QUASIPERFECT NUMBERS WITH THE SAME EXPONENT

Tomohiro Yamada
Center for Japanese Language and Culture
Osaka University, Aomatanihigashi, Minoo, Osaka, Japan
tyamada1093@gmail.com

Received: 11/9/16, Revised: 3/13/19, Accepted: 6/3/19, Published: 7/31/19

Abstract

We study some divisibility properties of quasiperfect numbers. We show that if $N = (p_1 p_2 \cdots p_t)^{2a} = m^2$ is quasiperfect, then $2a+1$ is divisible by 3 and N has at least one prime factor smaller than $e^{721.85}$. Moreover, we find some lower bounds concerning quasiperfect numbers of the form $N = m^2$ with m squarefree.

1. Introduction

A positive integer N is called perfect if $\sigma(N) = 2N$, where $\sigma(N)$ denotes the sum of divisors of N. As is well known, an even integer N is perfect if and only if $N = 2^{k-1}(2^k - 1)$ with $2^k - 1$ prime. In contrast, one of the oldest unsolved problems is whether there exists an odd perfect number or not. Moreover, it is also unknown whether there exists an odd multiperfect number, i.e., an integer N with $\sigma(N) = kN$ for some integer $k \geq 2$ or not.

Cattaneo [1] called a positive integer N quasiperfect if $\sigma(N) = 2N + 1$, and showed that such an integer must be an odd square, that any proper divisor m of N satisfies $\sigma(m) < 2m$, and that any divisor of $\sigma(N)$ is congruent to 1 or 3 modulo 8. Hagis and Cohen [9] showed that if N is quasiperfect, then $N > 10^{35}$ and N has at least 7 distinct prime factors.

It is well known that an odd perfect number must be of the form $q^b p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t}$ for some integers $a_1, a_2, \ldots, a_t, b$ and distinct primes $p_1, p_2, \ldots, p_t, q$ with $q \equiv b \equiv 1 \pmod 4$. In the special case $a_1 = a_2 = \cdots = a_t = a$, several results are known as follows:

(A) If $a_1 = a_2 = \cdots = a_t = a$, we know that $a \geq 9$ and $a \neq 10, 11, 12, 13, 14, 16, 17, 18, 19, 24, 62$, combining results in [3], [10], [13], [15], [16] and [20];

(B) If d divides $2a_i + 1$ for all i, then d cannot be one of $3, 35, 65$, combining results in [6], [15] and [16];

(C) In [21], the author showed that, for any given a, there exist only finitely many odd perfect numbers of the form $q^b(p_1p_2\cdots p_t)^{2a}$;

(D) Fletcher, Nielsen and Ochem [7] showed that an odd perfect number $q^b p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t}$ for which there exists a finite set S of primes such that each $2a_i + 1$ is divisible by a prime in S, must have a prime factor below an effectively computable constant C depending on S. In [23], the author gave an explicit upper bound for C.

Some results similar to (A), (B) above are known for quasiperfect numbers of the form $p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t}$ with $a_1 = a_2 = \cdots = a$ or $d \mid (2a_i + 1) (i = 1, 2, \ldots, t)$ for some integer $d > 1$. Cohen [2] showed that if an integer of the form $(p_1p_2\cdots p_t)^{2a}$ is quasiperfect, then a must be congruent to $1, 3, 5, 9$ or $11 \pmod{12}$, and if an integer of the form $p_1^{6a_1+2} p_2^{6a_2+2} \cdots p_t^{6a_t+2}$ is quasiperfect, then $t \geq 230876$.

We would like to begin by giving an extension of this result, which follows from an elementary argument.

Theorem 1. *If $N = p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t} = m^2$ is quasiperfect, then there exists a prime factor $p_j \equiv 1 \pmod 4$ for which $2a_j + 1$ does not have a divisor congruent to $5 \pmod 8$. Moreover, if $N = (p_1p_2\cdots p_t)^{2a} = m^2$ is quasiperfect, then:*

(a) Any prime factor of N must be congruent to 1 or $7 \pmod 8$.

