FINITE SEARCHES, CHOWLA'S COSINE PROBLEM, AND LARGE NEWMAN POLYNOMIALS

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Abstract

A length n cosine sum is an expression of the form $\cos a_1\theta + \cdots + \cos a_n\theta$ where $a_1 < \cdots < a_n$ are positive integers, and a length n Newman polynomial is an expression of the form $z^{a_1} + \cdots + z^{a_n}$ where $a_1 < \cdots < a_n$ are nonnegative integers. We define $-\lambda(n)$ to be the largest minimum of a length n cosine sum as $\{a_1, \ldots, a_n\}$ ranges over all sets of n positive integers, and we define $\mu(n)$ to be the largest minimum modulus on the unit circle of a length n Newman polynomial as $\{a_1, \ldots, a_n\}$ ranges over all sets of n nonnegative integers. Since there are infinitely many possibilities for the a_j, it is not obvious how to compute $\lambda(n)$ or $\mu(n)$ for a given n in finitely many steps. Campbell et al. found the value of $\mu(3)$ in 1983, and Goddard found the value of $\mu(4)$ in 1992. In this paper, we find the values of $\lambda(2)$ and $\lambda(3)$ and nontrivial bounds on $\mu(5)$. We also include further remarks on the seemingly difficult general task of reducing the computation of $\lambda(n)$ or $\mu(n)$ to a finite problem.

1. Introduction

We define a *length n cosine sum* to be any expression of the form

$$\cos a_1\theta + \cdots + \cos a_n\theta$$

where $a_1 < \cdots < a_n$ are positive integers. If $f(\theta)$ is a length n cosine sum, then $f(0) = n$ and $f(\theta) \geq -n$ for all θ. For $f(\theta) = \cos a_1\theta + \cdots + \cos a_n\theta$, we define

$$L(a_1, \ldots, a_n) = \min_{\theta} f(\theta) = \min_{0\leq\theta\leq\pi} f(\theta)$$

which is negative, since $f(\theta)$ has average value 0 on $[0, \pi]$. More specifically, we have

$$-n \leq L(a_1, \ldots, a_n) < 0.$$

We then define

$$-\lambda(n) = \sup L(a_1, \ldots, a_n)$$

where $\{a_1, \ldots, a_n\}$ ranges over all of the infinitely many sets of n positive integers. Then $0 \le \lambda(n) \le n$. There is also a construction showing $\lambda(n) = O(\sqrt{n})$ (see Section 2).

We define a *length n Newman polynomial* to be any expression of the form

$$z^{a_1} + \cdots + z^{a_n}$$

where $a_1 < \cdots < a_n$ are nonnegative integers. If $F(z)$ is a length n Newman polynomial and $|z| = 1$, then $0 \le |F(z)| \le n$. We define

$$M(a_1, \ldots, a_n) = \min_{|z|=1} |z^{a_1} + \cdots + z^{a_n}|$$

so we have

$$0 \le M(a_1, \ldots, a_n) \le n,$$

and in fact one can show $M(a_1, \ldots, a_n) \le \sqrt{n}$ by considering the L^2 norm of F. (We note that the case $n = 2$ is uninteresting because $z^{a_1} + z^{a_2} = z^{a_1}(1 + z^{a_2 - a_1})$ always has minimum modulus 0 at the $(a_2 - a_1)$th roots of -1.) We then define

$$\mu(n) = \sup M(a_1, \ldots, a_n)$$

where $\{a_1, \ldots, a_n\}$ ranges over all of the infinitely many sets of n nonnegative integers.

Then $\lambda(n)$ and $\mu(n)$ are well-defined functions of n because we are taking the supremum of a bounded set. However, since there are infinitely many possibilities for the a_j, it is not obvious how to compute $\lambda(n)$ or $\mu(n)$ for a given n in finitely many steps. Also, since $\lambda(n)$ and $\mu(n)$ are defined as suprema of infinite sets, it is not obvious if $\lambda(n)$ or $\mu(n)$ are always equal to $-L(a_1, \ldots, a_n)$ or $M(a_1, \ldots, a_n)$ for a specific choice of $a_1, \ldots, a_n$. In any case, some brute-force exploration of specific small examples leads to conjectures such as

$$\begin{aligned}
\lambda(2) &= -L(1,2) = 9/8 = 1.125, \\
\lambda(3) &= -L(1,2,3) \approx 1.315565, \\
\lambda(4) &= -L(1,2,3,4) \approx 1.519558, \\
\lambda(5) &= -L(1,2,4,5,6) \approx 1.627461, \\
\lambda(6) &= -L(1,2,4,6,7,8) \approx 1.591832,
\end{aligned}$$

and

$$\begin{aligned}
\mu(3) &= M(0,1,3) \approx 0.607346, \\
\mu(4) &= M(0,1,2,4) \approx 0.752394, \\
\mu(5) &= M(0,1,2,6,9) = 1.
\end{aligned}$$

