



**PRIMES  $p \equiv 1 \pmod{d}$  AND  $a^{(p-1)/d} \equiv 1 \pmod{p}$**

**Peng Gao**

*School of Mathematics and Systems Science, Beihang University, Beijing, China*  
penggao@buaa.edu.cn

**Liangyi Zhao**

*School of Mathematics and Statistics, University of New South Wales, Sydney, Australia*  
l.zhao@unsw.edu.au

*Received: 7/24/18, Revised: 12/28/18, Accepted: 5/23/19, Published: 6/3/19*

**Abstract**

Suppose that  $d \in \{2, 3, 4, 6\}$  and  $a \in \mathbb{Z}$  with  $a \neq -1$  and  $a$  is not square. Let  $P_{(a,d)}$  be the number of primes  $p$  not exceeding  $x$  such that  $p \equiv 1 \pmod{d}$  and  $a^{(p-1)/d} \equiv 1 \pmod{p}$ . In this paper, we study the mean value of  $P_{(a,d)}$ .

**1. Introduction**

Let  $\mathbb{P}$  denote the set of prime numbers and  $d > 0$  be a squarefree integer. Moreover, suppose that  $a \in \mathbb{Z}$  with  $a \neq -1$  and  $a$  is not a square. Set

$$\mathcal{P}_{(a,d)} = \{p \in \mathbb{P} : p \equiv 1 \pmod{d} \text{ and } a^{(p-1)/d} \equiv 1 \pmod{p}\}$$

and

$$P_{a,d}(x) = \#\{p \in \mathcal{P}_{(a,d)} : p \leq x\}.$$

It has long been known and studied [9, 5] that the estimation of  $P_{(a,d)}$  is connected with Artin's conjecture on primitive roots. This conjecture asserts that the number of primes not exceeding  $x$ , and for which  $a$  is a primitive root, is asymptotically  $c\pi(x)$  where  $c$  is some explicitly defined constant whose expression involves  $a$ .

Assuming that truth of the Riemann hypothesis for the Dedekind zeta function for the number field  $\mathbb{Q}[\sqrt[d]{a}, \sqrt[d]{1}]$ , C. Hooley [5] proved that

$$P_{(a,d)}(x) = \frac{\varepsilon(d)}{d\varphi(d)} \text{li}(x) + O(\sqrt{x} \log(dx)), \quad (1)$$

where  $\varepsilon(d)$  is defined in the following way. Writing  $a = lm^2$  with  $l$  square free,  $\varepsilon(d) = 2$  if  $l \equiv 1 \pmod{4}$  and  $2l|d$  and  $\varepsilon(d) = 1$  otherwise. Using (1), Hooley was

able to prove Artin’s conjecture conditionally. For a survey of this conjecture, we refer the reader to [9].

S. Li [8] studied  $P_{(a,d)}(x)$  on average and proved that

$$\frac{1}{y} \sum_{2 \leq a \leq y} P_{(a,d)}(x) = \frac{\pi(x; 1, d)}{d} + O\left(\frac{x E(x, y)}{\varphi(d) \log x}\right), \tag{2}$$

where

$$E(x, y) = y^{-1/21} \text{ if } y \leq x^{2/3} \quad \text{and} \quad E(x, y) = x^{-1/6} \log x \text{ if } y > x^{2/3}.$$

In this paper, we improve the bound (2) for  $d = 2, 3, 4$  and  $6$ .

In the case  $d = 2$ , we prove the following result.

**Theorem 1.** *We have*

$$\frac{1}{y} \sum_{2 \leq a \leq y} P_{(a,2)}(x) - \frac{\pi(x)}{2} \ll \begin{cases} x \sqrt{\frac{\log x}{y}}, & \text{if } \frac{x}{\log x} \geq y, \\ \sqrt{x} \log x, & \text{if } \frac{x}{\log x} < y. \end{cases}$$

Additionally, for  $d = 3, 4$  and  $6$ , we have the following.

