



ON EXACTLY K -DEFICIENT-PERFECT NUMBERS

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Abstract

For a positive integer n , let $\sigma(n)$ denote the sum of all positive divisors of n . A positive integer n is called an exactly k -deficient-perfect number if $\sigma(n) = 2n - d_1 - d_2 - \cdots - d_k$, where d_i ($1 \leq i \leq k$) are distinct proper divisors of n . In this paper, we determine all odd exactly 2-deficient-perfect numbers n with two distinct prime divisors.

1. Introduction

For a positive integer n , let $\sigma(n)$ denote the sum of all positive divisors of n . We call n perfect if $\sigma(n) = 2n$. It is well known that an even integer n is perfect if and only if $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes. It is not known whether there exists an odd perfect number. Numerous authors have defined a number of closely related concepts. For example, n is called deficient if $\sigma(n) < 2n$, and n is called abundant if $\sigma(n) > 2n$, etc.

In 2012, Pollack and Shevelev [2] introduced the concept of k -near-perfect numbers. For $k \geq 1$, n is called k -near-perfect if n is the sum of all of its proper divisors with at most k exceptions (called redundant divisors). A 1-near-perfect number with exactly 1 redundant divisor is called *near-perfect*. Pollack and Shevelev [2] presented an upper bound on the count of near-perfect numbers and proved that there are infinitely many k -near-perfect numbers n with exactly k redundant divisors for all large k . Recently, Li and Liao [1] gave two equivalent conditions of all even near-perfect numbers of the forms $2^\alpha p_1 p_2$ and $2^\alpha p_1^2 p_2$. For more results on near-perfect numbers, see [3, 4, 6].

A positive integer n is called an *exactly k -deficient-perfect number* if $\sigma(n) = 2n - d_1 - d_2 - \cdots - d_k$, where d_i ($1 \leq i \leq k$) are distinct proper divisors of n (called deficient divisors). In particular, a positive integer n is *deficient-perfect* with *deficient* divisor d if $\sigma(n) = 2n - d$, where d is a proper divisor of n . Tang, Ren

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and Feng [4] determined all deficient-perfect numbers with at most two distinct prime factors. In [5], Tang and Feng proved that there are no odd deficient-perfect numbers with three distinct prime factors.

Suppose that $n = q^\alpha$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = q^{\beta_1}, d_2 = q^{\beta_2}$, where q is a prime and α, β_1, β_2 are integers with $0 \leq \beta_1 < \beta_2 < \alpha$. Then

$$\sigma(q^\alpha) = 2q^\alpha - q^{\beta_1} - q^{\beta_2}.$$

That is,

$$(q-2)q^\alpha = (q-1)(q^{\beta_1} + q^{\beta_2}) - 1. \quad (1)$$

If $q = 2$, then we have $(q-1)(q^{\beta_1} + q^{\beta_2}) = 1$, which is impossible. Hence $q > 2$. From (1), we have

$$q^\alpha \leq (q-2)q^\alpha = (q-1)(q^{\beta_1} + q^{\beta_2}) - 1 \leq (q-1)(q^{\alpha-2} + q^{\alpha-1}) - 1.$$

Namely, $q^\alpha \leq q^\alpha - q^{\alpha-2} - 1$, a contradiction. Now, we have proved the following proposition.

Proposition 1. *If n is an exactly 2-deficient-perfect number, then n has at least two distinct prime divisors.*

In this paper, the following result is proved.

Theorem 1. *An odd integer n is an exactly 2-deficient-perfect number with two distinct prime factors if and only if one of the following holds.*

- (i) $n = 117$ with two deficient divisors $d_1 = 39$ and $d_2 = 13$;
- (ii) $n = 99$ with two deficient divisors $d_1 = 33$ and $d_2 = 9$;
- (iii) $n = 891$ with two deficient divisors $d_1 = 297$ and $d_2 = 33$;
- (iv) $n = 63$ with two deficient divisors $d_1 = 21$ and $d_2 = 1$;
- (v) $n = 21$ with two deficient divisors $d_1 = 7$ and $d_2 = 3$;
- (vi) $n = 3 \times 5^\beta$ with two deficient divisors $d_1 = 5^\beta$ and $d_2 = 1$;
- (vii) $n = 3^\alpha \times 5$ with two deficient divisors $d_1 = 3^\alpha$ and $d_2 = 3$, where $\alpha \geq 2$;
- (viii) $n = 3375$ with two deficient divisors $d_1 = 375$ and $d_2 = 135$.