In 1983, Campbell et al. [2] proved that $\mu(3) = M(0,1,3)$, and in 1992, Goddard [6] proved that $\mu(4) = M(0,1,2,4)$. To prove results of that type, one must reduce a potentially infinite search to a finite search. The main results of this paper are:

Theorem 1.1. *We have*

$$\lambda(2) = -L(1,2) = 9/8 = 1.125,$$

$$\lambda(3) = -L(1,2,3) = \frac{17+7\sqrt{7}}{27} \approx 1.315565.$$

Theorem 1.2. *We have* $1 \le \mu(5) \le 1 + \pi/6 \approx 1.5236$.

Theorem 1.1 is proved in Section 4 and Theorem 1.2 is proved in Section 6.

Various authors have investigated the growth rates of the functions $\lambda(n)$ and $\mu(n)$. Around the late 1940s, Ankeny and Chowla conjectured [3, 4] that $\lambda(n)$ approaches infinity with n. This was first proved by Uchiyama and Uchiyama [9] using results of Cohen [5]; their lower bound for $\lambda(n)$ was sublogarithmic. Over the years, better lower bounds for $\lambda(n)$ have been found. The best lower bound currently known is due to Ruzsa [8]; it is superlogarithmic but grows more slowly than any power of n. The best known upper bound for $\lambda(n)$ appears to be $O(\sqrt{n})$. Chowla conjectured [4] that this is the true rate of growth.

It seems that the growth of $\mu(n)$ is less studied than the growth of $\lambda(n)$. There is a construction showing $\mu(n)$ exceeds a power of n for infinitely many n; specifically, we have $\mu(n) \ge n^{0.14}$ when n is a power of 9 (See Section 2). Boyd [1] conjectured that $\mu(n) > 1$ for all $n \ge 6$ and that $\log \mu(n)/\log n$ approaches a positive constant as n approaches infinity. The current author [7] proved that $\mu(n) > 0$ for all $n > 2$. It appears that nobody has proved that $\mu(n)$ approaches infinity with n.

It would be interesting to show that for any n, the value of $\lambda(n)$ or $\mu(n)$ can be computed in a finite number of steps (even an impractically large finite number) and it would be interesting to know if $\lambda(n)$ and $\mu(n)$ are always equal to $-L(a_1,\dots,a_n)$ or $M(a_1,\dots,a_n)$ for a specific choice of the a_j. This appears to be difficult to prove.

2. Constructions Bounding $\lambda(n)$ and $\mu(n)$

There are some straightforward constructions that lead to bounds on $\lambda(n)$ and $\mu(n)$. These probably count as mathematical folklore. We include them here for completeness.

Lemma 2.1. *For each positive integer n, there exists a length n cosine sum $f(\theta)$ that satisfies $f(\theta) \ge -\sqrt{2n} - \frac{1}{2}$ for all θ. It follows that $\lambda(n) \le \sqrt{2n} + \frac{1}{2}$.*

Proof. Let $k = \lceil \sqrt{2n} \rceil$, so $\sqrt{2n} \le k < \sqrt{2n} + 1$. Let $\{b_1 < \ldots < b_k\}$ be a set of k nonnegative integers such that the $\binom{k}{2}$ positive differences $b_j - b_i$ are all distinct;

for instance, we can take $b_j = 2^{j-1}$. (A set with this property is sometimes called a 'Sidon set'; in this proof, we do not need our Sidon set to be optimal in any sense.) Then if $z = e^{i\theta}$, we have

$$0 \le \left|z^{b_1} + \cdots + z^{b_k}\right|^2 = (z^{b_1} + \cdots + z^{b_k})(z^{-b_1} + \cdots + z^{-b_k})$$
$$= k + \sum_{i<j} 2\mathrm{Re}(z^{b_j - b_i}) = k + 2\sum_{i<j}\cos(b_j - b_i)\theta.$$

Thus, if we define $g(\theta) = \sum_{i<j}\cos(b_j - b_i)\theta$, then $g(\theta)$ is a length $\binom{k}{2}$ cosine sum satisfying $g(\theta) \ge -k/2$ for all θ. Now observe

$$n - \sqrt{\frac{n}{2}} = \frac{\sqrt{2n}(\sqrt{2n}-1)}{2} \le \frac{k(k-1)}{2} < \frac{(\sqrt{2n}+1)\sqrt{2n}}{2} = n + \sqrt{\frac{n}{2}}.$$

If $\binom{k}{2} > n$, choose $h(\theta)$ to be a sum consisting of $\binom{k}{2} - n$ of the cosines in the sum $g(\theta)$, and choose $f(\theta) = g(\theta) - h(\theta)$. If $\binom{k}{2} < n$, choose $h(\theta)$ to be a sum of $n - \binom{k}{2}$ cosines not appearing in the sum $g(\theta)$, and choose $f(\theta) = g(\theta) + h(\theta)$. If $\binom{k}{2} = n$, define $h(\theta) = 0$ and choose $f(\theta) = g(\theta)$.

