



ON A DIOPHANTINE EQUATION WITH THREE PRIME VARIABLES

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Abstract

Let $[x]$ denote the integral part of the real number x , and N be a sufficiently large integer. In this paper, it is proved that, for $1 < c < \frac{3113}{2703}$, the Diophantine equation $N = [p_1^c] + [p_2^c] + [p_3^c]$ is solvable in prime variables p_1, p_2, p_3 , which constitutes an improvement upon that of Cai.

1. Introduction and Main Result

Let $[x]$ be the integral part of the real number x . In 1933–1934, Segal [13, 14] first considered the Waring’s problem with non-integer degrees, who showed that for any sufficiently large integer N and $c > 1$ being not an integer, there exists a integer $k_0 = k_0(c) > 0$ such that the equation

$$N = [x_1^c] + [x_2^c] + \cdots + [x_k^c]$$

is solvable for $k \geq k_0(c)$. Later, Segal’s bound for $k_0(c)$ was improved by Deshouillers [4] and by Arkhilov and Zhitkov [1], respectively. Let $G(c)$ be the least of the integers $k_0(c)$ such that every sufficiently large integer N can be written as a sum of not more than $k_0(c)$ numbers with the form $[n^c]$. In particular, Deshouillers [5] and Gritsenko [8] considered the case $k = 2$ and gave $G(c) = 2$ for $1 < c < 4/3$ and $1 < c < 55/41$, respectively.

In 1937, Vinogradov [15] solved asymptotic form of the ternary Goldbach problem. He proved that, for sufficiently large integer N satisfying $N \equiv 1 \pmod{2}$,

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the equation $N = p_1 + p_2 + p_3$ is solvable in primes p_1, p_2, p_3 . As an analogue of the ternary Goldbach problem, in 1995, Laporta and Tolev [12] investigated the solvability of the equation $N = [p_1^c] + [p_2^c] + [p_3^c]$ in prime variables p_1, p_2, p_3 . Define

$$\mathcal{R}_3(N) = \sum_{N=[p_1^c]+[p_2^c]+[p_3^c]} (\log p_1)(\log p_2)(\log p_3).$$

Laporta and Tolev [12] showed that the sum $\mathcal{R}_3(N)$ has an asymptotic formula for $1 < c < 17/16$, and gave

$$\mathcal{R}_3(N) = \frac{\Gamma^3(1+1/c)}{\Gamma(3/c)} N^{3/c-1} + O\left(N^{3/c-1} \exp(-(\log N)^{1/3-\delta})\right)$$

for any $0 < \delta < 1/3$. Later, Kumchev and Nedeva [11] improved the result of Laporta and Tolev [12], and enlarged the range of c to $12/11$. Afterwards, Zhai and Cao [17] refined the result of Kumchev and Nedeva [11], who extended the range of c to $258/235$. In 2018, Cai [3] enhanced the result of Zhai and Cao [17] and gave the upper bound of c as $137/119$.

In this paper, we shall continue to improve the result of Cai [3] and establish the following result.

Theorem 1. *Let $1 < c < \frac{3113}{2703}$, and N be a sufficiently large integer. Then we have*

$$\mathcal{R}_3(N) = \frac{\Gamma^3(1+1/c)}{\Gamma(3/c)} N^{3/c-1} + O\left(N^{3/c-1} \exp(-(\log N)^{1/3-\delta})\right)$$

for any $0 < \delta < 1/3$, where the implied constant in the O -term depends only on c .

Notation. Throughout this paper, we suppose that $1 < c < \frac{3113}{2703}$. Let p , with or without subscripts, always denote a prime number; and let ε always denote arbitrary small positive constant, which may not be the same at different occurrences. As usual, we use $[x]$, $\{x\}$ and $\|x\|$ to denote the integral part of x , the fractional part of x and the distance from x to the nearest integer, respectively. Also, we write $e(x) = e^{2\pi i x}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; and $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$.

We also define

$$\begin{aligned} P &= N^{1/c}, & \tau &= P^{1-c-\varepsilon}, & S(\alpha) &= \sum_{p \leqslant P} (\log p) e([p^c]\alpha), \\ S(\alpha, X) &= \sum_{X < p \leqslant 2X} (\log p) e([p^c]\alpha), & T(\alpha, X) &= \sum_{X < n \leqslant 2X} e([n^c]\alpha). \end{aligned}$$

2. Preliminary Lemmas

In this section, we shall state some preliminary lemmas, which are required in the proof of Theorem 1.

