this case, for $\delta > -1$, a map ψ with $\psi \circ T = T^* \circ \psi$ can be found. A calculation shows

$$\psi(t) = \frac{\delta+\delta^2 t}{2\delta+3+\delta t}.$$

Then the invariant density can be calculated in the usual way. Note that β is the central fixed point of T.

(2) We may also start with an increasing map

$$g(x) = \frac{A+Bx}{C+Dx}$$

and the conditions

$$g(0) = \beta, \ g(1) = \beta+2, 0 < \beta < 1$$

and

$$\frac{A+B\beta}{C+D\beta} - 1 = \beta.$$

In a similar way as before we find

$$Tx = \frac{1+(4+7\delta+2\delta^2)x}{2+\delta+(2\delta+\delta^2)x}, \ 0 \le x < \frac{1+\delta}{4+5\delta+\delta^2},$$

$$Tx = \frac{-1-\delta+(4+5\delta+\delta^2)x}{2+\delta+(2\delta+\delta^2)x}, \ \frac{1+\delta}{4+5\delta+\delta^2} \le x < \frac{3+2\delta}{4+3\delta},$$

$$Tx = \frac{-3-2\delta+(4+3\delta)x}{2+\delta+(2\delta+\delta^2)x}, \ \frac{3+2\delta}{4+3\delta} \le x < 1.$$

Again, we find a function ψ with $\psi \circ T = T^* \circ \psi$. It is

$$\psi(t) = \frac{\delta+\delta^2 t}{-2\delta-1+\delta t}$$

which serves to calculate an invariant density. Again, β is the central fixed point of T. □

4. A Further Variation

We consider butterfly maps with three branches, but the point $T0 = T1$ is a point of partition. These three branches will be labeled by the parameters λ, μ, and ν.

Theorem 4. *For butterfly maps with three increasing branches such that the point $T0 = T1$ is a point of partition, an explicit form of the invariant density can be given if the three parameters satisfy $P(\lambda, \mu, \nu) = 0$ for a polynomial $P(X, Y, Z)$.*

Proof. Here we use the partition 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1.

(1) For sake of completeness we first look at the case of decreasing branches. More precisely, let

$$V(\lambda)(\frac{2}{3}) = 0,\ V(\lambda)(0) = \frac{1}{3},$$

$$V(\mu)(1) = \frac{1}{3},\ V(\mu)(0) = \frac{2}{3},$$

$$V(\nu)(1) = \frac{2}{3},\ V(\nu)(\frac{2}{3}) = 1.$$

But this case is not really interesting because the interval $[\frac{2}{3}, 1]$ is an absorbing region.

(2) Therefore we now consider the increasing case:

$$V(\lambda)(\frac{2}{3}) = 0,\ V(\lambda)(1) = \frac{1}{3},$$

$$V(\mu)(0) = \frac{1}{3},\ V(\mu)(1) = \frac{2}{3},$$

$$V(\nu)(0) = \frac{2}{3},\ V(\nu)(\frac{2}{3}) = 1.$$

Then we give these branches as follows:

$$V(\lambda)x = \frac{-2 + 3x}{3 + \lambda - \lambda x},\ V(\mu)x = \frac{1 + (1 + 2\mu)x}{3 + 3\mu x},\ V(\nu)x = \frac{4 + (3 - 2\nu)x}{6 - 2\nu x}.$$

The parameters λ, μ, and ν have to obey some conditions to avoid certain types of fixed points, e. g., the map $V(\mu, \nu)$ has the fixed point $\frac{2}{3}$ which will not affect the existence of an invariant measure on $[0, 1]$ and the fixed point $\frac{5+4\mu}{2\nu-3\mu+2\mu\nu}$. This fixed point must be outside of the interval $[\frac{5+4\mu}{9+6\mu}, \frac{2}{3}[$. The reason for this will become clear in the course of our construction.
If our map admits an invariant density, it has two pieces which satisfy the following equations:

$$(1a)\ \ h_I(x) = h_I(V(\mu)x)\omega(\mu; x) + h_{II}(V(\nu)x)\omega(\nu; x),\ 0 \le x < \frac{2}{3},$$

$$(1b)\ \ h_{II}(x) = h_I(V(\lambda)x)\omega(\lambda; x) + h_I(V(\mu)x)\omega(\mu; x),\ \frac{2}{3} \le x < 1.$$