2. Proof of Theorem 1

Proof of Theorem 1. Suppose that $n = p_1^\alpha p_2^\beta$ is an exactly 2-deficient-perfect number with exactly two distinct deficient divisors d_1 and d_2 , where p_1 and p_2 are two primes with $2 < p_1 < p_2$. Then

$$\sigma(p_1^\alpha p_2^\beta) = 2p_1^\alpha p_2^\beta - d_1 - d_2. \quad (2)$$

If $p_1 > 3$, then

$$2 = \frac{\sigma(p_1^\alpha p_2^\beta)}{p_1^\alpha p_2^\beta} + \frac{d_1}{p_1^\alpha p_2^\beta} + \frac{d_2}{p_1^\alpha p_2^\beta} < \frac{5}{4} \cdot \frac{7}{6} + \frac{1}{5} + \frac{1}{7} = 1.8011\dots,$$

a contradiction. Hence $p_1 = 3$. Now (2) becomes

$$\sigma(3^\alpha \cdot p_2^\beta) = 2 \cdot 3^\alpha \cdot p_2^\beta - d_1 - d_2,$$

where $d_1 = 3^{s_1} \cdot p_2^{t_1}$ and $d_2 = 3^{s_2} \cdot p_2^{t_2}$ are two distinct proper divisors of n . Write $D_1 = 3^{\alpha-s_1} \cdot p_2^{\beta-t_1}$, $D_2 = 3^{\alpha-s_2} \cdot p_2^{\beta-t_2}$, and assume $D_1 < D_2$. Then we have

$$2 = \frac{\sigma(3^\alpha \cdot p_2^\beta)}{3^\alpha \cdot p_2^\beta} + \frac{1}{D_1} + \frac{1}{D_2}. \quad (3)$$

If $p_2 > 23$, then

$$2 = \frac{\sigma(3^\alpha p_2^\beta)}{3^\alpha p_2^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{29}{28} + \frac{1}{3} + \frac{1}{9} = 1.9980\dots,$$

a contradiction. Therefore, $p_2 \in \{5, 7, 11, 13, 17, 19, 23\}$. We consider five cases.

Case 1. $p_2 \in \{17, 19, 23\}$. Then $\{D_1, D_2\} \subset \{3, 9, p_2, 27, 3p_2, \dots\}$. If $D_2 \geq p_2$, then, by (3), we have

$$2 = \frac{\sigma(3^\alpha \cdot p_2^\beta)}{3^\alpha \cdot p_2^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{p_2}{p_2 - 1} + \frac{1}{3} + \frac{1}{p_2} \leq \frac{3}{2} \cdot \frac{17}{16} + \frac{1}{3} + \frac{1}{17} = 1.9857\dots,$$

a contradiction. So $D_1 = 3$ and $D_2 = 9$. Thus

$$\sigma(3^\alpha \cdot p_2^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{p_2^{\beta+1} - 1}{p_2 - 1} = 2 \cdot 3^\alpha \cdot p_2^\beta - 3^{\alpha-1} \cdot p_2^\beta - 3^{\alpha-2} \cdot p_2^\beta.$$

It follows that

$$3^{\alpha-2} = \frac{p_2^{\beta+1} - 1}{(28 - p_2)p_2^\beta - 27}.$$

Consequently, for $p_2 = 17, 19, 23$, we have

$$3^{\alpha-2} = 1 + \frac{6 \cdot 17^\beta + 26}{11 \cdot 17^\beta - 27} \in (1, 2),$$

$$3^{\alpha-2} = 2 + \frac{19^\beta + 53}{9 \cdot 19^\beta - 27} \in (2, 3),$$

$$3^{\alpha-2} = 4 + \frac{3 \cdot 23^\beta + 107}{5 \cdot 23^\beta - 27} \in (4, 5) \cup \{6\},$$

which are impossible.

Case 2. $p_2 = 13$. Then $\{D_1, D_2\} \subset \{3, 9, 13, 27, 39, \dots\}$. If $D_1 \geq 9$, then

$$2 = \frac{\sigma(3^\alpha \cdot 13^\beta)}{3^\alpha \cdot 13^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{13}{12} + \frac{1}{9} + \frac{1}{13} = 1.8130\dots,$$

a contradiction. If $D_2 \geq 27$, then

$$2 = \frac{\sigma(3^\alpha \cdot 13^\beta)}{3^\alpha \cdot 13^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{13}{12} + \frac{1}{3} + \frac{1}{27} = 1.9953\dots,$$

a contradiction. Hence $D_1 = 3$ and $D_2 \in \{9, 13\}$. We divide into the following two subcases.