(b) $2a + 1 \equiv 3 \pmod 8$ and, if q is a prime factor of $2a + 1$, then q must be congruent to 1 or $3 \pmod 8$.

(c) $2a + 1$ must be divisible by 3.

Proof. Since $\sigma(N) = 2N + 1 = 2m^2 + 1 \equiv 3 \pmod 8$, we have $\sigma(p_j^{2a_j}) \not\equiv 1 \pmod 8$ for some j, for which, by Cattaneo's result mentioned above, $\sigma(p_j^{2a_j}) \equiv 3 \pmod 8$ cannot have a prime factor congruent to 5 or 7 modulo 8. If d divides $2a_j + 1$, then $\sigma(p_j^{d-1})$ divides $\sigma(p_j^{2a_j}) \mid \sigma(N) = 2m^2 + 1$ and therefore has only prime factors congruent to 1 or 3 modulo 8. Since $\sigma(p_j^{2a_j}) \equiv 3 \pmod 8$, we must have $p_j \equiv 1 \pmod 4$. If $p_j \equiv 1 \pmod 8$, then $d \equiv 1$ or $3 \pmod 8$, and if $p_j \equiv 5 \pmod 8$, then $d \equiv 1$ or $7 \pmod 8$. Hence, d cannot be congruent to 5 modulo 8. This proves the former part of the theorem.

Next, assume that $N = (p_1p_2\cdots p_t)^{2a} = m^2$ is quasiperfect. By Cohen's result mentioned above, we must have $2a + 1 \equiv 3 \pmod 4$. Hence, N has no prime factor congruent to $3 \pmod 8$. In particular, $N = m^2$ is not divisible by 3 and therefore $\sigma(N) = 2m^2 + 1$ must be divisible by 3. Hence, there exists a prime factor p_k for which $\sigma(p_k^{2a})$ is divisible by 3. This implies $p_k \equiv a \equiv 1 \pmod 3$. So that $2a + 1$ must be divisible by 3 as stated in (c). Since $p_i^2 + p_i + 1$ divides $\sigma(N)$, $p_i^2 + p_i + 1$ must be congruent to 1 or 3 modulo 8 and therefore $p_i \equiv \pm 1 \pmod 8$ for any i, proving (a). By the former part of the theorem, $p_s \equiv 1 \pmod 8$ for some s. If

d divides $2a_s + 1$, then $\sigma(p_s^{d-1}) \mid \sigma(p_s^{2a}) \mid \sigma(N) = 2m^2 + 1$ and therefore $d \equiv 1$ or $3 \pmod 8$. Finally, Cohen's result mentioned above yields that a cannot be a multiple of 4 and therefore $2a + 1 \equiv 3 \pmod 8$, which proves (b). □

No result similar to (C) above has been known for quasiperfect numbers and neither has the author been able to prove such a result. Instead, in [23], the author proved that if $N = p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t}$ is quasiperfect and there exists a finite set S of primes such that each $2a_i + 1$ is divisible by a prime in S, then N must have a prime factor below $C_0 = \exp(2173.5\,|S|^2 \max\{8l, e^{13.3}\})$. More generally, the author proved that if $N = p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t}$ satisfies that $\sigma(N) \geq 2N$ has no prime factor congruent to 5 or 7 modulo 8, and there exists a finite set S of primes such that each $2a_i + 1$ is divisible by a prime in S, then N must have a prime factor below C_0. In this paper, we prove the following result.