**Theorem 2.** *Let  $\varepsilon > 0$  be given. If  $d \in \{3, 4, 6\}$ , then*

$$\frac{1}{y} \sum_{2 \leq a \leq y} P_{(a,d)}(x) - \frac{\pi(x; 1, d)}{d} \ll (xy)^\varepsilon E(x, y),$$

where  $E(x, y)$  is defined piece-wise in the following way:

$$E(x, y) = \begin{cases} x^{1/2}, & \text{if } x \leq y^{3/5}, \\ x^{4/3} y^{-1/2}, & \text{if } y^{3/5} < x \leq y^{6/7}, \\ x^{3/4}, & \text{if } y^{6/7} < x \leq y^{6/5}, \\ x^{7/6} y^{-1/2}, & \text{if } y^{6/5} < x \leq y^{3/2}, \\ x^{5/6}, & \text{if } y^{3/2} < x \leq y^{9/5}, \\ x^{10/9} y^{-1/2}, & \text{if } y^{9/5} < x \leq y^{108/55}, \\ x^{1/2} y^{7/10}, & \text{if } y^{108/55} < x \leq y^{11/5}, \\ x^{2/3} y^{1/3}, & \text{if } y^{11/5} < x \leq y^{5/2}, \\ xy^{-1/2}, & \text{if } y^{5/2} < x. \end{cases}$$

Note that the  $O$ -terms in Theorems 1 and 2 are smaller than that in (2) in all  $xy$ -ranges. Finally, we shall prove the following smoothed version of Theorem 1, assuming the truth of the generalized Riemann hypothesis.

**Theorem 3.** *Suppose that  $x, Y > 1, U = x^{1/8}Y^{1/4}$  and set*

$$D(Y) = \{d : Y \leq d \leq 2Y, d \text{ odd and square-free}\}.$$

*Let  $\Phi_Y(t)$  be a non-negative smooth function supported on the interval  $(1, 2)$  satisfying the condition  $\Phi_Y(t) = 1$  for  $t \in (1 + 1/U, 2 - 1/U)$  and*

$$\Phi_Y^{(j)}(t) \ll_j U^j \tag{3}$$

*for all integers  $j \geq 0$ . Assuming the truth of the generalized Riemann hypothesis for the Dirichlet  $L$ -functions, we have*

$$\begin{aligned} \frac{1}{\#D(Y)} \sum_{(a,2)=1} \mu^2(a) P_{(8a,2)}(x) \Phi_Y\left(\frac{a}{Y}\right) \\ = \frac{\pi(x)}{2} + O\left(\log \log x + \left(\frac{x^{7/8}}{Y^{1/4}} + \frac{x}{Y^{1/2}}\right) \log(xY)\right). \end{aligned} \tag{4}$$

Note that the strength of Theorem 3 is most prominent when  $Y$  is large compared to  $x$ .

## 2. Preliminaries

We shall need the following lemma due to Pólya and Vinogradov.

**Lemma 1.** *Let  $\chi(n)$  be a non-principal character modulo  $q$  and  $M \in \mathbb{Z}$  and  $N \in \mathbb{N}$ . Then we have*

$$\left| \sum_{M < n \leq M+N} \chi(n) \right| \leq 6\sqrt{q} \log q.$$

*Proof.* This is a standard result and a proof can be found in [6, Theorem 12.5].  $\square$

The following mean-value theorem for quadratic character sums will be useful for us as well.

**Lemma 2.** *Let  $\chi$  run over all real non-principal characters  $\chi(n) = \left(\frac{D}{n}\right)$  with  $|D| \leq X$ . We have*

$$\sum_{\chi} \left| \sum_{n \leq Y} \chi(n) \right|^2 \ll XY \log^2 X.$$

*Proof.* We refer the reader to [7, Lemma 5] for a proof of this lemma.  $\square$

**Lemma 3.** *Assume the truth of the generalized Riemann hypothesis for the Dirichlet L-functions. Then for any non-principal Dirichlet character  $\chi \pmod{q}$ , we have*

$$\sum_{p \leq X} \chi(p) \ll X^{1/2} \log(qX).$$

*Proof.* This follows readily from [6, Proposition 5.25]. □

**Lemma 4 (Large sieve inequality for certain fixed order characters).** *Let  $(a_m)_{m \in \mathbb{N}}$  be an arbitrary sequence of complex number and  $k \in \{3, 4, 6\}$ . Then, for any  $\varepsilon > 0$ ,*

$$\sum_{Q < q \leq 2Q} \sum_{\substack{\chi \pmod{q} \\ \chi^k = \chi_0, \chi \neq \chi_0}}^* \left| \sum_{M < m \leq 2M}^* a_m \chi(m) \right|^2 \ll (QM)^\varepsilon \Delta \sum_{M < m \leq 2M} |a_m|^2, \quad (5)$$

where

$$\Delta = \min \left\{ Q^{5/3} + M, Q^{4/3} + Q^{1/2}M, Q^{11/9} + Q^{2/3}M, Q + Q^{1/3}M^{5/3} + M^{12/5} \right\}.$$