We find an invariant density by using the device of jump transformations and dual algorithm (see [8]). We introduce the jump transformation S by setting

$$Sx = T^2x,\, x \in B(\lambda,\nu) = B(\lambda),\, Sx = Tx, x \in B(\mu) \cup B(\nu).$$

Let g_I, g_{II} be the invariant density on $[0, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$, respectively. Then we find the equations

$$(2a)\, g_I(x) = g_I(V(\lambda,\nu)x)\omega(\lambda,\nu;x) + g_I(V(\mu)x)\omega(\mu;x) + g_{II}(V(\nu)x)\omega(\nu;x),$$

$$(2b)\, g_{II}(x) = g_I(V(\mu)x)\omega(\mu;x).$$

We insert the expression for g_{II} into equation $(2a)$ and obtain

$$(3) g_I(x) = g_I(V(\lambda,\nu)x)\omega(\lambda,\nu;x) + g_I(V(\mu)x)\omega(\mu;x) + g_I(V(\mu,\nu)x)\omega(\mu,\nu;x).$$

This equation can be seen as a Kuzmin equation for a map $R : [0, \frac{2}{3}] \to [0, \frac{2}{3}]$ with the partition 0, $\frac{1}{3}, \frac{5+4\mu}{9+6\mu}$, and $\frac{2}{3}$. The map R is a map with three branches which have been studied in [7]. The inverse branches are given as

$$V(\lambda,\nu)x = \frac{(9-2\nu)x}{18+2\lambda-(6\nu+3\lambda)x},$$

$$V(\mu)x = \frac{1+(1+2\mu)x}{3+3\mu x},$$

$$V(\mu,\nu)x = \frac{10+8\mu+(3-4\nu+6\mu-4\mu\nu)x}{18+12\mu+(-6\nu+9\mu-6\mu\nu)x}.$$

According to [7], we can write down an explicit form of the invariant density if the condition

$$\det \begin{pmatrix} -6\nu-3\lambda & -9-2\nu-2\lambda & 0 \\ 3\mu & -2+2\mu & 1 \\ -6\nu+9\mu-6\nu\mu & -15+4\nu-6\mu-4\mu\nu & 10+8\mu \end{pmatrix} = 0$$

is satisfied.
Then one can construct a map $\psi(t) = \frac{a+bt}{c+dt}$ such that $\psi \circ R = R^* \circ \psi$ where R^* denotes the *dual map* to R.

Another possibility is to check if an *exceptional dual* for the map R can be constructed (see [7], [4]). One has to find an interval with endpoints α and β (including the case ∞) such that the following conditions are satisfied:

$$V(\lambda,\nu)^*\alpha = \alpha,\, V(\lambda,\nu)^*\beta = V(\mu,\nu)^*\alpha,\, V(\lambda,\nu)^*\beta = V(\mu)^*\alpha,\, V(\mu)^*\beta = \beta.$$

According to the theory of jump transformations, we find

$$h_I(x) = g_I(x),\, h_{II}(x) = g_{II}(x) + g_I(V(\lambda)x)\omega(\lambda;x).$$

Now it is easy to verify the equations (1a) and (1b):

$$h_I(V(\mu)x)\omega(\mu;x) + h_{II}(V(\nu)x)\omega(\nu;x) =$$

$$h_I(V(\mu)x)\omega(\mu;x) + h_I(V(\mu,\nu)x)\omega(\mu,\nu;x) + h_I(V(\lambda,\nu)x)\omega(\lambda,\nu;x) = h_I(x),$$

$$h_I(V(\lambda)x)\omega(\lambda;x) + h_I(V(\mu)x)\omega(\mu;x) = g_I(V(\lambda)x)\omega(\lambda;x) + g_{II}(x) = h_{II}(x).$$

□

Examples. These examples illustrate the case of increasing branches of theorem 4.