Subcase 2.1. $D_1 = 3, D_2 = 9$. Then

$$\sigma(3^\alpha \cdot 13^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{13^{\beta+1} - 1}{12} = 2 \cdot 3^\alpha \cdot 13^\beta - 3^{\alpha-1} \cdot 13^\beta - 3^{\alpha-2} \cdot 13^\beta.$$

That is,

$$\frac{9 \cdot 3^{\alpha-1} - 1}{5 \cdot 3^{\alpha-1} - 13} = 13^\beta \geq 13.$$

It follows that $\alpha - 1 \leq 1$. Consequently, we obtain the unique solution $\alpha = 2, \beta = 1$. Namely, $n = 117$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = 39$ and $d_2 = 13$.

Subcase 2.2. $D_1 = 3, D_2 = 13$. Then

$$\sigma(3^\alpha \cdot 13^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{13^{\beta+1} - 1}{12} = 2 \cdot 3^\alpha \cdot 13^\beta - 3^{\alpha-1} \cdot 13^\beta - 3^\alpha \cdot 13^{\beta-1}.$$

It follows that

$$13^{\beta-1} = \frac{3 \cdot 3^\alpha - 1}{11 \cdot 3^\alpha - 169}.$$

If $\alpha \leq 2$, then

$$\frac{3 \cdot 3^\alpha - 1}{11 \cdot 3^\alpha - 169} < 0,$$

a contradiction.

If $\alpha \geq 3$, then

$$0 < \frac{3 \cdot 3^\alpha - 1}{11 \cdot 3^\alpha - 169} < 1,$$

a contradiction.

Case 3. $p_2 = 11$. Then $\{D_1, D_2\} \subset \{3, 9, 11, 27, 33, 81, 99, \dots\}$. If $D_1 \geq 9$, then

$$2 = \frac{\sigma(3^\alpha \cdot 11^\beta)}{3^\alpha \cdot 11^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{11}{10} + \frac{1}{9} + \frac{1}{11} = 1.8520\dots,$$

a contradiction. If $D_2 \geq 81$, then

$$2 = \frac{\sigma(3^\alpha \cdot 11^\beta)}{3^\alpha \cdot 11^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{11}{10} + \frac{1}{3} + \frac{1}{81} = 1.9956\dots,$$

a contradiction. Hence $D_1 = 3$ and $D_2 \in \{9, 11, 27, 33\}$. We consider four subcases.

Subcase 3.1. $D_1 = 3, D_2 = 9$. Then

$$\sigma(3^\alpha \cdot 11^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{11^{\beta+1} - 1}{10} = 2 \cdot 3^\alpha \cdot 11^\beta - 3^{\alpha-1} \cdot 11^\beta - 3^{\alpha-2} \cdot 11^\beta.$$

It follows that

$$3^{\alpha-2} = \frac{11^{\beta+1} - 1}{17 \cdot 11^\beta - 27}.$$

But

$$\frac{1}{3} < \frac{11^{\beta+1} - 1}{17 \cdot 11^\beta - 27} < 1,$$

a contradiction.

Subcase 3.2. $D_1 = 3, D_2 = 11$. Then

$$\sigma(3^\alpha \cdot 11^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{11^{\beta+1} - 1}{10} = 2 \cdot 3^\alpha \cdot 11^\beta - 3^{\alpha-1} \cdot 11^\beta - 3^\alpha \cdot 11^{\beta-1}.$$

It follows that

$$11^{\beta-1} = \frac{3^{\alpha+1} - 1}{49 \cdot 3^{\alpha-1} - 121}.$$

If $\alpha - 1 \geq 2$, then

$$0 < \frac{3^{\alpha+1} - 1}{49 \cdot 3^{\alpha-1} - 121} < 1,$$

a contradiction. So $\alpha - 1 \leq 1$. Consequently, we obtain the unique solution $\alpha = 2, \beta = 1$. Namely, $n = 99$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = 33$ and $d_2 = 9$.