The implied constant in (5) depends on  $\varepsilon$ , the star on the sum over  $\chi$  restricts the sum to primitive characters and the asterisks attached to the sum over  $m$  indicates that  $m$  ranges over square-free integers.

*Proof.* For  $k = 4$  and  $k = 6$ , the outer sums on the left-hand side of (5) include quadratic characters. We use [4, Theorem 1] to bound the contribution of those characters. For the other characters, we use the results in [3, 1]. For  $k = 4$ , this lemma is an improvement of [3, Theorem 1.2] and the proof goes along the same line as Section 6 of [3]. The only difference is, instead of using [3, Theorem 1.1], one uses [2, Theorem 1.3] at the appropriate places. For  $k = 3$  and  $k = 6$ , this is [1, Theorem 1.4] and [1, Theorem 1.5], respectively. □

### 3. Proof of Theorem 1

By considering the group structure of  $(\mathbb{Z}/p\mathbb{Z})^*$ , we easily conclude that  $a^{(p-1)/2} \equiv 1 \pmod{p}$  if and only if  $p \nmid a$  and  $a$  is a square modulo  $p$ . So

$$P_{(a,2)}(x) = \frac{1}{2} \sum_{2 < p \leq x} \left( \chi_0(a) + \left( \frac{a}{p} \right) \right),$$

where  $\chi_0$  is the principal character modulo  $p$ . Therefore, we get

$$S := \frac{1}{y} \sum_{2 \leq a \leq y} P_{(a,2)}(x) = \frac{1}{2y} \sum_{2 < p \leq x} \sum_{a \leq y} \chi_0(a) + \frac{1}{2y} \sum_{2 < p \leq x} \sum_{a \leq y} \left( \frac{a}{p} \right) = S_1 + S_2, \quad (6)$$

say. We easily have

$$\begin{aligned}
 S_1 &= \frac{1}{2y} \sum_{2 < p \leq x} \left( [y] - \left[ \frac{y}{p} \right] \right) = \frac{1}{2y} \sum_{2 < p \leq x} \left( y - \frac{y}{p} + O(1) \right) \\
 &= \frac{\pi(x)}{2} + O\left( \log \log x + \frac{\pi(x)}{y} \right),
 \end{aligned}
 \tag{7}$$

where we have used the well-known bound

$$\sum_{p \leq x} \frac{1}{p} \ll \log \log x.
 \tag{8}$$

Now

$$S_2 \ll \frac{1}{2y} \sum_{2 < p \leq x} \sum_{a \leq y} \left( \frac{a}{p} \right) \ll \frac{1}{2y} \sum_{2 < p \leq x} \left| \sum_{a \leq y} \left( \frac{a}{p} \right) \right|.$$

Applying Lemma 1, we get that

$$S_2 \ll \frac{1}{2y} \sum_{p \leq x} \sqrt{p} \log p \ll \frac{x^{3/2}}{y},
 \tag{9}$$

by the prime number theorem. On the other hand, using Cauchy-Schwarz inequality, we conclude that

$$\sum_{p \leq x} \left| \sum_{a \leq y} \left( \frac{a}{p} \right) \right| \leq \pi(x)^{1/2} \left( \sum_{p \leq x} \left| \sum_{a \leq y} \left( \frac{a}{p} \right) \right|^2 \right)^{1/2} \ll xy^{1/2} \log^{1/2} x,$$

where the last inequality comes from Lemma 2 and the prime number theorem. Hence,

$$S_2 \ll \frac{1}{y} xy^{1/2} \log^{1/2} x = \frac{x \log^{1/2} x}{y^{1/2}}.
 \tag{10}$$

Finally, combining (6), (7), (9) and (10), we arrive at

$$S = \frac{\pi(x)}{2} + O\left( \log \log x + \frac{\pi(x)}{y} + \min\left( \frac{x^{3/2}}{y}, \frac{x \log^{1/2} x}{y^{1/2}} \right) \right).$$

The two terms in the minimum above are equal when  $x/\log x = y$ . The theorem follows by an easy analysis.