(a)

$$\lambda = -\frac{13}{5}, \mu = 0, \nu = \frac{1}{2}$$

$$V(\lambda)x = \frac{-10+15x}{2+13x},\ V(\mu)x = \frac{1+x}{3},\ V(\nu)x = \frac{4+2x}{6-x}$$

$$V(\lambda,\nu)x = \frac{5x}{8+3x},\ V(\mu)x = \frac{1+x}{3},\ V(\mu,\nu)x = \frac{10+x}{18-3x}$$

$$\psi(t) = \frac{1-2t}{1+t}$$

$$g_I(x) = h_I(x) = \frac{1}{5-x} + \frac{1}{1+x},\ g_{II}(x) = \frac{1}{14-x} + \frac{1}{4+x},$$

$$h_{II}(x) = \frac{1}{14-x} + \frac{1}{4+x} - \frac{5}{2+5x} + \frac{7}{-2+7x}.$$

(b)

$$\lambda = -3, \mu = -\frac{3}{4}, \nu = \frac{3}{2}$$

$$V(\lambda)x = \frac{-2+3x}{3x},\ V(\mu)x = \frac{4-2x}{12-9x},\ V(\nu)x = \frac{4}{6-3x}$$

$$V(\lambda,\nu)x = \frac{x}{2},\ V(\mu)x = \frac{4-2x}{12-9x},\ V(\mu,\nu)x = \frac{4-3x}{9-9x}$$

$$\psi(t) = -t$$

$$g_I(x) = h_I(x) = \frac{3}{2-3x},\ g_{II}(x) = \frac{1}{1-x} - \frac{3}{4-3x},$$

$$h_{II}(x) = \frac{1}{1-x} - \frac{3}{4-3x} + \frac{1}{2-x} + \frac{1}{x}.$$

(c) In this example we use an exceptional dual!

$$\text{Let } \gamma = \frac{-1+\sqrt{7}}{2} \text{ and } \lambda = -\frac{-3+8\gamma}{3}, \mu = \frac{1}{2}, \nu = \frac{3-8\gamma}{6}.$$

$$V(\lambda)x = \frac{-6+9x}{6+8\gamma+(3-8\gamma)x},\ V(\mu)x = \frac{2+4x}{6+3x},\ V(\nu)x = \frac{12+(6+8\gamma)x}{18+(-3+8\gamma)x}$$
$$V(\lambda,\nu)x = \frac{x}{2},\ V(\mu)x = \frac{2+4x}{6+3x},\ V(\mu,\nu)x = \frac{14+(3+8\gamma)x}{24+12\gamma x}$$

The exceptional dual is given on $[0,\gamma]$ by the maps

$$V(\lambda,\nu)^*y = \frac{y}{2},\ V(\mu,\nu)^*y = \frac{12\gamma+(3+8\gamma)y}{24+14y},\ V(\mu)^*y = \frac{3+4y}{6+2y}$$
$$g_I(x) = h_I(x) = \frac{\gamma}{1+\gamma x},\ g_{II}(x) = \frac{3+4\gamma}{6+2\gamma+(3+4\gamma)x} - \frac{1}{2+x}$$
$$h_{II}(x) = \frac{3+4\gamma}{6+2\gamma+(3+4\gamma)x} - \frac{3-8\gamma}{6+8\gamma+(3-8\gamma)x}.$$

(d) In this example we use again an exceptional dual!

$$\lambda = -\frac{3}{2}, \mu = 0, \nu = -3$$
$$V(\lambda)x = \frac{-4+6x}{3+3x},\ V(\mu)x = \frac{1+x}{3},\ V(\nu)x = \frac{4+9x}{6+6x}$$
$$V(\lambda,\nu)x = \frac{2x}{2+3x},\ V(\mu)x = \frac{1+x}{3},\ V(\mu,\nu)x = \frac{10+15x}{18+18x}$$