Then $h(\theta)$ is a sum of at most $\sqrt{n/2}$ cosines, so we have $\pm h(\theta) \ge -\sqrt{n/2}$. We then have

$$f(\theta) = g(\theta) \pm h(\theta) \ge -\frac{k}{2} - \sqrt{\frac{n}{2}} > -\frac{\sqrt{2n}+1}{2} - \sqrt{\frac{n}{2}} = -\sqrt{2n} - \frac{1}{2}$$

which completes the proof. □

Lemma 2.2. *Let F be a length n Newman polynomial satisfying $|F(z)| \ge K_1$ for all $|z| = 1$, and let G be a length m Newman polynomial satisfying $|G(z)| \ge K_2$ for all $|z| = 1$. Then there exists a length nm Newman polynomial H satisfying $|H(z)| \ge K_1 K_2$ for all $|z| = 1$.*

Proof. Suppose

$$F(z) = z^{a_1} + \cdots + z^{a_n},$$
$$G(z) = z^{b_1} + \cdots + z^{b_m}.$$

If k is a sufficiently large positive integer, then the product

$$H(z) = F(z^k)G(z) = (z^{ka_1} + \cdots + z^{ka_n})(z^{b_1} + \cdots + z^{b_m})$$

has the property that the nm exponents $ka_i + b_j$ are all distinct, and hence $H(z)$ is a Newman polynomial of length nm. If $|z| = 1$, then $|H(z)| = \left|F(z^k)\right| |G(x)| \ge K_1 K_2$. This completes the proof. □

As mentioned in [1], the length 9 Newman polynomial

$$F(z) = 1 + z + z^2 + z^3 + z^4 + z^7 + z^8 + z^{10} + z^{12}$$

has unusually high minimum modulus on the unit circle. Specifically, $|F(z)| \geq 1.362$ for all z on the unit circle. Then, by repeated application of Lemma 2.2, we can construct for each k a Newman polynomial $H(z)$ of length 9^k with the property that $|H(z)| \geq 1.362^k$ for all z on the unit circle. Since $1.362 > 9^{0.14}$, this means that for infinitely many values of n, we have a length n Newman polynomial $H(z)$ that satisfies $|H(z)| \geq n^{0.14}$ on the unit circle.

For example, there is a Newman polynomial of length $9^3 = 729$ that satisfies $|H(z)| \geq 1.362^3 \approx 2.53$ on the unit circle, so $\mu(729) \geq 2.53$. It would be interesting to know (for example) the least n such that $\mu(n) \geq 2$.

3. Some Notation and Lemmas

In this section, we establish some notation and some useful lemmas.

Let $f(\theta) = \cos a_1\theta + \cdots + \cos a_n\theta$, let $g = \gcd(a_1, \ldots, a_n)$, and let $a'_j = a_j/g$. Then $\cos a'_1\theta + \cdots + \cos a'_n\theta$ is a length n cosine sum taking on the same values (and hence having the same minimum) as $f(\theta)$. Therefore, in the definition of $\lambda(n)$, it suffices to consider only those $\{a_1, \ldots, a_n\}$ for which $\gcd(a_1, \ldots, a_n) = 1$.

Let $F(z) = z^{a_1} + \cdots + z^{a_n}$, where $a_1 < \cdots < a_n$ are nonnegative integers. Define $b_j = a_j - a_1$. If $|z| = 1$, then

$$F_1(z) = F(z)/z^{a_1} = 1 + z^{b_2} + \cdots + z^{b_n}$$

has the same modulus as $F(z)$. Next, define $g = \gcd(b_2, \ldots, b_n)$ and $b'_j = b_j/g$. If

$$F_2(z) = 1 + z^{b'_2} + \cdots + z^{b'_n}$$

then $F_2(z^g) = F_1(z)$, so $F_1(z)$ and $F_2(z)$ have the same set of outputs as z ranges over the unit circle. Therefore, in the definition of $\mu(n)$, it suffices to consider only those $\{a_1, \ldots, a_n\}$ of the form $\{0, a_2, \ldots, a_n\}$ where $\gcd(a_2, \ldots, a_n) = 1$. Furthermore, there is one more symmetry we exploit. If