#### 4. Proof of Theorem 2

For  $d = 3, 4$  and  $6$ , an argument similar to that in the previous section gives that

$$\frac{1}{y} \sum_{a \leq y} P_{(a,d)} = \frac{1}{dy} \sum_{\substack{p \leq x \\ p \equiv 1 \pmod d}} \sum_{a \leq y} \chi_0(a) + \frac{1}{dy} \sum_{\substack{p \leq x \\ p \equiv 1 \pmod d}} \sum_{a \leq y} \sum_{\substack{\chi \pmod p \\ \chi^d = \chi_0, \chi \neq \chi_0}} \chi(a) =: S_1 + S_2,$$

say. As before,

$$S_1 = \frac{\pi(x; 1, d)}{d} + O\left(\log \log x + \frac{\pi(x; 1, d)}{y}\right). \tag{11}$$

To estimate  $S_2$ , we note that for all  $a \in \mathbb{N}$ ,  $a$  can be written uniquely as  $l^2m$  with  $l, m \in \mathbb{N}$  and  $m$  square-free. Hence, using Cauchy's inequality,

$$S_2^2 \ll \frac{\pi(x; 1, d)}{y^2} \sum_{\substack{p \leq x \\ p \equiv 1 \pmod d}} \sum_{\substack{\chi \pmod p \\ \chi^d = \chi_0, \chi \neq \chi_0}} \left| \sum_{m \leq y}^* \left( \sum_{l \leq \sqrt{y/m}} \chi(l^2) \right) \chi(m) \right|^2,$$

where the asterisks on the sum over  $m$  indicates that  $m$  runs over square-free integers. Now we will apply Lemma 5 to  $S_2$ . Aided by table in [1, p. 897], we get that

$$S_2 \ll (xy)^\varepsilon \sqrt{\pi(x; 1, d)} \times \begin{cases} 1 & \text{if } x \leq y^{3/5} \\ x^{5/6}/y^{1/2} & \text{if } y^{3/5} < x \leq y^{6/7} \\ x^{1/4} & \text{if } y^{6/7} < x \leq y^{6/5} \\ x^{2/3}/y^{1/2} & \text{if } y^{6/5} < x \leq y^{3/2} \\ x^{1/3} & \text{if } y^{3/2} < x \leq y^{9/5} \\ x^{11/18}/y^{1/2} & \text{if } y^{9/5} < x \leq y^{108/55} \\ y^{7/10} & \text{if } y^{108/55} < x < y^{11/5} \\ x^{1/6}y^{1/3} & \text{if } y^{11/5} < x \leq y^{5/2} \\ x^{1/2}/y^{1/2} & \text{if } y^{5/2} < x \end{cases}. \tag{12}$$

Now a quick analysis will give that the  $O$ -terms in (11) can be absorbed into the bounds in (12) in all ranges. The result thus follows.

### 5. Proof of Theorem 3

Set

$$D(Y) = \{d : Y \leq d \leq 2Y, d \text{ odd and square-free}\}.$$

It is easy to see that

$$\#D(Y) \sim \frac{4Y}{\pi^2}. \tag{13}$$

Now let  $\Phi_Y(t)$  be a non-negative smooth function supported on the interval  $(1, 2)$  satisfying the conditions  $\Phi_Y(t) = 1$  for  $t \in (1 + 1/U, 2 - 1/U)$  and such that

$$\Phi_Y^{(j)}(t) \ll_j U^j \tag{14}$$

for all integers  $j \geq 0$ . We need to find an asymptotic formula for

$$\begin{aligned}
 S &= \frac{1}{\#D(Y)} \sum_{(a,2)=1} \mu^2(a) P_{(8a,2)}(x) \Phi_Y \left( \frac{a}{Y} \right) \\
 &= \frac{1}{2\#D(Y)} \sum_{2 < p \leq x} \sum_{(a,2)=1} \mu^2(a) \chi_0(8a) \Phi_Y \left( \frac{a}{Y} \right) \\
 &\quad + \frac{1}{2\#D(Y)} \sum_{2 < p \leq x} \sum_{(a,2)=1} \mu^2(a) \left( \frac{8a}{p} \right) \Phi_Y \left( \frac{a}{Y} \right).
 \end{aligned} \tag{15}$$

Let  $S_1$  and  $S_2$ , respectively, denote the two sums above. We shall study them separately.