The exceptional dual is given on $[0,\infty[$ by the maps

$$V(\mu)^*y = \frac{y}{3+y}, V(\mu,\nu)^*y = \frac{18+15y}{18+10y},\ V(\lambda,\nu)^*y = \frac{3+2y}{2}$$
$$g_I(x) = h_I(x) = \frac{1}{x},\ g_{II}(x) = \frac{1}{1+x},\ h_{II}(x) = \frac{3}{-2+3x}.$$

5. Outlook: Maps With Four Branches

We start with the increasing map $g(x) = \frac{1+(6+7\mu)x}{2+2\mu x}$, $\mu > -1$ and consider the map $Tx = g(x) \bmod 1$. Note that $T0 = \frac{1}{2} = T1$. This map leads to a fibred system T with the partition

$$0 < \frac{1}{6+5\mu} < \frac{3}{6+3\mu} = \frac{1}{2+\mu} < \frac{5}{6+\mu} < 1.$$

The map T is given piecewise as

$$T_k x = g(x) - k,\ \ k = 0,1,2,3.$$

If $\mu = 0$, then T is piecewise linear and $h(x) = 1$ (Lebesgue measure) is the invariant density.

Theorem 5. *For $\mu > -1$ and $\mu \neq 0$ it is not possible to construct an exceptional dual.*

Proof. We try to find $\alpha, \xi, \theta, \eta, \beta$, and μ such that

$$T_0^*\alpha = \xi = T_1^*\xi,\ T_0^*\xi = \beta = T_1^*\theta,\ T_2^*\theta = \alpha = T_3^*\eta,\ T_2^*\eta = \eta = T_3^*\beta.$$

From

$$\frac{2\mu + (6+7\mu)\xi}{2+\xi} = \beta = \frac{2\mu + (6+5\mu)\theta}{2-\theta},$$

one calculates the relation

$$\theta = \frac{\xi}{1+\xi},$$

and from

$$\frac{2\mu + (6+3\mu)\theta}{2-3\theta} = \alpha = \frac{2\mu + (6+\mu)\eta}{2-5\eta},$$

the relation

$$\theta = \frac{\eta}{1-\eta}.$$

The fixed points ξ and η satisfy the quadratic equations

$$(Q_1) \quad \xi^2 + (4+5\mu)\xi + 2\mu = 0,$$

$$(Q_2) \quad 3\eta^2 + (4+3\mu)\eta + 2\mu = 0.$$

If we insert $\xi = \frac{\eta}{1-2\eta}$ into equation (Q_1) we find

$$(-7-2\mu)\eta^2 + (4-3\mu)\eta + 2\mu = 0.$$

If the quadratic equations for η are irreducible, we see that $3 = -7-2\mu$ and $4+3\mu = 4-3\mu$, a contradiction.

If these equations are reducible then

$$\eta = -\frac{2\mu^2 + 10\mu}{3\mu^2 + 10\mu + 20}.$$

We insert $\eta = \frac{\xi}{1+2\xi}$ into equation (Q_2) and obtain

$$(11+14\mu)\xi^2 + (4+11\mu)\xi + 2\mu = 0.$$

Since these equations now are reducible we obtain

$$\xi = -\frac{14\mu^2 + 10\mu}{35\mu^2 + 50\mu + 20}.$$

We substitute this value into the relation $\eta = \frac{\xi}{1+2\xi}$ and obtain

$$\eta = -\frac{14\mu^2 + 10\mu}{7\mu^2 + 30\mu + 20}.$$

We equate both values for η and observe $\mu \neq 0$. Then we get the equation

$$7\mu^2 + 10\mu + 10 = 0,$$

which has no real solution. □

We also consider the decreasing map $g(x) = \frac{7+(\mu-6)x}{2+2\mu x}$, $\mu > -1$ and study shortly the map $Tx = g(x) \bmod 1$. This map leads to a fibred system T with the partition

$$0 < \frac{1}{6+5\mu} < \frac{3}{6+3\mu} = \frac{1}{2+\mu} < \frac{5}{6+\mu} < 1.$$

The map T is given piecewise as

$$T_k x = g(x) - k, \ \ k = 0, 1, 2, 3.$$

If $\mu = 0$ then T is piecewise linear and again $h(x) = 1$ (Lebesgue measure) is the invariant density.