Lemma 1. Let $L, Q \geq 1$ and z_ℓ be complex numbers. Then we have

$$\left| \sum_{L < \ell \leq 2L} z_\ell \right|^2 \leq \left(2 + \frac{L}{Q} \right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q} \right) \sum_{L < \ell+q, \ell-q \leq 2L} z_{\ell+q} \overline{z_{\ell-q}}.$$

Proof. See Lemma 2 of Fouvry and Iwaniec [6]. \square

Lemma 2. Suppose that $f(x) : [a, b] \rightarrow \mathbb{R}$ has continuous derivatives of arbitrary order on $[a, b]$, where $1 \leq a < b \leq 2a$. Suppose further that

$$|f^{(j)}(x)| \asymp \lambda_1 a^{1-j}, \quad j \geq 1, \quad x \in [a, b].$$

Then for any exponential pair (κ, λ) , we have

$$\sum_{a < n \leq b} e(f(n)) \ll \lambda_1^\kappa a^\lambda + \lambda_1^{-1}.$$

Proof. See (3.3.4) of Graham and Kolesnik [7]. \square

Lemma 3. Let x be a non-integer, $\alpha \in (0, 1)$, and $H \geq 3$. Then we have

$$e(-\alpha\{x\}) = \sum_{|h| \leq H} c_h(\alpha) e(hx) + O\left(\min\left(1, \frac{1}{H\|x\|}\right)\right),$$

where

$$c_h(\alpha) = \frac{1 - e(-\alpha)}{2\pi i(h + \alpha)}.$$

Proof. See Lemma 12 of Buriev [2] or Lemma 3 of Kumchev and Nedeva [11]. \square

Lemma 4. Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \geq 64Z^2U$, $Z \geq 4U^2$, $V^3 \geq 32X$. Assume further that $F(n)$ is a complex-valued function such that $|F(n)| \leq 1$. Then the sum

$$\sum_{X < n \leq 2X} \Lambda(n) F(n)$$

may be decomposed into $O(\log^{10} X)$ sums, each of which either of Type I:

$$\sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} F(mk)$$

with $K \gg Z$, where $a(m) \ll m^\varepsilon$, $MK \asymp X$, or of Type II:

$$\sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} b(k) F(mk)$$

with $U \ll M \ll V$, where $a(m) \ll m^\varepsilon$, $b(k) \ll k^\varepsilon$, $MK \asymp X$.

Proof. See Lemma 3 of Heath–Brown [9]. \square

Lemma 5. *For any $\varepsilon > 0$, the pair $(\frac{32}{205} + \varepsilon, \frac{269}{410} + \varepsilon)$ is an exponential pair.*

Proof. See the Corollary of Theorem 1 of Huxley [10]. \square

Lemma 6. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta(\alpha - 1)(\beta - 1) \neq 0$. Define the bilinear sums of Type I as follows*

$$S_I(M, K) := \sum_{M < m \leq 2M} \sum_{k \in \mathcal{I}(m)} a(m) e\left(F \frac{m^\alpha k^\beta}{M^\alpha K^\beta}\right),$$

where $F > 0, M \geq 1, K \geq 1, |a(m)| \ll 1$, and $\mathcal{I}(m)$ is a subinterval of $(K, 2K]$. Then for any exponential pair (κ, λ) , we have

$$S_I(M, K) \ll ((F^{1+2\kappa} M^{4+4\kappa} K^{3+2\lambda})^{\frac{1}{6+4\kappa}} + M^{\frac{1}{2}} K + MK^{\frac{1}{2}} + F^{-1} MK) \log(2+FMK).$$