Recalling that the support of  $\Phi_Y$  lies inside the interval  $(1, 2)$  and the bound (3), we have

$$\begin{aligned}
 S_1 &= \frac{1}{2\#D(Y)} \sum_{2 < p \leq x} \left( \sum_{\substack{(a,2)=1 \\ Y \leq a \leq 2Y}} \mu^2(a) \chi_0(8a) + O\left(\frac{Y}{U}\right) \right) \\
 &= \frac{1}{2\#D(Y)} \sum_{2 < p \leq x} \sum_{\substack{(a,2p)=1 \\ Y \leq a \leq 2Y}} \mu^2(a) + O\left(\frac{\pi(x)}{U}\right).
 \end{aligned} \tag{16}$$

Using Möbius inversion, we get

$$\begin{aligned}
 \sum_{2 < p \leq x} \sum_{\substack{(a,2p)=1 \\ Y \leq a \leq 2Y}} \mu^2(a) &= \sum_{2 < p \leq x} \sum_{\substack{(a,2)=1 \\ Y \leq a \leq 2Y}} \mu^2(a) \sum_{d|(a,p)} \mu(d) \\
 &= \sum_{2 < p \leq x} \sum_{d|p} \mu(d) \sum_{\substack{(a,2)=1, d|a \\ Y \leq a \leq 2Y}} \mu^2(a).
 \end{aligned} \tag{17}$$

Now it is easy to see that the above can be recast as

$$\sum_{2 < p \leq x} \sum_{\substack{Y \leq a \leq 2Y \\ (a,2)=1}} \mu^2(a) + O\left( \sum_{p \leq x} \sum_{\substack{Y \leq a \leq 2Y \\ p|a}} 1 \right) = \pi(x) \#D(Y) + O\left( Y \sum_{p \leq x} \frac{1}{p} \right). \tag{18}$$

Now combining (16), (17), (18) and (8), we arrive at

$$S_1 = \frac{\pi(x)}{2} + O\left( \frac{Y}{\#D(Y)} \log \log x + \frac{\pi(x)}{U} \right). \tag{19}$$

The treatment of  $S_2$  will be more complicated. Let  $z$  be a positive number to be chosen later. We start with

$$\mu^2(a) = M_z(a) + R_z(a), \text{ where } M_z(a) = \sum_{\substack{l^2|a \\ l \leq z}} \mu(l) \text{ and } R_z(a) = \sum_{\substack{l^2|a \\ l > z}} \mu(l).$$

We split  $S_2$  as

$$S_2 = S_{2,1} + S_{2,2} \tag{20}$$

where

$$S_{2,1} = \frac{1}{2\#D(Y)} \sum_{2 < p \leq x} \sum_{(a,2)=1} M_z(a) \left(\frac{8a}{p}\right) \Phi_Y\left(\frac{a}{Y}\right)$$

and

$$S_{2,2} = \frac{1}{2\#D(Y)} \sum_{2 < p \leq x} \sum_{(a,2)=1} R_z(a) \left(\frac{8a}{p}\right) \Phi_Y\left(\frac{a}{Y}\right).$$

We first deal with  $S_{2,2}$ . Interchanging summations gives us

$$S_{2,2} \ll \frac{1}{\#D(Y)} \sum_{l > z} \sum_{\substack{m \geq 1 \\ (m,2)=1}} \Phi_Y\left(\frac{l^2 m}{Y}\right) \left| \sum_{2 < p \leq x} \left(\frac{8l^2 m}{p}\right) \right|.$$

Using Lemma 3, we get that the above is

$$S_{2,2} \ll \frac{1}{\#D(Y)} \sum_{l > z} \sum_{m \geq 1} \Phi_Y\left(\frac{l^2 m}{Y}\right) x^{1/2} \log(l^2 m x) \ll \frac{\#D(Y)}{Y} \frac{x^{1/2} \log(x Y z)}{z}. \tag{21}$$