$$F(z) = 1 + z^{a_2} + \cdots + z^{a_{n-1}} + z^{a_n}$$

then we define

$$H(z) = z^{a_n} F(z^{-1}) = z^{a_n} + z^{a_n - a_2} + \cdots + z^{a_n - a_{n-1}} + 1,$$

and note that $|F(z)|$ and $|H(z)|$ have the same set of outputs as z ranges over the unit circle. So we are free to choose between $F(z)$ and $H(z)$ and can hence assume $a_{n-1} \geq a_n - a_2$.

For the above reasons, we make the following definitions. Define

$$\mathbb{N}'_n = \{(a_1, \ldots, a_n) \mid 0 < a_1 < \cdots < a_n,\ \gcd(a_1, \ldots, a_n) = 1\},$$
$$\mathbb{N}''_n = \{(0, a_2, \ldots, a_n) \mid 0 < a_2 < \cdots < a_n,\ \gcd(a_2, \ldots, a_n) = 1,\ a_{n-1} \geq a_n - a_2\}$$

and note that if $L(a_1, \ldots, a_n)$ and $M(a_1, \ldots, a_n)$ are as defined in Section 1, then we have

$$-\lambda(n) = \sup L(a_1, \ldots, a_n)$$

where the supremum is taken over all $(a_1, \ldots, a_n) \in \mathbb{N}'_n$, and we have

$$\mu(n) = \sup M(0, a_2, \ldots, a_n)$$

where the supremum is taken over all $(0, a_2, \ldots, a_n) \in \mathbb{N}''_n$.

We also define $\mathbb{T}$ to be $\mathbb{R}$ mod 2π. Following are some definitions and lemmas regarding subsets of $\mathbb{T}$.

Definition. An *equispaced subset* of $\mathbb{T}$ of *order* m is any subset of $\mathbb{T}$ of the form

$$\left\{\theta_0 + \frac{2k\pi}{m} \;\middle|\; k \in \mathbb{Z}\right\}.$$

Note that if we fix $\xi \in \mathbb{T}$, then the set $\{\theta \in \mathbb{T} \mid m\theta = \xi\}$ is an equispaced set of order m.

Lemma 3.1. *Let S_1 and S_2 be equispaced subsets of $\mathbb{T}$ of order m_1 and m_2 respectively, and let $g = \gcd(m_1, m_2)$. Then there exists $\theta_1 \in S_1$ and $\theta_2 \in S_2$ such that $|\theta_1 - \theta_2| \leq \frac{\pi g}{m_1 m_2}$.*

Proof. Suppose

$$S_1 = \left\{\xi_1 + \frac{2k\pi}{m_1} \;\middle|\; k \in \mathbb{Z}\right\} \qquad \text{and} \qquad S_2 = \left\{\xi_2 + \frac{2k\pi}{m_2} \;\middle|\; k \in \mathbb{Z}\right\}.$$

The real number $\frac{m_1 m_2}{2\pi}(\xi_1 - \xi_2)$ must be within $\frac{g}{2}$ of an integer multiple of g, call it ag. Also, ag can be written in the form $km_1 - \ell m_2$ for some integers k and ℓ. Thus we have

$$\left|\frac{m_1 m_2}{2\pi}(\xi_1 - \xi_2) - (km_1 - \ell m_2)\right| \leq \frac{g}{2},$$

which, multiplying by $\frac{2\pi}{m_1 m_2}$ and rearranging, gives us

$$\left|\left(\xi_1 + \frac{2\pi\ell}{m_1}\right) - \left(\xi_2 + \frac{2\pi k}{m_2}\right)\right| \leq \frac{\pi g}{m_1 m_2},$$

completing the proof of the lemma. □

The following lemma is straightforward.

Lemma 3.2. *If θ satisfies $|\theta - \pi| = \varepsilon$, then $\cos\theta \le -1 + \frac{1}{2}\varepsilon^2$.*

We will also need other bounds on the cosine function, which need not be the best bounds possible. One can show the following.