Subcase 3.3. $D_1 = 3, D_2 = 27$. Then

$$\sigma(3^\alpha \cdot 11^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{11^{\beta+1} - 1}{10} = 2 \cdot 3^\alpha \cdot 11^\beta - 3^{\alpha-1} \cdot 11^\beta - 3^{\alpha-3} \cdot 11^\beta.$$

It follows that $(11^{\beta+1} - 81)(3^{\alpha-3} - 1) = 80$. If $\beta \geq 2$, then $11^{\beta+1} - 81 > 80$, a contradiction. So $\beta = 1$. Consequently, we obtain the unique solution $\alpha = 4, \beta = 1$. Namely, $n = 891$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = 297$ and $d_2 = 33$.

Subcase 3.4. $D_1 = 3, D_2 = 33$. Then

$$\sigma(3^\alpha \cdot 11^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{11^{\beta+1} - 1}{10} = 2 \cdot 3^\alpha \cdot 11^\beta - 3^{\alpha-1} \cdot 11^\beta - 3^{\alpha-1} \cdot 11^{\beta-1}.$$

It follows that $(3^{\alpha+1} - 121)(11^{\beta-1} - 1) = 120$. If $\beta \geq 4$, then $11^{\beta-1} - 1 > 120$, a contradiction. So $\beta \leq 3$. If $\beta = 3$, then $11^{\beta-1} - 1 = 120$. Thus $3^{\alpha+1} - 121 = 1$, i.e., $3^{\alpha+1} = 122$, which is impossible. If $\beta = 2$, then $11^{\beta-1} - 1 = 10$. Thus $3^{\alpha+1} - 121 = 12$, i.e., $3^{\alpha+1} = 133$, which is impossible. If $\beta = 1$, then $(3^{\alpha+1} - 121)(11^{\beta-1} - 1) = 0$, a contradiction.

Case 4. $p_2 = 7$. Then $\{D_1, D_2\} \subset \{3, 7, 9, 21, 27, 49, \dots\}$. If $D_1 \geq 7$ and $D_2 \geq 21$, then we have

$$2 = \frac{\sigma(3^\alpha \cdot 7^\beta)}{3^\alpha \cdot 7^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{7}{6} + \frac{1}{7} + \frac{1}{21} = 1.9404\dots,$$

a contradiction. Hence either $D_1 = 3$, or $D_1 = 7$ and $D_2 = 9$. There are the following two subcases.

Subcase 4.1. $D_1 = 3$. Recall that $D_2 = 3^{\alpha-s_2} \cdot 7^{\beta-t_2}$, we have

$$\sigma(3^\alpha \cdot 7^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{7^{\beta+1} - 1}{6} = 2 \cdot 3^\alpha \cdot 7^\beta - 3^{\alpha-1} \cdot 7^\beta - 3^{s_2} \cdot 7^{t_2}.$$

It follows that

$$(3^\alpha - 7) \cdot (7^\beta - 3) = 20 - 12 \cdot 3^{s_2} \cdot 7^{t_2}. \quad (4)$$

If $s_2 = t_2 = 0$, then $(3^\alpha - 7) \cdot (7^\beta - 3) = 20 - 12 = 8$. If $\beta = 1$, then $7^\beta - 3 = 4$. Thus $3^\alpha - 7 = 2$ and then $\alpha = 2$. We obtain a solution, that is, $n = 63$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = 21$ and $d_2 = 1$.

If $s_2 > 0$ or $t_2 > 0$, then $20 - 12 \cdot 3^{s_2} \cdot 7^{t_2} < 0$. Since $7^\beta - 3 > 0$, it follows from (4) that $3^\alpha - 7 < 0$. Thus $\alpha = 1$. By (4), we have

$$-4(7^\beta - 3) = 20 - 12 \cdot 3^{s_2} \cdot 7^{t_2}.$$

That is,

$$7^\beta - 3 = -5 + 3^{s_2+1} \cdot 7^{t_2}.$$

So

$$7^\beta = -2 + 3^{s_2+1} \cdot 7^{t_2}.$$

Hence $t_2 = 0$, otherwise $7 \mid -2$, a contradiction. Now we have $7^\beta = -2 + 3^{s_2+1}$. Noting that $0 \leq s_2 \leq \alpha = 1$, and $t_2 = 0$, we have $s_2 = 1$, otherwise $s_2 = t_2 = 0$, a contradiction with $s_2 > 0$ or $t_2 > 0$. Thus $\beta = 1$. Now we obtain another solution, namely, $n = 21$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = 7$ and $d_2 = 3$.