We now evaluate the inner sum of  $S_{2,1}$  following a method of K. Soundararajan in [10] by applying the Poisson summation formula to the sum over  $d$ . For all odd integers  $k$  and all integers  $m$ , we introduce the Gauss-type sums

$$\tau_m(k) := \sum_{a \pmod{k}} \left(\frac{a}{k}\right) e\left(\frac{am}{k}\right) =: \left(\frac{1+i}{2} + \left(\frac{-1}{k}\right) \frac{1-i}{2}\right) G_m(k).$$

We quote [10, Lemma 2.3] which determines  $G_m(k)$ .

**Lemma 5.** *If  $(k_1, k_2) = 1$  then  $G_m(k_1 k_2) = G_m(k_1) G_m(k_2)$ . Suppose that  $p^a$  is the largest power of  $p$  dividing  $m$  (put  $a = \infty$  if  $m = 0$ ). Then for  $b \geq 1$  we have*

$$G_m(p^b) = \begin{cases} 0 & \text{if } b \leq a \text{ is odd,} \\ \varphi(p^b) & \text{if } b \leq a \text{ is even,} \\ -p^a & \text{if } b = a + 1 \text{ is even,} \\ \left(\frac{m/p^a}{p}\right) p^a \sqrt{p} & \text{if } b = a + 1 \text{ is odd,} \\ 0 & \text{if } b \geq a + 2. \end{cases}$$

For a Schwartz function  $F$ , let

$$\tilde{F}(\xi) = \frac{1+i}{2} \hat{F}(\xi) + \frac{1-i}{2} \hat{F}(-\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x)) F(x) dx. \tag{22}$$

Moreover, we shall need Lemma 2.6 of [10], quoted below, which determines the inner sum of  $S_{2,1}$ .



**Lemma 6.** *Let  $\Phi$  be a non-negative, smooth function supported in  $(1, 2)$ . For any odd integer  $k$ ,*

$$\sum_{\gcd(d,2)=1} M_z(d) \left(\frac{d}{k}\right) \Phi\left(\frac{d}{X}\right) = \frac{X}{2k} \left(\frac{2}{k}\right) \sum_{\substack{\alpha \leq z \\ (\alpha, 2k)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_m (-1)^m G_m(k) \tilde{\Phi}\left(\frac{mX}{2\alpha^2 k}\right),$$

where  $\tilde{\Phi}$  is as defined in (22).

Applying Lemmas 5 and 6, we transform  $S_{2,1}$  into

$$\begin{aligned} S_{2,1} &= \frac{Y}{4\#D(Y)} \sum_{2 < p \leq x} \frac{1}{p} \sum_{\substack{\alpha \leq z \\ (\alpha, 2p)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_m (-1)^m G_m(p) \tilde{\Phi}\left(\frac{mY}{2\alpha^2 p}\right) \\ &= \frac{Y}{4\#D(Y)} \sum_{2 < p \leq x} \frac{1}{\sqrt{p}} \sum_{\substack{\alpha \leq z \\ (\alpha, 2p)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_{m \neq 0} (-1)^m \binom{m}{p} \tilde{\Phi}\left(\frac{mY}{2\alpha^2 p}\right) \quad (23) \\ &= S_{m=\square} + S_{m \neq \square}, \end{aligned}$$

where  $S_{m=\square}$  is the part of  $S_{2,1}$  with  $m$  being a square and  $S_{m \neq \square}$  the complementary sum. We shall also use the notations  $m = \square$  and  $m \neq \square$  to mean, respectively,  $m$  is a square and  $m$  is not a square.