Theorem 6. *For $\mu > -1$ and $\mu \neq 0$ it is not possible to construct an exceptional dual.*

Proof. In a similar way as before we try to find $\alpha, \xi, \theta, \eta, \beta$, and μ such that

$$T_2^*\theta = \alpha = T_3^*\xi,\ T_1^*\theta = \beta = T_0^*\eta,\ T_3^*\alpha = \eta = T_2^*\xi,\ T_1^*\eta = \xi = T_0^*\beta.$$

As before we find $\xi = \frac{\theta}{1+\theta}$ and $\eta = \frac{\theta}{1-\theta}$. We use

$$W_-(\xi) = \frac{\xi}{1-2\xi} = \eta$$

and

$$W_+(\eta) = \frac{\eta}{1+2\eta} = \xi.$$

The maps $T_0^* \circ T_0^* \circ W_-$ and $W_+ \circ T_3^* \circ T_3^*$ both have ξ as a fixed point. The corresponding linear maps have the eigenvalue equations

$$\lambda^2 - \lambda(40 + 32\mu - 3\mu^2) + 144(1+\mu)^2 = 0,$$

$$\lambda^2 - \lambda(40 + 48\mu + 5\mu^2) + 144(1+\mu)^2 = 0.$$

The comparison of these equations leads to $\mu = 0$ (the linear case) or $\mu = -2$, which is not an admissible parameter.
The same result can be derived from a comparison of the maps $W_- \circ T_0^* \circ T_0^*$ and $T_3^* \circ T_3^* \circ W_+$. □

We consider two further examples of maps with four branches. An invariant density can be found for some special values of parameters by constructing an *exceptional dual* (this is a dual map T^* which is not differentiably isomorphic).

(a) We consider the partition $0 < \frac{1}{4} < \frac{1}{2} < \frac{3}{4} < 1$ and the map T with the four inverse branches

$$V_\lambda(x) = \frac{1-2x}{4+\lambda x},\ V_\mu(x) = \frac{2+(2+\mu)x}{8+2\mu x},$$

$$V_\nu(x) = \frac{3-\nu x}{4+(2-2\nu)x},\ V_\tau(x) = \frac{3+(2+\tau)x}{4+\tau x}.$$

The map T admits the blocks

$$\lambda\lambda, \lambda\mu; \mu\lambda, \mu\mu, \mu\nu, \mu\tau; \nu\lambda, \nu\mu, \nu\nu, \nu\tau; \tau\lambda, \tau\mu.$$

A dual map T^* must admit the blocks

$$\lambda\lambda, \lambda\mu, \lambda\nu, \lambda\tau; \mu\lambda, \mu\mu, \mu\nu, \mu\tau; \nu\mu, \nu\nu; \tau\mu, \tau\nu.$$

Therefore no map $\psi(t) = \frac{c+dt}{a+bt}$ can exist such that $\psi \circ T = T^* \circ \psi$. We look for a dual with partition $\alpha < \xi < \theta < \eta < \beta$. One has to solve eight equations for nine parameters:

$$\alpha = \frac{\lambda - 2\beta}{4+\beta},\ \frac{\lambda-2\alpha}{4+\alpha} = \xi = \frac{2\mu+(2+\mu)\alpha}{8+2\alpha},\ \frac{2\mu+(2+\mu)\beta}{8+2\beta} = \theta = \frac{2-2\nu-\nu\eta}{4+3\eta},$$

$$\frac{2-2\nu-\nu\xi}{4+3\xi} = \eta = \frac{\tau+(2+\tau)\xi}{4+3\xi},\ \frac{\tau+(2+\tau)\eta}{4+3\eta} = \beta.$$