Theorem 2. *If $N = (p_1 p_2 \cdots p_t)^{2a} = m^2$ is quasiperfect, then N must have a prime factor below $C = e^{721.85} < 3.129477 \cdot 10^{313}$.*

We shall give an outline of our proof here. From Theorem 1, we can see that each $p_i^2 + p_i + 1$ divides $\sigma(N) = 2m^2 + 1$ and therefore cannot have a prime factor $\equiv 7, 13 \pmod{24}$, which is implicit in the Note of Lemma 3 of [2]. Using some sieve argument, we shall prove that the number of prime $p \leq X$ such that $p^2 + p + 1$ has no prime factor $\equiv 7, 13 \pmod{24}$ is $< cX/\log^{3/2} X$ with an explicit constant c.

Our sieve argument is based on the argument used in [22] to prove that an odd perfect number $q^b p_1^{2a_1} p_2^{2a_2} \cdots p_t^{2a_t}$ with all p_i in a given finite set S must have a prime factor below an effectively computable constant C depending on S. This method was refined by Fletcher, Nielsen and Ochem [7] and the author [23] to prove (D).

Finally, we would like to show the following lower bounds concerning quasiperfect numbers of the form $N = m^2$ with m squarefree.

Theorem 3. *If $N = m^2$ with m squarefree is quasiperfect, then $N > e^{17840573219}$ and N must have at least 406550054 distinct prime factors, one of which is $\geq$ 9457308739.*

Our results lead us to conjecture that there exists no quasiperfect number of the form $N = (p_1 p_2 \cdots p_t)^{2a}$.

2. Proof of Theorem 2

From (a) of Theorem 1 and a remark in the introduction, we see that if p is a prime factor of N, then $p \equiv \pm 1 \pmod 8$ and $p^2 + p + 1$ cannot have a prime factor congruent to 7 or 13 (mod 24). Letting $Q^{\pm}$ be the set of prime numbers $p \equiv \pm 1 \pmod 8$ such that $p^2 + p + 1$ has no prime factor congruent to 7 or 13 (mod 24), we can say that any prime factor of N must be contained in either Q^+ or Q^-.

We would like to introduce some notations in order to apply sieve methods. For a given prime p, we let Ω_p denote a set of congruent classes modulo p and define $\rho(p)$ to be the number of such congruent classes. Under this setting, let $S(x, \Omega, y)$ denote the set of integers not exceeding x which does not belong to Ω_p for any prime $p \leq y$.

Hereafter, $\pm$ denotes a sign which is possibly different at each occurrence but the same in the same context. For our purpose, we put $\Omega_p^{\pm} = \{n \mid (8n \pm 1)((8n \pm 1)^2 + (8n \pm 1) + 1) \equiv 0 \pmod{p}\}$ for primes p congruent to 7 or 13 (mod 24) and $\Omega_p^{\pm} = \{n \mid 8n \pm 1 \equiv 0 \pmod{p}\}$ for other odd primes p. Then, we see that $\rho^{\pm}(2) = 0$, $\rho^{\pm}(p) = 3$ for primes $p \equiv 7, 13 \pmod{24}$ and $\rho^{\pm}(p) = 1$ for the other primes p. Moreover, it is clear that if $p = 8n \pm 1$ belongs to $Q^{\pm}$ and $y < n \leq x$, then n must be contained in $S(x, \Omega^{\pm}, y)$ and therefore

$$\pi^{\pm}(8x \pm 1) \leq y + \left|S(x, \Omega^{\pm}, y)\right|, \tag{1}$$

where $\pi^{\pm}(X)$ denotes the number of primes not exceeding X that belongs to $Q^{\pm}$.

In order to estimate $|S(x, \Omega^{\pm}, y)|$, we use the sieve argument mentioned in the introduction. Let us introduce the following notation:

$$B(z) = B^{\pm}(z) = \frac{1}{\log z} \sum_{p \leq z} \frac{\rho^{\pm}(p) \log p}{p}, \tag{2}$$

$$V(z) = V^{\pm}(z) = \prod_{p \leq z} \left(1 - \frac{\rho^{\pm}(p)}{p}\right) \tag{3}$$

and

$$\psi_0^{\pm}(v, u) = 1 - \exp(-\psi_1(B^{\pm}(x^{\frac{1}{v}}), v/u)), \tag{4}$$

where

$$\psi_1(K, t) = \max\left\{0, t \log \frac{t}{K} - t + K\right\}. \tag{5}$$

Now, we have the following sieve inequality.