Now

$$S_{m=\square} = \frac{Y}{2\#D(Y)} \sum_{2 < p \leq x} \frac{1}{\sqrt{p}} \sum_{\substack{\alpha \leq z \\ (\alpha, 2p)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_{\substack{m=1 \\ (m,p)=1}}^{\infty} (-1)^m \tilde{\Phi}\left(\frac{m^2 Y}{2\alpha^2 p}\right). \quad (24)$$

Note that trivially estimating the integral in (22) gives

$$\tilde{\Phi}(\xi) \ll 1,$$

and integration by parts yields

$$\tilde{\Phi}(\xi) = \frac{-1}{2\pi\xi} \left( \int_1^{1+1/U} + \int_{2-1/U}^2 \right) \Phi'(x) (\sin(2\pi\xi x) - \cos(2\pi\xi x)) dx \ll \frac{1}{|\xi|}.$$

These bounds for  $\tilde{\Phi}$  give

$$\sum_{m=1}^{\infty} \left| \tilde{\Phi}\left(\frac{m^2 Y}{2\alpha p}\right) \right| \ll \sum_{m \leq (2\alpha^2 p/Y)^{1/2}} 1 + \sum_{m > (2\alpha^2 p/Y)^{1/2}} \frac{2\alpha^2 p}{m^2 Y} \ll \alpha \sqrt{\frac{p}{Y}}.$$

So inserting the above estimate into (24), we get

$$S_{m=\square} \ll \frac{Y^{1/2} x}{\#D(Y)} \log z. \quad (25)$$

The estimation of  $S_{m \neq \square}$  is more complicated. Re-arranging the orders of summation and using  $\left(\frac{4\alpha^2}{p}\right)$  to detect the condition  $(2\alpha, p) = 1$ , we get

$$S_{m \neq \square} = \frac{Y}{4\#D(Y)} \sum_{\substack{\alpha \leq z \\ (\alpha, 2)=1}} \frac{\mu(\alpha)}{\alpha^2} \sum_{m \neq 0, \square} (-1)^m \sum_{p \leq x} \left(\frac{4\alpha^2 m}{p}\right) \frac{1}{\sqrt{p}} \tilde{\Phi}\left(\frac{mY}{2\alpha^2 p}\right).$$

Using integration by parts and Lemma 3, we get that the inner-most sum in the above displayed equation is

$$\begin{aligned} &\ll \log(\alpha(|m| + 2)x) \left| \tilde{\Phi}\left(\frac{mY}{2\alpha^2 x}\right) \right| + \int_1^x \frac{\log(\alpha(|m| + 2)V)}{V} \left| \tilde{\Phi}\left(\frac{mY}{2\alpha^2 V}\right) \right| dV \\ &\quad + \int_1^x \frac{Y \log(\alpha(|m| + 2)V)}{\alpha^2 V^2} \left| m\tilde{\Phi}'\left(\frac{mY}{2\alpha^2 V}\right) \right| dV. \end{aligned}$$

Thus

$$S_{m \neq \square} \ll R_1 + R_2 + R_3, \tag{26}$$

where

$$R_1 = \frac{Y}{\#D(Y)} \sum_{\alpha \leq z} \frac{1}{\alpha^2} \sum_m \log(\alpha(|m| + 2)X) \left| \tilde{\Phi}\left(\frac{mY}{2\alpha^2 x}\right) \right|,$$

$$R_2 = \frac{Y}{\#D(Y)} \sum_{\alpha \leq z} \frac{1}{\alpha^2} \int_1^x \sum_m \frac{\log(\alpha(|m| + 2)V)}{V} \left| \tilde{\Phi}\left(\frac{mY}{2\alpha^2 V}\right) \right| dV$$

and

$$R_3 = \frac{Y}{\#D(Y)} \sum_{\alpha \leq z} \frac{Y}{\alpha^4} \int_1^x \sum_m \frac{\log(\alpha(|m| + 2)V)}{V^2} \left| m\tilde{\Phi}'\left(\frac{mY}{2\alpha^2 V}\right) \right| dV.$$

Now observe that

$$\begin{aligned} &\sum_m \log(\alpha(|m| + 2)x) \left| \tilde{\Phi}\left(\frac{mY}{2\alpha^2 x}\right) \right| \\ &\ll \sum_{m \leq 2\alpha^2 x/Y} \log(\alpha(|m| + 2)x) + \sum_{m > 2\alpha^2 x/Y} \log(\alpha(|m| + 2)x) \frac{\alpha^4 x^2 U}{m^2 Y^2} \tag{27} \\ &\ll \frac{\alpha^2 x}{Y} \log(\alpha x) + \frac{\log(\alpha x) \alpha^2 x}{Y} U \ll \frac{\log(\alpha x) \alpha^2 x}{Y} U \end{aligned}$$

and similarly, using (14),

$$\sum_m \log(\alpha(|m| + 2)V) \left| m\tilde{\Phi}'\left(\frac{mY}{2\alpha^2 V}\right) \right| \ll \log(\alpha V) \frac{\alpha^4 V^2 U^2}{Y^2}. \tag{28}$$

