Abstract
While studying the location of the zeros of the Eisenstein series $E_k(z)$, Rankin considered the determinants $\Delta_n$ of an associated Hankel matrix. He observed that the first few possess remarkable factorizations, and expressed the hope that a general theorem explaining these factorizations could be found. In this note we provide such a theorem by giving an explicit formula for $\Delta_n$ using work of Kaneko and Zagier on Atkin polynomials.

1. Introduction
The zeros of Eisenstein series were studied by R. Rankin in [5], where he showed that for $k = 28, 30, 32, 34$ and $38$ the zeros of $E_k$ lie on the unit circle. Soon after R. Rankin’s result, F.K.C. Rankin and Swinnerton-Dyer [4] proved that the zeros of $E_k$ lie on the unit circle for all even $k \geq 4$. In this note we confirm an observation made in [5] about the determinants $\Delta_n = |H_n|$ of the Hankel matrix given by

$$H_n = \begin{pmatrix} g_0 & g_1 & \cdots & g_n \\ g_1 & g_2 & \cdots & g_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_{n+1} & \cdots & g_{2n} \end{pmatrix},$$

where the $g_v$ are defined as follows. Let

$$E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k}$$

and let $\Delta(z)$ be the unique normalized weight 12 cusp form. Let $j$ be the modular invariant defined by $j(z) := \frac{E_4^3(z)}{\Delta(z)}$ and with Fourier expansion

$$j(z) = q^{-1} \sum_{n=0}^{\infty} a_n q^n,$$
where \( q = e^{2 \pi iz} \). For a function \( F \) that is meromorphic on a fundamental domain \( \mathcal{F} \), write \( R(F) \) for the sum of the residues of \( F \) at points of \( \mathcal{F} \) and

\[
j^\nu(z) = q^{-\nu} \sum_{n=0}^{\infty} a_n^{(\nu)} q^n.
\]

Then \( g_\nu \) is defined by

\[
g_\nu := 2 \pi i R(j^\nu E_2) = a_0^{(\nu)} - 24 \sum_{m=1}^{\nu} a_{\nu-m}^{(\nu)} \sigma(m),
\]

where \( \sigma(n) = \sum_{d|n} d \) and \( E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \). The first few values are

\[
g_0 = 1, \quad g_1 = 720, \quad g_2 = 911520, \quad g_3 = 1301011200, \quad g_4 = 1958042030400.
\]

This is sequence A030185 in [6]. Rankin gave the values up to \( \Delta_{13} \) in a table by their prime factorization; the last of these is of size approximately 2.79 \cdot 10^{483} (the computations in Rankin’s paper were made by Mr. Stephen Muir of the Atlas Computer Laboratory in Chilton, Didcot). Rankin went on to say that “... they possess remarkable factorizations; each of them is a highly composite number expressible as a product of powers of small primes. These results are given in \( \S 4 \) in the hope that they may stimulate someone to prove a general theorem about these determinants.”

Here we prove such a theorem.

**Theorem 1.** For \( n \geq 1 \), let \( H_n \) be as in (1) and \( \Delta_n = |H_n| \). Then,

\[
\Delta_n = 2^{n^2+4n \cdot 3^n+2n \cdot 5^n \cdot 7^n \cdot 13^n} \prod_{r=2}^{n} \left( \frac{(12r-13)(12r-7)(12r-5)(12r+1)}{(2r-1)^2(r-1)^r} \right)^{n-r+1}.
\]

Note that the largest prime that can appear in the factorization of \( \Delta_n \) is at most \( 12n + 1 \).

**2. Proof of Theorem 1**

Recall that if \( V \) is the space of polynomials in one variable over a field \( K \), and \( \phi : V \to K \) is a linear functional, then one can consider the scalar product on \( V \) defined by \( (f, g) = \phi(fg) \). One can also consider the family (which for generic \( \phi \) exists and is unique) of monic polynomials which are mutually orthogonal with respect to the scalar product.

Atkin [2, page 3] defined a sequence of polynomials \( A_n(j) \in \mathbb{Q}[j] \), one for each degree \( n \), as the orthogonal polynomials with respect to a scalar product. The
particular scalar product used by Atkin is defined in several equivalent ways in [2, Proposition 3], one of them being

\[(f, g) := \text{constant term of } fgE_2 \text{ as a Laurent series in } q.\]

Then from the definition (2) we see that

\[g_\nu = (j^\nu, 1).\]

**Proof of Theorem 1.** A recursion for the Atkin polynomials \(A_n\) is given by ([2, equation (18)]):

\[A_{n+1}(j) = (j - (\lambda_{2n} + \lambda_{2n+1}))A_n(j) - \lambda_{2n-1}\lambda_{2n}A_{n-1}(j),\]  