Lemma 1. *For any real $v \geq u \geq 2$, we have*

$$\left|S(x, \Omega^{\pm}, x^{\frac{1}{u}})\right| \leq \frac{(x + x^{\frac{2}{u}}) V^{\pm}(x^{\frac{1}{u}})}{\psi_0^{\pm}(v, u)}. \tag{6}$$

Proof. This is essentially a special case of Lemma 2.2 of [23]. However, we would like to give the proof again for self-containedness.

Let $g(m) = g^{\pm}(m)$ be the multiplicative function over the squarefree integers m with $g^{\pm}(p) = \rho^{\pm}(p)/(p - \rho^{\pm}(p))$ for each prime p. We define $G(w) = \sum_{n \leq w} g(n)$ and $G_z(w) = \sum_{n \leq w, p|n \Rightarrow p \leq z} g(n)$. Observing that $0 \leq \rho(p) < p$ for any prime p, we use Theorem 7.14 in [11] to obtain

$$\left|S(x, \Omega, x^{\frac{1}{u}})\right| \leq \frac{(x + x^{\frac{2}{u}})}{G(x^{\frac{1}{u}})}. \tag{7}$$

Now Theorem 2.2.1 in [8] gives (6) with $\sqrt{D} = x^{1/u}$ and $B = \sup_t B(t)$ in place of $B(z)$. But we can see that this theorem still holds with B replaced by $B(z)$, whether the supremum B exists or not. Indeed, it follows from the argument in p.p. 53–54 in [8] that

$$1 - V(P(z))G_z(x^{1/u}) \leq \exp\left(-c\frac{\log x}{u\log z} + B(z)(e^c - 1)\right) \tag{8}$$

for any constant $c \geq 0$. Choosing $c = \log(v/u) - \log B(z)$, we obtain the lemma. □

Now we need to estimate $B(z)$ and $V(z)$. To this end, we need some explicit estimates for a sum and a product involving primes in arithmetic progressions. Our start points are the following error estimates in the prime number theorem for arithmetic progressions with difference 24, which can be obtained from some known explicit versions of the prime number theorem for arithmetic progressions and a recent numerical result for the Generalized Riemann Hypothesis of L-functions.

Lemma 2. *As in [4], we set $R = 9.645908801$. Let ℓ be any integer coprime to 6. Then,*

$$\frac{1}{z}\left|\psi(z; 24, \ell) - \frac{z}{8}\right| < \begin{cases} c_1 := 6.28782 \cdot 10^{-4} & \textit{for } z \geq 10^{10}, \\ c_2 := 2.04555 \cdot 10^{-5} & \textit{for } z \geq e^{30}, \\ c_3 := 1.24271 \cdot 10^{-7} & \textit{for } z \geq e^{60}. \end{cases} \tag{9}$$

Moreover, for any real $z \geq e^{250R}$, we have

$$\left|\psi(z; 24, \ell) - \frac{z}{8}\right| < \frac{0.001197z}{\log z} \tag{10}$$

and

$$\left|\psi(z; 24, \ell) - \frac{z}{8}\right| < \frac{2.88643z}{\log^2 z}. \tag{11}$$

Proof. Platt [17] confirmed that $L(s, \chi)$ has no nontrivial zero with $\Re s \neq 1/2$ and $|\Im s| \leq 10^8/24$ for any character χ modulo 24. From Theorem 3.6.3 of [18], we see that any nontrivial zero s of $L(s, \chi)$ satisfies $1 - \operatorname{Re} s \geq 1/R\log(24\,|\operatorname{Im} s|\,/68.9385)$ for any character χ modulo 24. Now (9) follows from Theorem 5.1.1 of [18] applied with $H_\chi = 10^8/24$ and $C_1(\chi) = 68.9385$ for each character χ modulo 24.