From (27), it follows that

$$R_1 \ll \frac{xU}{\#D(Y)} \sum_{\alpha \leq z} \log(\alpha x) \ll \frac{xU}{\#D(Y)} z \log(zx) \tag{29}$$

and

$$R_2 \ll \frac{U}{\#D(Y)} \sum_{\alpha \leq z} \int_1^x \log(\alpha V) dV \ll \frac{U}{\#D(Y)} xz \log(zx). \tag{30}$$

Using (28)

$$R_3 \ll \frac{U^2}{\#D(Y)} \sum_{\alpha \leq z} \int_1^x \log(\alpha V) dV \ll \frac{U^2}{\#D(Y)} xz \log(zx). \tag{31}$$

Combining (26), (29), (30) and (31), we arrive at

$$S_{m \neq \square} \ll \frac{xz}{\#D(Y)} U^2 \log(zx).$$

From this, together with (25) and (23), it follows that

$$S_{2,1} \ll \frac{Y^{1/2}x}{\#D(Y)} \log z + \frac{xz}{\#D(Y)} U^2 \log(zx).$$

So from the above, together with (21), (20), (19), (15) and (13), we get

$$S = \frac{\pi(x)}{2} + O\left(\log \log x + \frac{\pi(x)}{U} + \frac{x \log z}{Y^{1/2}} + \frac{xz}{Y} U^2 \log(zx) + \frac{x^{1/2} \log(xYz)}{z}\right). \tag{32}$$

Now it still remains to optimize  $z$ . To this end, we set

$$z = \frac{Y^{1/2}}{Ux^{1/4}}.$$

So (32) becomes

$$S = \frac{\pi(x)}{2} + O\left(\log \log x + \frac{\pi(x)}{U} + \frac{x \log Y}{Y^{1/2}} + \frac{Ux^{3/4}}{Y^{1/2}} \log(xY)\right).$$

Now recall  $U = x^{1/8}Y^{1/4}$  and  $\pi(x) \sim x/\log x$ . The proof of Theorem 3 is complete.

**Acknowledgments.** P. G. is supported in part by NSFC grants 11371043 and 11871082 and L. Z. by the FRG grant PS43707 and the Faculty Silverstar Fund PS49334. Parts of this work were done when P. G. visited the University of New South Wales (UNSW) in June 2017. He wishes to thank UNSW for the invitation, financial support and warm hospitality during his pleasant stay. Finally, both authors would like to thank the anonymous referee for his/her careful reading of the manuscript and many helpful comments.

**References**

- [1] S. Baier and M. P. Young, Mean values with cubic characters, *J. Number Theory* **130** (2010), 879-903.
- [2] V. Blomer, L. Goldmakher and B. Louvel,  $L$ -functions with  $n$ -th order twists, *Int. Math. Res. Not. IMRN* **2014** (2014), 1925-1955.
- [3] S. Baier and M. P. Young, Large sieve inequalities for quartic character sums, *Q. J. Math.* **63** (2012), 891-917.
- [4] D. R. Heath-Brown, A mean value estimate for real character sums, *Acta Arith.* **72** (1995), 235-275.
- [5] C. Hooley, On Artin's conjecture, *J. Reine Angew. Math.* **225** (1995), 209-220.
- [6] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society, Providence, 2004.
- [7] M. Jutila, On the mean value of  $L(1/2, \chi)$  for real characters, *Analysis* **1** (1981), 149-161.
- [8] S. Li, Notes on primes  $P \equiv 1 \pmod{D}$  and  $A^{P-1/D} \equiv 1 \pmod{P}$ , *Integers* **15** (2015), A25, 8pp.
- [9] M. R. Murty, Artin's conjecture for primitive roots, *Math. Intelligencer* **10** (1988), 59-67.
- [10] K. Soundararajan, Nonvanishing of quadratic Dirichlet  $L$ -functions at  $s = \frac{1}{2}$ , *Ann. of Math.* (2) **152** (2000), 447-488.