Proof. See Theorem 2 of Wu [16]. \square

Lemma 7. *For any real number θ , there holds*

$$\min\left(1, \frac{1}{H\|\theta\|}\right) = \sum_{h=-\infty}^{+\infty} a_h e(h\theta),$$

where

$$a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{h^2}\right).$$

Proof. See p.245 of Heath–Brown [9]. \square

Lemma 8. *Let $1 < c < \frac{3113}{2703}, P^{\frac{5}{6}} \ll X \ll P, H = X^{\frac{2047}{27030}}$ and $c_h(\alpha)$ denote complex numbers such that $|c_h(\alpha)| \ll (1 + |h|)^{-1}$. Then, for any $\alpha \in (\tau, 1 - \tau)$, if $M \ll X^{\frac{7781}{13515}}$, we have*

$$\mathcal{S}_I(\alpha) := \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} e((h + \alpha)(mk)^c) \ll X^{\frac{24983}{27030} + \varepsilon},$$

where $a(m) \ll m^\varepsilon$ and $MK \asymp X$.

Proof. Obviously, we have

$$|\mathcal{S}_I(\alpha)| \ll X^\varepsilon \max_{|\xi| \in (\tau, H+1)} \sum_{M < m \leq 2M} \left| \sum_{K < k \leq 2K} e(\xi(mk)^c) \right|. \quad (1)$$

If $M \ll X^{\frac{3959}{9010}}$, then we use Lemma 2 to estimate the inner sum over k in (1) with the exponential pair $(\kappa, \lambda) = AB(0, 1) = (\frac{1}{6}, \frac{2}{3})$ and derive that

$$\begin{aligned} S_I(\alpha) &\ll X^\varepsilon \max_{|\xi| \in (\tau, H+1)} \sum_{M < m \leq 2M} \left((|\xi|X^c K^{-1})^{\frac{1}{6}} K^{\frac{2}{3}} + \frac{K}{|\xi|X^c} \right) \\ &\ll X^\varepsilon \max_{|\xi| \in (\tau, H+1)} \left(|\xi|^{\frac{1}{6}} X^{\frac{c}{6}} K^{\frac{1}{2}} M + \frac{MK}{|\xi|X^c} \right) \\ &\ll X^\varepsilon (H^{\frac{1}{6}} X^{\frac{c}{6} + \frac{1}{2}} M^{\frac{1}{2}} + X^{1-c} \tau^{-1}) \ll X^{\frac{24983}{27030} + \varepsilon}. \end{aligned}$$

If $X^{\frac{3959}{9010}} \ll M \ll X^{\frac{7781}{13515}}$, we use Lemma 6 to estimate the inner sum over k in (1) with the exponential pair $AB(0, 1) = (\frac{1}{6}, \frac{2}{3})$ and obtain that

$$\begin{aligned} S_I(\alpha) &\ll X^\varepsilon \max_{|\xi| \in (\tau, H+1)} \left((|\xi|X^c)^{\frac{1}{5}} M^{\frac{7}{10}} K^{\frac{13}{20}} + M^{\frac{1}{2}} K + MK^{\frac{1}{2}} + |\xi|^{-1} X^{-c} MK \right) \\ &\ll X^\varepsilon \left(H^{\frac{1}{5}} M^{\frac{1}{20}} X^{\frac{c}{5} + \frac{13}{20}} + XM^{-\frac{1}{2}} + X^{\frac{1}{2}} M^{\frac{1}{2}} + X^{1-c} \tau^{-1} \right) \ll X^{\frac{24983}{27030} + \varepsilon}, \end{aligned}$$

which completes the proof of Lemma 8. \square

Lemma 9. Let $1 < c < \frac{3113}{2703}, P^{\frac{5}{6}} \ll X \ll P, H = X^{\frac{2047}{27030}}$ and $c_h(\alpha)$ denote complex numbers such that $|c_h(\alpha)| \ll (1 + |h|)^{-1}$. Then, for any $\alpha \in (\tau, 1 - \tau)$, if there holds $X^{\frac{2047}{13515}} \ll M \ll X^{\frac{355}{901}}$, then we have

$$S_{II}(\alpha) := \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} b(k) e((h + \alpha)(mk)^c) \ll X^{\frac{24983}{27030} + \varepsilon},$$

where $a(m) \ll m^\varepsilon, b(k) \ll k^\varepsilon$ and $MK \asymp X$.