Moreover, we can apply Theorem 5 of [4] with $H = 1000, C_1(k) = 32$ (we note that H and $C_1(k)$ in this theorem can be taken smaller) and $X_4 \leq \sqrt{250}$ for any character modulo 24 to obtain (10) and (11). This proves the lemma. □

Remark 1. Kadiri [12] claimed to prove that any nontrivial zero s of $L(s, \chi)$ satisfies $1 - \operatorname{Re} s \geq 1/6.397\log(24\,|\operatorname{Im} s|)$ for all characters χ modulo 24. This would give better estimates.

Using these error estimates, we obtain the following bounds.

Lemma 3. *For any real $z \geq e^{60}$, we have*

$$\sum_{p\leq z, p\equiv 7,13 \pmod{24}} \frac{\log p}{p} < \frac{\log z}{4} \tag{12}$$

and

$$\prod_{p\leq z, p\equiv 7,13 \pmod{24}} \left(1-\frac{1}{p}\right)^{-1} > 0.952711 \log^{\frac{1}{4}} z. \tag{13}$$

Proof. In this proof of the lemma, we let ℓ be an integer congruent to 7 or 13 modulo 24. We begin by proving (12) for $z \geq e^{60}$. Lemma 2 gives

$$\begin{aligned} \int_{10^{10}}^{\infty} \frac{|E_\psi(t;24,l)|}{t^2}dt <&(30-10\log 10)c_1 + 30c_2 + (250R-60)c_3 + \frac{2.88643}{250R} \\ <&0.008+0.002 = 0.01, \end{aligned} \tag{14}$$

where $E_\psi(t;k,\ell) = \psi(t,k,\ell) - t/\varphi(k)$. By Theorem 5.2.1 of [18], we have

$$\int_{10^8}^{10^{10}} \frac{|E_\psi(t;24,l)|}{t^2}dt < 1.745 \int_{10^8}^{10^{10}} \frac{dt}{t^{3/2}} = \frac{3.49}{10^4} - \frac{3.49}{10^5} < 0.00032. \tag{15}$$

Some calculations yield that

$$\sum_{p<10^8, p\equiv 7 \pmod{24}} \frac{\log p}{p} < \log 10 - 0.101846 \tag{16}$$

and

$$\sum_{p<10^8, p\equiv 13 \pmod{24}} \frac{\log p}{p} < \log 10 - 0.202137. \tag{17}$$

Moreover, other calculations give $\psi(10^8;24,7) > 12499496$ and $\psi(10^8;24,13) > 12499441$. We use partial summation and then apply these inequalities and (11) to obtain that, for $z \geq 10^8$ and $\ell = 7$ or 13,

$$\begin{aligned} \sum_{\substack{p\leq z,\\ p\equiv \ell \pmod{24}}} \frac{\log p}{p} &\leq \sum_{\substack{p\leq 10^8,\\ p\equiv \ell \pmod{24}}} \frac{\log p}{p} + \sum_{\substack{10^8<n\leq z,\\ n\equiv \ell \pmod{24}}} \frac{\Lambda(n)}{n} \\ &= \sum_{\substack{p\leq 10^8,\\ p\equiv \ell \pmod{24}}} \frac{\log p}{p} + \frac{\psi(z;24,\ell)}{z} - \frac{\psi(10^8;24,\ell)}{10^8} + \int_{10^8}^{z} \frac{\psi(t;24,\ell)}{t^2}dt \\ &< \frac{\log z}{8} - 0.101846 + \frac{0.01}{\log z} + 0.0004 + 0.01 \\ &< \frac{\log z}{8}, \end{aligned} \tag{18}$$

which immediately gives (12) for $z \geq e^{60}$ (and even for $z \geq 10^8$).