(3)

where the numbers \(\lambda_n\) are defined by the continued fraction expansion

\[\sum_{k=0}^{\infty} g_k x^k = \frac{g_0}{1 - \frac{\lambda_1 x}{1 - \frac{\lambda_2 x}{1 - \cdots}}}.\]  

(4)

By [3, Theorem 29], we can use the recurrence in (3) to give a formula for \(\Delta_n\) in terms of the \(\lambda_n\):

\[\Delta_n = \det_{0 \leq i, r \leq n} (g_{i+r}) = \prod_{r=1}^{n} (\lambda_{2r-1}\lambda_{2r})^{n-r+1}.\]  

(5)

Equation (19) of [2] gives an explicit formula for the \(\lambda_n\):

\[\lambda_1 = 720, \quad \lambda_n = 12 \left(6 + \frac{(-1)^n}{n-1}\right) \left(6 + \frac{(-1)^{n-1}}{n}\right) \text{ for } n > 1.\]  

(6)

For \(r > 1\) this gives

\[\lambda_{2r-1}\lambda_{2r} = 12 \left(6 + \frac{(-1)^{2r-1}}{(2r-1)-1}\right) \left(6 + \frac{(-1)^{2r-1}}{2r-1}\right) 12 \left(6 + \frac{(-1)^{2r}}{(2r)-1}\right) \left(6 + \frac{(-1)^{2r}}{2r}\right) \frac{36(12r - 13)(12r - 7)(12r - 5)(12r + 1)}{(2r - 1)^2(r - 1)r}.\]

Plugging this formula into equation (5) and simplifying yields the result. \(\square\)

Let \(\nu_p(m)\) be the highest power of \(p\) that divides a non-zero integer \(m\). From Theorem 1 one can obtain \(\nu_p(\Delta_n)\) for any prime \(p\). In the case \(p = 2\) it has a simple expression.

**Corollary 1.** We have

\[\nu_2(\Delta_n) = 4n - s_2(n) + 2 \sum_{r=1}^{n} s_2(r),\]

where \(s_2(r)\) is the sum of the digits of \(r\) in base 2.
Proof. From Theorem 1 we see that
\[
\nu_2(\Delta_n) = n^2 + 4n - \nu_2 \left( \prod_{r=2}^{n} ((r-1)r)^{n-r+1} \right),
\]
so we only need to show that \(\nu_2 \left( \prod_{r=2}^{n} ((r-1)r)^{n-r+1} \right) = n^2 - 2 \sum_{r=1}^{n} s_2(r) + s_2(n)\). Using the fact that \(\nu_2(r) = 1 + s_2(r-1) - s_2(r)\) we obtain
\[
\begin{align*}
\nu_2 \left( \prod_{r=2}^{n} ((r-1)r)^{n-r+1} \right) \\
= \sum_{r=2}^{n} (n-r+1)(\nu_2(r) + \nu_2(r-1)) \\
= 2 \sum_{r=2}^{n} (n-r+1) - \sum_{r=2}^{n} (n-r+1)s_2(r) + \sum_{r=1}^{n-2} (n-r-1)s_2(r) \\
= n^2 - s_2(n) - 2s_2(n-1) + \sum_{r=1}^{n-2} s_2(r)(n-r-1 - (n-r+1)) \\
= n^2 - 2 \sum_{r=1}^{n} s_2(r) + s_2(n). \quad \Box
\end{align*}
\]

We point out that the sequence \((\sum_{r=1}^{n} s_2(r))_n\) is sequence A000788 in [6].

As seen in [2], there are many ways to approach the Atkin polynomials. In this spirit, we briefly explain another way in which one could obtain a closed formula for \(\Delta_n\). From Section 4 of [1] we have
\[
\Delta_n = ||A_0(j)||^2 \cdot ||A_1(j)||^2 \cdots ||A_n(j)||^2 = \prod_{i=0}^{n}(A_i, A_i).
\]
(7)

For \(n \geq 1\), ([2, Proposition 6])
\[
(A_n, A_n) = -12^{6n+1} \left( \frac{-1/12}{(2n-1)!(2n)!} \right)^n \frac{(5/12)_n(7/12)_n(13/12)_n}{(2n-1)!(2n)!},
\]
(8)
where \((x)_n = x(x+1)\cdots(x+n-1)\) and \((A_0, A_0) = (1, 1) = 1\). Thus,
\[
\Delta_n = \prod_{i=1}^{n} -12^{6i+1} \frac{(-1/12)_i(5/12)_i(7/12)_i(13/12)_i}{(2i-1)!(2i)!}.
\]
(9)

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References