Proof. Let $Q = X^{\frac{2047}{13515}} (\log X)^{-1}$. From Lemma 1 and Cauchy's inequality, we

derive that

$$\begin{aligned}
|\mathcal{S}_{II}(\alpha)| &\ll \sum_{|h| \leq H} |c_h(\alpha)| \left| \sum_{K < k \leq 2K} b(k) \sum_{M < m \leq 2M} a(m) e((h + \alpha)(mk)^c) \right| \\
&\ll \sum_{|h| \leq H} |c_h(\alpha)| \left(\sum_{K < k \leq 2K} |b(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{K < k \leq 2K} \left| \sum_{M < m \leq 2M} a(m) e((h + \alpha)(mk)^c) \right|^2 \right)^{\frac{1}{2}} \\
&\ll K^{\frac{1}{2} + \varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left(\sum_{K < k \leq 2K} \frac{M}{Q} \sum_{0 \leq q < Q} \left(1 - \frac{q}{Q} \right) \right. \\
&\quad \times \left. \sum_{M+q < m \leq 2M-q} a(m+q) \overline{a(m-q)} e((h + \alpha)k^c \Delta_c(m, q)) \right)^{\frac{1}{2}} \\
&\ll K^{\frac{1}{2} + \varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left(\frac{M}{Q} \sum_{K < k \leq 2K} \left(M^{1+\varepsilon} + \sum_{1 \leq q < Q} \left(1 - \frac{q}{Q} \right) \right. \right. \\
&\quad \times \left. \left. \sum_{M+q < m \leq 2M-q} a(m+q) \overline{a(m-q)} e((h + \alpha)k^c \Delta_c(m, q)) \right) \right)^{\frac{1}{2}} \\
&\ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left(\frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q < Q} \sum_{M < m \leq 2M} \left| \sum_{K < k \leq 2K} e((h + \alpha)k^c \Delta_c(m, q)) \right| \right)^{\frac{1}{2}}, \tag{2}
\end{aligned}$$

where $\Delta_c(m, q) = (m+q)^c - (m-q)^c$. Thus, it is sufficient to estimate the sum

$$S_0 := \sum_{K < k \leq 2K} e((h + \alpha)k^c \Delta_c(m, q)).$$

By Lemma 2 with the exponential pair $(\kappa, \lambda) = BA^2B(0, 1) = (\frac{2}{7}, \frac{4}{7})$, we have

$$S_0 \ll (|h + \alpha| X^{c-1} q)^{\frac{2}{7}} K^{\frac{4}{7}} + \frac{1}{|h + \alpha| X^{c-1} q}.$$

Putting the above estimate into (2), we obtain that

$$\begin{aligned}
& \mathcal{S}_{II}(\alpha) \\
& \ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left(\frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q < Q} \sum_{M < m \leq 2M} \right. \\
& \quad \times \left. \left((|h + \alpha| X^{c-1} q)^{\frac{2}{7}} K^{\frac{4}{7}} + \frac{1}{|h + \alpha| X^{c-1} q} \right) \right)^{\frac{1}{2}} \\
& \ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left(\frac{X^2}{Q} + \frac{X}{Q} \left(H^{\frac{2}{7}} X^{\frac{2}{7}(c-1)} M K^{\frac{4}{7}} Q^{\frac{9}{7}} + X^{1-c} M \tau^{-1} \log Q \right) \right)^{\frac{1}{2}} \\
& \ll X^{1+\varepsilon} Q^{-\frac{1}{2}} \sum_{|h| \leq H} |c_h(\alpha)| \ll X^{1+\varepsilon} Q^{-\frac{1}{2}} \sum_{|h| \leq H} \frac{1}{1+|h|} \ll X^{\frac{24983}{27030}+\varepsilon},
\end{aligned}$$

which completes the proof of Lemma 9. \square

Lemma 10. *For $\alpha \in (\tau, 1 - \tau)$, there holds*

$$S(\alpha) \ll P^{\frac{24983}{27030}+\varepsilon}.$$

Proof. First, we have

$$S(\alpha) = \mathcal{U}(\alpha) + O(P^{1/2}),$$

where

$$\mathcal{U}(\alpha) = \sum_{n \leq P} \Lambda(n) e([n^c] \alpha).$$

By a splitting argument, it is sufficient to prove that, for $P^{5/6} \ll X \ll P$ and $\alpha \in (\tau, 1 - \tau)$, there holds

$$\mathcal{U}^*(\alpha) := \sum_{X < n \leq 2X} \Lambda(n) e([n^c] \alpha) \ll X^{\frac{24983}{27030}+\varepsilon}.$$