If $4+\alpha \neq 0$ and $4+3\xi \neq 0$ one gets the equations

$$\alpha = \frac{2\lambda-2\mu}{6+\mu},\ \beta = \frac{-2\lambda+8\mu+\lambda\mu}{2\lambda+12},$$

$$\theta = \frac{8\mu^2+\lambda\mu^2+4\lambda\mu+40\mu-4\lambda}{2\lambda\mu+16\mu+12\lambda+96},\ 2-2\nu-\nu\xi = \tau+2\xi+\tau\xi.$$

We solve one special case, namely $\theta = 0$. We then get

$$\beta = -\frac{2\mu}{2+\mu},\ \eta = \frac{2-2\nu}{\nu}.$$

These equations lead to

$$\alpha = \frac{2\lambda+\lambda\mu+4\mu}{8+2\mu},\ \xi = \frac{-8+10\nu+2\nu^2}{6-6\nu+\nu^2},\ \tau = \frac{28-36\nu+8\nu^2}{-2+4\nu-\nu^2}.$$

From

$$\alpha = \frac{2\lambda+\lambda\mu+4\mu}{8+2\mu} = \frac{2\lambda-2\mu}{6+\mu},$$

we get

$$\lambda = \frac{40\mu + 8\mu^2}{4 - 4\mu - \mu^2}.$$

Now we choose $\mu = -1$ and obtain with $\nu = \frac{9-\sqrt{33}}{4}$ (a root of $2\nu^2 - 9\nu + 6 = 0$) the values

$$\alpha = -\frac{10}{7},\ \xi = -\frac{2}{3}, \theta = 0,\ \eta = \frac{-3+\sqrt{33}}{6},\ \beta = 2,$$

$$\lambda = -\frac{32}{7},\ \mu = -1,\ \nu = \frac{9-\sqrt{33}}{4},\ \tau = 1 + \sqrt{33}.$$

The invariant density then is

$$h_I(x) = \frac{2}{1+2x} + \frac{10}{7-10x} \text{ for } 0 \leq x < \frac{1}{2},$$

$$h_{II}(x) = \frac{2-2\nu}{\nu + (2-2\nu)x} + \frac{2}{3-2x} \text{ for } \frac{1}{2} < x \leq 1.$$

We outline the connection with the jump map S which avoids the non-full cylinders labeled λ and τ. If g denotes the invariant density for S then

$$h_I(x) = g(x) + g(V_\tau x)\omega(\tau; x) + g(V_\lambda x)\omega(\lambda; x) + g(V_{\lambda\lambda} x)\omega(\lambda\lambda; x),$$

$$+ g(V_{\tau\lambda} x\omega(\tau\lambda; \omega(\tau\lambda; x) + g(V_{\lambda\lambda\lambda} x)\omega(\lambda\lambda\lambda; x) + g(V_{\tau\lambda\lambda} x)\omega(\tau\lambda\lambda; x) + ...$$

$$h_{II}(x) = g(x).$$

(b) We consider again the partition $0 < \frac{1}{4} < \frac{1}{2} < \frac{3}{4} < 1$ and the map T with the four inverse branches

$$V_\lambda(x) = \frac{-1+2x}{4+\lambda-\lambda x},\ V_\mu(x) = \frac{2+2\mu x}{8+2\mu x},$$

$$V_\nu(x) = \frac{4+(2+3\nu)x}{8+4\nu x},\ V_\tau(x) = \frac{3+(2+\tau)x}{4+\tau x}.$$

This map admits the blocks

$$\lambda\nu, \lambda\tau; \mu\lambda, \mu\mu, \mu\nu, \mu\tau; \nu\lambda, \nu\mu, \nu\nu, \nu\tau; \tau\lambda, \tau\mu.$$

Then a dual map T^* must admit the blocks

$$\lambda\mu, \lambda\nu, \lambda\tau; \mu\mu, \mu\nu, \mu\tau; \nu\lambda, \nu\mu, \nu\nu; \tau\lambda, \tau\mu, \tau\nu.$$