Next we prove (13) for $z \geq e^{60}$. By (9) we have

$$\int_z^{e^{250R}} \frac{c_3(1+\log t)}{t\log^2 t}dt = c_3\left(\frac{1}{\log z} - \frac{1}{250R} + \log(250R) - \log\log z\right) \tag{19}$$

for $e^{60} \leq z < e^{250R}$ and, by (10), we have

$$\int_z^{\infty} \frac{0.001197(1+\log t)}{t\log^3 t}dt = 0.001197\left(\frac{1}{\log z} + \frac{1}{2\log^2 z}\right) < 4.9659\times 10^{-7} \tag{20}$$

for $z \geq e^{250R}$.

Henceforth, we let $z \geq e^{60}$ as in the lemma. Using (19) and (20), we have

$$\begin{aligned}
&\sum_{\substack{p\leq z,\\ p\equiv \ell \pmod{24}}} \frac{1}{p}\\
=&\frac{1}{8}\log\log z + M(24,\ell) + \frac{E_\theta(z;24,\ell)}{z\log z} - \int_z^\infty \frac{(1+\log t)E_\theta(t;24,\ell)}{t^2\log^2 t}dt \\
>&\frac{1}{8}\log\log z + M(24,\ell) - \frac{c_3}{60} - 4.61034\times 10^{-6} - 4.9659\times 10^{-7},
\end{aligned} \tag{21}$$

where $E_\theta(z;k,\ell) = \theta(z;k,\ell) - z/\varphi(k)$ and $M(k,\ell)$ denotes the limit

$$\lim_{x\to\infty} \sum_{\substack{p\leq x,\\ p\equiv \ell \pmod{k}}} \frac{1}{p} - \frac{\log\log x}{\varphi(k)}. \tag{22}$$

Using $M(24,7) > 0.0038975\cdots$ and $M(24,13) > -0.068154106\cdots$, as is given by [14], we obtain

$$\sum_{\substack{p\leq z,\\ p\equiv 7,13 \pmod{24}}} \frac{1}{p} > \frac{1}{8}\log\log z - 0.064266, \tag{23}$$

and therefore

$$\begin{aligned}
\prod_{\substack{p\leq z,\\ p\equiv 7,13 \pmod{24}}} \left(1-\frac{1}{p}\right)^{-1} &= \exp\left[\sum_{\substack{p\leq z,\\ p\equiv 7,13 \pmod{24}}} \left(\frac{1}{p} + \frac{1}{2p^2} + \cdots\right)\right] \\
&> \exp\left(0.015815 + \sum_{\substack{p\leq z,\\ p\equiv 7,13 \pmod{24}}} \frac{1}{p}\right) \\
&> 0.952711\log^{\frac{1}{4}} z,
\end{aligned} \tag{24}$$

which proves the lemma. □

Now $V(P(z))$ and $B(z)$ can be easily estimated from Lemma 3 and known error estimates for the ordinary prime number theorem. Indeed, combining (13) and the estimate

$$\prod_{p\le z}\left(1-\frac{1}{p}\right)<\frac{e^{-\gamma}}{\log z}\left(1+\frac{1}{5\log^3 z}\right) \tag{25}$$

for $z\ge 2278382$ given in Theorem 5.9 of [5], we conclude that

$$\begin{aligned} V(P(z)) &< \prod_{2<p\le z}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le z,\\ p\equiv 7,13 \pmod{24}}}\left(1-\frac{1}{p}\right)^2 \\ &<\frac{2e^{-\gamma}}{(0.952711)^2\log^{\frac{3}{2}} z}\left(1+\frac{1}{5\log^2 z}\right) \\ &<\frac{1.23273}{\log^{\frac{3}{2}} z}, \end{aligned} \tag{26}$$

for $z\ge e^{60}$. Moreover, by (3.22) of [19], we have

$$\sum_{p\le z}\frac{\log p}{p}<\log z-1.33258+\frac{1}{2\log z}<\log z \tag{27}$$

for $z>319$ and, together with (12), we immediately obtain

$$\sum_{p\le z}\frac{\rho(p)\log(p)}{p}<\frac{3}{2}\log z \tag{28}$$

or, equivalently, $B(z)<1.5$ for $z\ge e^{60}$.