By Lemma 3 with $H = X^{\frac{2047}{27030}}$, we have

$$\begin{aligned}
\mathcal{U}^*(\alpha) &= \sum_{X < n \leq 2X} \Lambda(n) e(n^c \alpha - \{n^c\} \alpha) = \sum_{X < n \leq 2X} \Lambda(n) e(n^c \alpha) e(-\{n^c\} \alpha) \\
&= \sum_{X < n \leq 2X} \Lambda(n) e(n^c \alpha) \left(\sum_{|h| \leq H} c_h(\alpha) e(h n^c) + O\left(\min\left(1, \frac{1}{H \|n^c\|}\right)\right) \right) \\
&= \sum_{|h| \leq H} c_h(\alpha) \sum_{X < n \leq 2X} \Lambda(n) e((h + \alpha)n^c) + O\left(\log X \cdot \sum_{X < n \leq 2X} \min\left(1, \frac{1}{H \|n^c\|}\right)\right). \tag{3}
\end{aligned}$$

By Lemma 7 and Lemma 2 with the exponential pair $(\kappa, \lambda) = B(0, 1) = (\frac{1}{2}, \frac{1}{2})$, we derive that

$$\begin{aligned}
& \sum_{X < n \leq 2X} \min\left(1, \frac{1}{H\|n^c\|}\right) \\
&= \sum_{X < n \leq 2X} \sum_{\ell=-\infty}^{+\infty} a_\ell e(\ell n^c) \ll \sum_{\ell=-\infty}^{+\infty} |a_\ell| \left| \sum_{X < n \leq 2X} e(\ell n^c) \right| \\
&\ll \frac{X \log 2H}{H} + \sum_{1 \leq \ell \leq H} \frac{1}{\ell} \left| \sum_{X < n \leq 2X} e(\ell n^c) \right| + \sum_{\ell > H} \frac{H}{\ell^2} \left| \sum_{X < n \leq 2X} e(\ell n^c) \right| \\
&\ll \frac{X \log 2H}{H} + \sum_{1 \leq \ell \leq H} \frac{1}{\ell} \left((X^{c-1} \ell)^{\frac{1}{2}} X^{\frac{1}{2}} + \frac{1}{\ell X^{c-1}} \right) \\
&\quad + \sum_{\ell > H} \frac{H}{\ell^2} \left((X^{c-1} \ell)^{\frac{1}{2}} X^{\frac{1}{2}} + \frac{1}{\ell X^{c-1}} \right) \\
&\ll X^{\frac{24983}{27030}} \log X + H^{\frac{1}{2}} X^{\frac{c}{2}} + X^{1-c} \ll X^{\frac{24983}{27030}} \log X. \tag{4}
\end{aligned}$$

Taking $U = X^{\frac{2047}{13515}}$, $V = X^{\frac{355}{901}}$, and $Z = [X^{\frac{5734}{13515}}] + \frac{1}{2}$ in Lemma 4, it is easy to see that the sum

$$\sum_{|h| \leq H} c_h(\alpha) \sum_{X < n \leq 2X} \Lambda(n) e((h + \alpha)n^c)$$

can be represented as $O(\log^{10} X)$ sums, each of which either of Type I

$$\mathcal{S}_I(\alpha) = \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} e((h + \alpha)(mk)^c)$$

with $K \gg Z$, $a(m) \ll m^\varepsilon$, $MK \asymp X$, or of Type II

$$\mathcal{S}_{II}(\alpha) = \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} b(k) e((h + \alpha)(mk)^c)$$

with $U \ll M \ll V$, $a(m) \ll m^\varepsilon$, $b(k) \ll k^\varepsilon$, $MK \asymp X$. For the Type I sums, by noting the fact that $K \gg Z$ and $MK \asymp X$, we deduce that $M \ll X^{\frac{7781}{13515}}$. From Lemma 8, we have $\mathcal{S}_I(\alpha) \ll X^{\frac{24983}{27030} + \varepsilon}$. For the Type II sums, by Lemma 9, we have $\mathcal{S}_{II}(\alpha) \ll X^{\frac{24983}{27030} + \varepsilon}$. Therefore, we conclude that

$$\sum_{|h| \leq H} c_h(\alpha) \sum_{X < n \leq 2X} \Lambda(n) e((h + \alpha)n^c) \ll X^{\frac{24983}{27030} + \varepsilon}. \tag{5}$$