Again, no map $\psi(t) = \frac{c+dt}{a+bt}$ can exist such that $\psi \circ T = T^* \circ \psi$. We look for a dual with partition $\alpha < \xi < \theta < \eta < \beta$. One has to solve eight equations for nine parameters:

$$\alpha = \frac{-\lambda+2\xi}{4\lambda-\xi},\ \frac{-\lambda+2\beta}{4\lambda-\beta} = \xi = \frac{2\mu+(2+\mu)\xi}{8+2\xi},\ \frac{2\mu+(2+\mu)\beta}{8+2\beta} = \theta = \frac{4\nu+(2+3\nu)\alpha}{8+4\alpha},$$

$$\frac{4\nu + (2+3\nu)\eta}{8+4\eta} = \eta = \frac{\tau + (2+\tau)\alpha}{4+3\alpha},\ \frac{\tau + (2+\tau)\eta}{4+3\eta} = \beta.$$

The two fixed points ξ and η provide a good start! The equations which relate to them are:

$$2\xi^2 + \xi(6-\mu) - 2\mu = 0,$$
$$4\eta^2 + \eta(6-3\nu) - 4\nu = 0.$$

We want to keep $\alpha = 0$ and $\beta = 1$ and find $\lambda = 2\xi$, $\tau = 4\eta$, and $\mu = \nu = 1$. Then we have

$$\xi = \frac{-5+\sqrt{41}}{4},\ \eta = \frac{-3+\sqrt{73}}{8}.$$

The invariant density should satisfy

$$h_I(x) = h_I(V_\mu x)\omega(\mu; x) + h_{II}(V_\nu x)\omega(\nu; x) + h_{II}(V_\tau x)(\omega(\tau; x),\ 0 \le x < \frac{1}{2},$$

$$h_I(x) = h_I(V_\lambda x)\omega(\lambda; x) + h_I(V_\mu x)\omega(\mu; x) + h_{II}(V_\nu x)\omega(\nu; x),\ \frac{1}{2} < x \le 1$$

and can be given as

$$h_I(x) = \frac{1-\xi}{(1+x)(1+\xi x)},\ h_{II}(x) = \frac{\eta}{1+\eta x}.$$

Again, we outline the connection with the jump map S which avoids the non-full cylinders labeled λ and τ. If g denotes the invariant density for S then

$$h_I(x) = g(x) + g(V_\tau x)\omega(\tau; x) + g(V_{\lambda\tau} x)\omega(\lambda\tau; x) + g(V_{\tau\lambda\tau} x)\omega(\tau\lambda\tau; x) + ...$$
$$h_{II}(x) = g(x) + g(V_\lambda x)\omega(\lambda; x) + g(V_{\tau\lambda} x)\omega(\tau\lambda; x) + g(V_{\lambda\tau\lambda} x)\omega(\lambda\tau\lambda; x) + ...$$

Acknowledgment. My thanks go to the referee whose remarks contributed essential improvements to my paper.

References

[1] K. Dajani, C. Kraaikamp, and N. D. S. Langeveld, Continued fraction expansions with variable numerators, *Ramanujan J.* **37** (2015), no. 3, 617–639.

[2] O. Jenkinson and M. Pollicott, Ergodic properties of the Bolyai-Rényi expansion, *Indag. Math. (N.S.)* **11** (2000), 399–418.

[3] F. Schweiger, Invariant measures for continued fractions with variable numerators, *Integers* **17** (2017), #A56.

[4] F. Schweiger, Invariant measures for Moebius maps with three branches, *J. Number Theory* **184** (2018), 206–215.

[5] F. Schweiger, *Continued Fractions and Their Generalizations: A Short History of f-Expansions*, Boston, Massachusetts, Docent Press, 2017.

[6] F. Schweiger, Invariant measures of piecewise fractional linear maps and piecewise quadratic maps, *Int. J. Number Theory* **14** (2018), 1559-1572.

[7] F. Schweiger, Differentiable equivalence of fractional linear maps, *Dynamics & Stochastics*, 237–247. IMS Lecture Notes Monogr. Ser., **48**, 2006.

[8] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford Science Publications, New York, The Clarendon Press, Oxford University Press, 1995.