Equations (26) and (28) allow us to take $B=1.5, u=2.0173$ and $v=7.58$ in the sieve inequality (6), which gives

$$\left|S(x,\Omega^{\pm},y)\right|\le\frac{37.13862x}{\log^{\frac{3}{2}} x} \tag{29}$$

for $x>e^{719.77}$ and therefore, by (1),

$$\pi^{\pm}(X)\le\frac{4.65574X}{\log^{\frac{3}{2}} X} \tag{30}$$

for $X>C=e^{721.85}$.

Now we have

$$\begin{aligned}
\prod_{p\geq C, p\in Q^+\cup Q^-} \frac{p}{p-1} &< \exp\left[\sum_{p\geq C, p\in Q^+\cup Q^-}\left(\frac{1}{p}+\frac{1}{2p^2}+\cdots\right)\right] \\
&< \exp\left(\frac{1}{C}+\sum_{p\geq C, p\in Q^+\cup Q^-}\frac{1}{p}\right) \\
&< \exp\left(\frac{2}{C}+\int_C^\infty \frac{9.31148dt}{t\log^{\frac{3}{2}} t}\right) \\
&= \exp\left(\frac{2}{C}+\frac{18.62296}{\log^{\frac{1}{2}} C}\right) < 2.
\end{aligned} \tag{31}$$

However, as mentioned in the beginning of this section, any prime factor of N must belong to $Q^+\cup Q^-$. Hence, if N has no prime factor below C, then $\sigma(N) < N\prod_{p|N} p/(p-1) < 2N$, which contradicts the assumption that $\sigma(N) = 2N+1$. This proves Theorem 2.

3. Proof of Theorem 3

Assume that $N = m^2$ with m squarefree is quasiperfect and let P be the greatest prime factor of N. As noted in the previous section, any prime factor of N belongs to $Q^+\cup Q^-$.

Calculation gives

$$\prod_{p<2^{29}, p\in Q^+\cup Q^-} \frac{p^2+p+1}{p^2} < 1.75014319434, \tag{32}$$

$$\prod_{p<2^{29}, p\in Q^+\cup Q^-} p > e^{62460825.5} \tag{33}$$

and

$$\pi^+(2^{29})+\pi^-(2^{29}) = 3285696. \tag{34}$$

By Theorem 6.12 of [5], we have

$$\prod_{2^{29}<p\leq P}\frac{p}{p-1} < \frac{\log P}{29\log 2}\left(1+\frac{1}{121945\log^3 2}\right)\left(1+\frac{1}{5\log^3 P}\right). \tag{35}$$

It follows from (32) and (35) that

$$\prod_{p\leq 9457308738, p\in Q^+\cup Q^-} \frac{p^2+p+1}{p^2} < 2, \tag{36}$$

and therefore we must have $P \geq 9457308739$.

Now, let $P_0 = 9457308739$. Then we must have

$$m \geq \left(\prod_{p < 2^{29}, p \in Q^+ \cup Q^-} p \right) \left(\prod_{2^{29} < p \leq P_0} p \right). \tag{37}$$

Theorem 4.2 of [5] gives

$$\prod_{2^{29} < p \leq P_0} p = \exp(\theta(P) - \theta(2^{29})) > \exp\left(P_0 - \frac{P_0}{100 \log^2 P_0} - 536842885.9 \right) \tag{38}$$

and, with the aid of (37), we see that $m > e^{8920286609.5}$ and $N > e^{17840573219}$.