From (3)–(5), we derive the desired result. This completes the proof of Lemma 10. \square

Lemma 11. *For $\alpha \in (0, 1)$, we have*

$$\mathcal{T}(\alpha, X) \ll X^{\frac{269c+538}{1217} + \varepsilon} + \frac{1}{\alpha X^{c-1}}.$$

Proof. Taking $H_1 = X^{\frac{679-269c}{1217}}$, and by Lemma 3, we deduce that

$$\begin{aligned} \mathcal{T}(\alpha, X) &= \sum_{X < n \leq 2X} e((n^c - \{n^c\})\alpha) \\ &= \sum_{X < n \leq 2X} e(n^c \alpha) \left(\sum_{|h| \leq H_1} c_h(\alpha) e(hn^c) + O\left(\min\left(1, \frac{1}{H_1 \|n^c\|}\right)\right)\right) \\ &= \sum_{|h| \leq H_1} c_h(\alpha) \sum_{X < n \leq 2X} e((h + \alpha)n^c) + O\left(\sum_{X < n \leq 2X} \min\left(1, \frac{1}{H_1 \|n^c\|}\right)\right). \end{aligned} \quad (6)$$

From Lemma 7, we get

$$\begin{aligned} \sum_{X < n \leq 2X} \min\left(1, \frac{1}{H_1 \|n^c\|}\right) &= \sum_{X < n \leq 2X} \sum_{k=-\infty}^{+\infty} a_k e(kn^c) \\ &\ll \sum_{k=-\infty}^{+\infty} |a_k| \left| \sum_{X < n \leq 2X} e(kn^c) \right|. \end{aligned} \quad (7)$$

According to Lemma 5, the pair $BA(\frac{32}{205} + \varepsilon, \frac{269}{410} + \varepsilon) = (\frac{269}{948} + \varepsilon, \frac{269}{474} + \varepsilon)$ is an exponential pair. Then we shall use Lemma 2 with the exponential pair $(\frac{269}{948} + \varepsilon, \frac{269}{474} + \varepsilon)$ to estimate the sum over n on the right-hand side in (7), and derive that

$$\begin{aligned} &\sum_{X < n \leq 2X} \min\left(1, \frac{1}{H_1 \|n^c\|}\right) \\ &\ll \frac{X \log 2H_1}{H_1} + \sum_{1 \leq k \leq H_1} \frac{1}{k} \left| \sum_{X < n \leq 2X} e(kn^c) \right| + \sum_{k > H_1} \frac{H_1}{k^2} \left| \sum_{X < n \leq 2X} e(kn^c) \right| \\ &\ll \frac{X \log 2H_1}{H_1} + \sum_{1 \leq k \leq H_1} \frac{1}{k} \left((X^{c-1} k)^{\frac{269}{948} + \varepsilon} X^{\frac{269}{474} + \varepsilon} + \frac{1}{k X^{c-1}} \right) \\ &\quad + \sum_{k > H_1} \frac{H_1}{k^2} \left((X^{c-1} k)^{\frac{269}{948} + \varepsilon} X^{\frac{269}{474} + \varepsilon} + \frac{1}{k X^{c-1}} \right) \\ &\ll X^{\frac{269c+538}{1217}} \log X + H_1^{\frac{269}{948} + \varepsilon} X^{\frac{269(c+1)}{948} + \varepsilon} + X^{1-c} \ll X^{\frac{269c+538}{1217} + \varepsilon}. \end{aligned} \quad (8)$$