Subcase 4.2. $D_1 = 7, D_2 = 9$. Then

$$\sigma(3^\alpha \cdot 7^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{7^{\beta+1} - 1}{6} = 2 \cdot 3^\alpha \cdot 7^\beta - 3^\alpha \cdot 7^{\beta-1} - 3^{\alpha-2} \cdot 7^\beta.$$

It follows that $(3^{\alpha-1} - 49)(7^{\beta-1} - 9) = 440$. If $\beta - 1 \geq 4$, then $7^{\beta-1} - 9 > 440$, a contradiction. So $0 \leq \beta - 1 \leq 3$. By direct calculation, we know that $(3^{\alpha-1} - 49)(7^{\beta-1} - 9) = 440$ has no solution for $0 \leq \beta - 1 \leq 3$.

Case 5. $p_2 = 5$. Then $\{D_1, D_2\} \subset \{3, 5, 9, 15, 25, 27, 45, 75, 81, \dots\}$. If $D_1 \geq 9$ and $D_2 \geq 75$, then

$$2 = \frac{\sigma(3^\alpha \cdot 5^\beta)}{3^\alpha \cdot 5^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{5}{4} + \frac{1}{9} + \frac{1}{75} = 1.9441\dots,$$

a contradiction. Similarly, if $D_1 \geq 15$, then

$$2 = \frac{\sigma(3^\alpha \cdot 5^\beta)}{3^\alpha \cdot 5^\beta} + \frac{1}{D_1} + \frac{1}{D_2} < \frac{3}{2} \cdot \frac{5}{4} + \frac{1}{15} + \frac{1}{25} = 1.9816\dots,$$

a contradiction. Hence, $D_1 = 3$ or $D_1 = 5$ or $D_1 = 9$, $D_2 \in \{15, 25, 27, 45\}$. Now, we consider the following six subcases.

Subcase 5.1. $D_1 = 3$. Recall that $D_2 = 3^{\alpha-s_2} \cdot 5^{\beta-t_2}$, we have

$$\sigma(3^\alpha \cdot 5^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{5^{\beta+1} - 1}{4} = 2 \cdot 3^\alpha \cdot 5^\beta - 3^{\alpha-1} \cdot 5^\beta - 3^{s_2} \cdot 5^{t_2}.$$

It follows that

$$(3^{\alpha-1} - 1) \cdot (5^{\beta+1} - 9) = 8(1 - 3^{s_2} \cdot 5^{t_2}). \quad (5)$$

Since $3^{\alpha-1} - 1 \geq 0$ and $5^{\beta+1} - 9 > 0$, it follows that $1 - 3^{s_2} \cdot 5^{t_2} \geq 0$. Thus $s_2 = t_2 = 0$. By (5), we have $\alpha = 1$. Therefore, $n = 3 \cdot 5^\beta$ ($\beta \geq 1$) are exactly 2-deficient-perfect numbers with two deficient divisors $d_1 = 5^\beta$ and $d_2 = 1$.

Subcase 5.2. $D_1 = 5$. Recall that $D_2 = 3^{\alpha-s_2} \cdot 5^{\beta-t_2}$, we have

$$\sigma(3^\alpha \cdot 5^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{5^{\beta+1} - 1}{4} = 2 \cdot 3^\alpha \cdot 5^\beta - 3^\alpha \cdot 5^{\beta-1} - 3^{s_2} \cdot 5^{t_2}.$$

It follows that

$$(3^{\alpha+1} - 25) \cdot (5^{\beta-1} - 1) = 8(3 - 3^{s_2} \cdot 5^{t_2}). \quad (6)$$

If $s_2 = t_2 = 0$, then, by (6), we have

$$(3^{\alpha+1} - 25) \cdot (5^{\beta-1} - 1) = 16. \quad (7)$$

If $\beta - 1 \geq 2$, then $5^{\beta-1} - 1 > 16$, a contradiction. So $\beta - 1 = 0, 1$. It is easy to see that (7) has no solution for $\beta - 1 = 0, 1$.

If $s_2 = 1$ and $t_2 = 0$, then $8(3 - 3^{s_2} \cdot 5^{t_2}) = 0$. By (6), we have $\beta - 1 = 0$. Therefore, $n = 3^\alpha \cdot 5$ ($\alpha > 1$) are exactly 2-deficient-perfect numbers with two deficient divisors $d_1 = 3^\alpha$ and $d_2 = 3$ (here $\alpha = 1$ is excluded, otherwise $d_1 = d_2 = 3$).