Finally, we observe that, using (34) and Theorem 5.1 of [5],

$$\begin{aligned} &\pi^+(P) + \pi^-(P) \geq \pi(P_0) - \pi(2^{29}) + \pi^+(2^{29}) + \pi^-(2^{29}) \\ \geq &\frac{P}{\log P}\left(1 + \frac{1}{\log P} + \frac{2}{\log^2 P} + \frac{7.32}{\log^3 P}\right) - \pi(2^{29}) + 3285696 > 406550053.02 \end{aligned} \tag{39}$$

and therefore N must have at least 406550054 prime factors. This completes the proof of Theorem 3.

References

[1] Paolo Cattaneo, Sui numeri quasiperfetti, *Boll. Un. Mat. Ital.* (3) **6** (1951), 59–62.

[2] Graeme L. Cohen, The nonexistence of quasiperfect numbers of certain forms, *Fibonacci Quart.* **20** (1982), 81–84.

[3] G. L. Cohen and R. J. Williams, Extensions of some results concerning odd perfect numbers, *Fibonacci Quart.* **23** (1985), 70–76.

[4] P. Dusart, Estimates of $\theta(x; k, l)$ for large values of x, *Math. Comp.* **71** (2001), 1137–1166.

[5] P. Dusart, Explicit estimates of some functions over primes, *Ramanujan J.*, **45** (2018), 227–251.

[6] Ronald Evans and Jonathan Pearlman, Nonexistence of odd perfect numbers of a certain form, *Fibonacci Quart.* **45** (2007), 122–127.

[7] S. Adam Fletcher, Pace P. Nielsen and Pascal Ochem, Sieve methods for odd perfect numbers, *Math. Comp.* **81** (2012), 1753–1776.

[8] G. Greaves, *Sieves in Number Theory*, Springer-Verlag, Berlin, 2001.

[9] Peter Hagis, Jr. and Graeme L. Cohen, Some results concerning quasiperfect numbers, *J. Aust. Math. Soc. (Ser. A)* **33** (1982), 275–286.

[10] P. Hagis Jr. and Wayne L. McDaniel, A new result concerning the structure of odd perfect numbers, *Proc. Amer. Math. Soc.* **32** (1972), 13–15.

[11] H. Iwaniec and Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.

[12] Habiba Kadiri, An explicit zero-free region for the Dirichlet L-functions, preprint http://arxiv.org/abs/0510570.

[13] H.-J. Kanold, Untersuchungen über ungerade vollkommene Zahlen, *J. Reine Angew. Math.* **183** (1941), 98–109.

[14] A. Languasco and A. Zaccagnini, Computing the Mertens and Meissel-Mertens constants for sums over arithmetic progressions, *Experiment. Math.* **19** (2010), 279–284. The complete set of results is available online http://www.math.unipd.it/ languasc/Mertens-comput.html.

[15] W. L. McDaniel, The non-existence of odd perfect numbers of a certain form, *Arch. Math. (Basel)* **21** (1970), 52–53.

[16] W. L. McDaniel and P. Hagis Jr., Some results concerning the non-existence of odd perfect numbers of the form $p^a M^{2\beta}$, *Fibonacci Quart.* **13** (1975), 25–28.

[17] David J. Platt, Numerical computations concerning the GRH, *Math. Comp.* **85** (2016), 3009–3027.

[18] O. Ramare and R. Rumely, Primes in arithmetic progressions, *Math. Comp.* **65** (1996), 397–425.

[19] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[20] R. Steuerwald, Verschärfung einen notwendigen Bedingung für die Existenz einen ungeraden vollkommenen Zahl, *S.-B. Bayer. Akad. Wiss.* 1937, 69–72.

[21] T. Yamada, Odd perfect numbers of a special form, *Colloq. Math.* **103** (2005), 303–307.

[22] Tomohiro Yamada, On the divisibility of odd perfect numbers by a high power of a prime, preprint, https://arxiv.org/abs/math/0511410.

[23] Tomohiro Yamada, On the divisibility of odd perfect numbers by a high power of a prime, II, submitted, preprint, https://arxiv.org/abs/1609.08795.