Similarly, for the first term in (6), we have

$$\begin{aligned}
& \sum_{|h| \leq H_1} c_h(\alpha) \sum_{X < n \leq 2X} e((h + \alpha)n^c) \\
&= c_0(\alpha) \sum_{X < n \leq 2X} e(\alpha n^c) + \sum_{1 \leq |h| \leq H_1} c_h(\alpha) \sum_{X < n \leq 2X} e((h + \alpha)n^c) \\
&\ll \frac{1}{\alpha X^{c-1}} + \sum_{1 \leq |h| \leq H_1} \frac{1}{h} \left(((h + \alpha)X^{c-1})^{\frac{269}{948} + \varepsilon} X^{\frac{269}{474} + \varepsilon} + \frac{1}{(h + \alpha)X^{c-1}} \right) \\
&\ll \frac{1}{\alpha X^{c-1}} + H_1^{\frac{269}{948} + \varepsilon} X^{\frac{269(c+1)}{948} + \varepsilon} + X^{1-c} \\
&\ll \frac{1}{\alpha X^{c-1}} + X^{\frac{269c+538}{1217} + \varepsilon}.
\end{aligned} \tag{9}$$

Combining (6)–(9), we obtain the desired result. This completes the proof of Lemma 11. \square

3. Proof of Theorem 1

By the definition of $\mathcal{R}_3(N)$, we have

$$\begin{aligned}
\mathcal{R}_3(N) &= \int_0^1 S^3(\alpha) e(-N\alpha) d\alpha = \int_{-\tau}^{1-\tau} S^3(\alpha) e(-N\alpha) d\alpha \\
&= \int_{-\tau}^{\tau} S^3(\alpha) e(-N\alpha) d\alpha + \int_{\tau}^{1-\tau} S^3(\alpha) e(-N\alpha) d\alpha \\
&= \mathcal{R}_3^{(1)}(N) + \mathcal{R}_3^{(2)}(N),
\end{aligned} \tag{10}$$

say. By the argument of Laporta and Tolev [12], we can know that, for $1 < c < 3/2$, there holds

$$\mathcal{R}_3^{(1)}(N) = \frac{\Gamma^3(1 + 1/c)}{\Gamma(3/c)} N^{3/c-1} + O\left(N^{3/c-1} \exp(-(log N)^{1/3-\delta})\right) \tag{11}$$

for any $0 < \delta < 1/3$, where the implied constant in the O -term depends only on c . Thus, it is sufficient to estimate $\mathcal{R}_3^{(2)}(N)$. First, we have

$$S(\alpha) = \sum_{p \leq P^{5/6}} (\log p) e([p^c]\alpha) + \sum_{P^{5/6} < p \leq P} (\log p) e([p^c]\alpha). \tag{12}$$

By a splitting argument and (12), we deduce that

$$\begin{aligned}
\mathcal{R}_3^{(2)}(N) &\ll (\log P) \max_{P^{5/6} \ll X \ll P} \left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| + P^{\frac{5}{6}} \int_0^1 |S(\alpha)|^2 d\alpha \\
&\ll (\log P) \max_{P^{5/6} \ll X \ll P} \left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| + P^{\frac{11}{6}} \log P.
\end{aligned} \tag{13}$$

For $P^{5/6} \ll X \ll P$, we have

$$\begin{aligned} & \left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| \\ &= \left| \sum_{X < p \leq 2X} (\log p) \int_{\tau}^{1-\tau} S^2(\alpha) e(([p^c] - N)\alpha) d\alpha \right| \\ &\leq \sum_{X < p \leq 2X} (\log p) \left| \int_{\tau}^{1-\tau} S^2(\alpha) e(([p^c] - N)\alpha) d\alpha \right| \\ &\ll (\log X) \sum_{X < n \leq 2X} \left| \int_{\tau}^{1-\tau} S^2(\alpha) e(([n^c] - N)\alpha) d\alpha \right|. \end{aligned}$$