If $s_2 \geq 2$ or $t_2 \geq 1$, then $8(3 - 3^{s_2} \cdot 5^{t_2}) \leq -16$. Since $5^{\beta-1} - 1 \geq 0$, it follows from (6) that $3^{\alpha+1} - 25 < 0$. Thus $\alpha = 1$. So $s_2 \leq 1$ and $t_2 \geq 1$. Now (6) becomes

$$(-16) \cdot (5^{\beta-1} - 1) = 8(3 - 3^{s_2} \cdot 5^{t_2}).$$

That is,

$$-2 \cdot 5^{\beta-1} = 1 - 3^{s_2} \cdot 5^{t_2}.$$

Since $t_2 \geq 1$, it follows that $\beta - 1 = 0$. Otherwise, $5 \mid 1$, a contradiction. Thus $3^{s_2} \cdot 5^{t_2} = 3$, a contradiction with $t_2 \geq 1$.

Subcase 5.3. $D_1 = 9, D_2 = 15$. Then

$$\sigma(3^\alpha \cdot 5^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{5^{\beta+1} - 1}{4} = 2 \cdot 3^\alpha \cdot 5^\beta - 3^{\alpha-2} \cdot 5^\beta - 3^{\alpha-1} \cdot 5^{\beta-1}.$$

It follows that

$$5^{\beta-1} = \frac{27 \cdot 3^{\alpha-2} - 1}{19 \cdot 3^{\alpha-2} - 25}.$$

If $\alpha - 2 \leq 0$, then

$$5^{\beta-1} = \frac{27 \cdot 3^{\alpha-2} - 1}{19 \cdot 3^{\alpha-2} - 25} < 0,$$

a contradiction.

If $\alpha - 2 = 1$, then

$$5^{\beta-1} = \frac{27 \cdot 3^{\alpha-2} - 1}{19 \cdot 3^{\alpha-2} - 25} = \frac{5}{2},$$

a contradiction.

If $\alpha - 2 \geq 2$, then

$$1 < \frac{27 \cdot 3^{\alpha-2} - 1}{19 \cdot 3^{\alpha-2} - 25} < 2,$$

a contradiction.

Subcase 5.4. $D_1 = 9, D_2 = 25$. Then $\alpha \geq 2, \beta \geq 2$ and

$$\sigma(3^\alpha \cdot 5^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{5^{\beta+1} - 1}{4} = 2 \cdot 3^\alpha \cdot 5^\beta - 3^{\alpha-2} \cdot 5^\beta - 3^\alpha \cdot 5^{\beta-2}.$$

It follows that

$$3^{\alpha-2} = \frac{125 \cdot 5^{\beta-2} - 1}{47 \cdot 5^{\beta-2} - 27}. \quad (8)$$

Since

$$2 < \frac{125 \cdot 5^{\beta-2} - 1}{47 \cdot 5^{\beta-2} - 27} < 9,$$

it follows from (8) that $\alpha - 2 = 1$. Again, by (8), we have $\beta - 2 = 1$. So $\alpha = 3$ and $\beta = 3$. Namely, $n = 3375 = 3^3 \times 5^3$ is an exactly 2-deficient-perfect number with two deficient divisors $d_1 = 375$ and $d_2 = 135$.

Subcase 5.5. $D_1 = 9, D_2 = 27$. Then

$$\sigma(3^\alpha \cdot 5^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{5^{\beta+1} - 1}{4} = 2 \cdot 3^\alpha \cdot 5^\beta - 3^{\alpha-2} \cdot 5^\beta - 3^{\alpha-3} \cdot 5^\beta.$$

It follows that

$$(5^{\beta+1} - 81)(3^{\alpha-3} - 1) = 80. \quad (9)$$

If $\beta \geq 3$, then $5^{\beta+1} - 81 > 80$, a contradiction. It is easy to see that (9) cannot hold for $\beta = 1, 2$.

Subcase 5.6. $D_1 = 9, D_2 = 45$. Then

$$\sigma(3^\alpha \cdot 5^\beta) = \frac{3^{\alpha+1} - 1}{2} \cdot \frac{5^{\beta+1} - 1}{4} = 2 \cdot 3^\alpha \cdot 5^\beta - 3^{\alpha-2} \cdot 5^\beta - 3^{\alpha-2} \cdot 5^{\beta-1}.$$

It follows that

$$(3^{\alpha-1} - 25)(5^{\beta-1} - 9) = 224. \quad (10)$$

If $\beta \geq 5$, then $5^{\beta-1} - 9 > 224$, a contradiction. It is easy to see that (10) cannot hold for $1 \leq \beta \leq 4$.

This completes the proof of Theorem 1. \square

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