By Cauchy's inequality, we deduce that

$$\begin{aligned} & \left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| \\ &\ll X^{\frac{1}{2}+\varepsilon} \left(\sum_{X < n \leq 2X} \left| \int_{\tau}^{1-\tau} S^2(\alpha) e(([n^c] - N)\alpha) d\alpha \right|^2 \right)^{\frac{1}{2}} \\ &= X^{\frac{1}{2}+\varepsilon} \left(\sum_{X < n \leq 2X} \int_{\tau}^{1-\tau} S^2(\alpha) e(([n^c] - N)\alpha) d\alpha \cdot \int_{\tau}^{1-\tau} \overline{S^2(\beta) e(([n^c] - N)\beta)} d\beta \right)^{\frac{1}{2}} \\ &= X^{\frac{1}{2}+\varepsilon} \left(\int_{\tau}^{1-\tau} \overline{S^2(\beta) e(-N\beta)} d\beta \int_{\tau}^{1-\tau} S^2(\alpha) T(\alpha - \beta, X) e(-N\alpha) d\alpha \right)^{\frac{1}{2}} \\ &\ll X^{\frac{1}{2}+\varepsilon} \left(\int_{\tau}^{1-\tau} |S(\beta)|^2 d\beta \int_{\tau}^{1-\tau} |S^2(\alpha) T(\alpha - \beta, X)| d\alpha \right)^{\frac{1}{2}}. \end{aligned} \tag{14}$$

For the inner integral in (14), we have

$$\begin{aligned} & \int_{\tau}^{1-\tau} |S^2(\alpha) T(\alpha - \beta, X)| d\alpha \\ &\ll \left(\int_{(\tau, 1-\tau) \cap \{\alpha: |\alpha - \beta| \leq X^{-c}\}} + \int_{(\tau, 1-\tau) \cap \{\alpha: |\alpha - \beta| > X^{-c}\}} \right) |S^2(\alpha) T(\alpha - \beta, X)| d\alpha. \end{aligned} \tag{15}$$

For the first term on the right-hand side of (15), we use Lemma 10 and the trivial estimate $T(\alpha - \beta, X) \ll X$ to deduce that

$$\begin{aligned} & \int_{(\tau, 1-\tau) \cap \{\alpha: |\alpha - \beta| \leq X^{-c}\}} |S^2(\alpha) T(\alpha - \beta, X)| d\alpha \\ &\ll X \cdot \sup_{\alpha \in (\tau, 1-\tau)} |S(\alpha)|^2 \times \int_{|\alpha - \beta| \leq X^{-c}} d\alpha \ll P^{\frac{49966}{27030} + \varepsilon} X^{1-c}. \end{aligned} \tag{16}$$

For the second term on the right-hand side of (15), by Lemma 10 and Lemma 11, we obtain

$$\begin{aligned}
& \int_{(\tau, 1-\tau) \cap \{\alpha: |\alpha-\beta| > X^{-c}\}} |S^2(\alpha) \mathcal{T}(\alpha - \beta, X)| d\alpha \\
& \ll \int_{(\tau, 1-\tau) \cap \{\alpha: |\alpha-\beta| > X^{-c}\}} |S(\alpha)|^2 \left(X^{\frac{269c+538}{1217} + \varepsilon} + \frac{1}{|\alpha-\beta| X^{c-1}} \right) d\alpha \\
& \ll X^{\frac{269c+538}{1217} + \varepsilon} \times \int_0^1 |S(\alpha)|^2 d\alpha + \sup_{\alpha \in (\tau, 1-\tau)} |S(\alpha)|^2 \times \int_{|\alpha-\beta| > X^{-c}} \frac{d\alpha}{|\alpha-\beta| X^{c-1}} \\
& \ll X^{\frac{269c+538}{1217} + \varepsilon} P \log P + P^{\frac{49966}{27030} + \varepsilon} X^{1-c}. \tag{17}
\end{aligned}$$

Combining (15) and (17), we conclude that

$$\int_{\tau}^{1-\tau} |S^2(\alpha) \mathcal{T}(\alpha - \beta, X)| d\alpha \ll X^{\frac{269c+538}{1217} + \varepsilon} P \log P + P^{\frac{49966}{27030} + \varepsilon} X^{1-c}. \tag{18}$$

Inserting (18) into (14), we obtain

$$\left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| \ll X^{\frac{1}{2} + \varepsilon} \left(X^{\frac{269c+538}{2434}} P + P^{\frac{24983}{27030}} X^{\frac{1-c}{2}} \right) \ll P^{3-c-\varepsilon}, \tag{19}$$

provided that $1 < c < \frac{3113}{2703}$. From (13) and (19), we obtain

$$\mathcal{R}_3^{(2)}(N) \ll P^{3-c-\varepsilon}. \tag{20}$$

By (10), (11) and (20), we get the desired result. This completes the proof of Theorem 1.

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