Lemma 3.3. *If θ satisfies $\left|\theta - \frac{2\pi}{3}\right| = \varepsilon \le \frac{\pi}{6}$, then*

$$\cos\theta \le -\frac{1}{2} + \frac{3}{\pi}\varepsilon$$

and if θ satisfies $\left|\theta - \frac{4\pi}{3}\right| = \varepsilon \le \frac{\pi}{6}$, then

$$\cos\theta \le -\frac{1}{2} + \frac{3}{\pi}\varepsilon.$$

Definition. If $S \subset \mathbb{T}$ is an equispaced set of order m, and $f(\theta)$ and $g(\theta)$ are real-valued functions on $\mathbb{T}$, then we define

$$\big\langle f(\theta), g(\theta)\big\rangle_S = \frac{1}{m}\sum_{\theta\in S} f(\theta)g(\theta),$$

the average value of $f(\theta)g(\theta)$ over S, which we can think of as a kind of dot product of $f(\theta)$ and $g(\theta)$.

One can verify that this dot product has the following properties:

- $\big\langle f_1(\theta) + f_2(\theta), g(\theta)\big\rangle_S = \big\langle f_1(\theta), g(\theta)\big\rangle_S + \big\langle f_2(\theta), g(\theta)\big\rangle_S$
- $\big\langle f(\theta), g_1(\theta) + g_2(\theta)\big\rangle_S = \big\langle f(\theta), g_1(\theta)\big\rangle_S + \big\langle f(\theta), g_2(\theta)\big\rangle_S$
- $\big\langle 1, 1\big\rangle_S = 1$
- $\big\langle 1, \cos k\theta\big\rangle_S = \big\langle \cos k\theta, 1\big\rangle_S = 0$ if k is *not* a multiple of m
- $\big\langle \cos k\theta, \cos k\theta\big\rangle_S = \frac{1}{2}$ if $2k$ is *not* a multiple of m
- $\big\langle \cos k\theta, \cos \ell\theta\big\rangle_S = 0$ if $k+\ell$ and $k-\ell$ are *not* multiples of m

A function that is nonnegative on $\mathbb{T}$ can be used as a 'weight function'. Some examples of nonnegative weight functions are:

$$1 - \cos k\theta,$$
$$2(1-\cos k\theta)^2 = 3 - 4\cos k\theta + \cos 2k\theta,$$

as well as any sum of such functions.

Lemma 3.4. *Let $w(\theta)$ be a nonnegative weight function on $\mathbb{T}$, let $g(\theta)$ be any real-valued function on $\mathbb{T}$, and let $S \subset \mathbb{T}$ be an equispaced set. If we have $\big\langle w(\theta), g(\theta)\big\rangle_S \le 0$, then $g(\theta) \le 0$ for some $\theta \in \mathbb{T}$.*

4. The Values of $\lambda(2)$ and $\lambda(3)$

To prove $\lambda(2) = -L(1,2) = 9/8$, let $(a,b) \in \mathbb{N}'_2$, so $\gcd(a,b) = 1$. We must show that $\cos a\theta + \cos b\theta \leq -9/8$ for some θ.

If $b \leq 2$, then $\cos a\theta + \cos b\theta = \cos\theta + \cos 2\theta$, which one can verify has minimum value $-9/8$. So suppose $b \geq 3$. Define

$$S_1 = \{\theta \in \mathbb{T} \mid a\theta = \pi\} \qquad \text{and} \qquad S_2 = \{\theta \in \mathbb{T} \mid b\theta = \pi\},$$

which are equispaced sets of order a and b respectively. By Lemma 3.1, there exist $\theta_1 \in S_1$ and $\theta_2 \in S_2$ such that $|\theta_1 - \theta_2| \leq \frac{\pi}{ab}$. Then $\cos b\theta_2 = \pi$ and $|a\theta_2 - \pi| = |a\theta_2 - a\theta_1| \leq \frac{\pi}{b}$, so by Lemma 3.2, we have

$$\cos a\theta_2 \leq -1 + \frac{\pi^2}{2b^2} \leq -1 + \frac{\pi^2}{18} \approx -0.45,$$

implying $\cos a\theta_2 + \cos b\theta_2 \leq -2 + \frac{\pi^2}{18} \approx -1.45 < -9/8$. This completes the evaluation of $\lambda(2)$.

Next, we will show that $\lambda(3) = -L(1,2,3)$. First, observe that trigonometric identities allow us to write

$$\begin{aligned}\cos\theta + \cos 2\theta + \cos 3\theta &= \cos\theta + (2\cos^2\theta - 1) + (4\cos^3\theta - 3\cos\theta) \\ &= 4\cos^3\theta + 2\cos^2\theta - 2\cos\theta - 1\end{aligned}$$

which is a polynomial in $\cos\theta$ of degree 3. One can verify that

$$\min_{-1\leq c\leq 1} 4c^3 + 2c^2 - 2c - 1 = -\frac{17 + 7\sqrt{7}}{27} \approx -1.315565.$$

That is, $-L(1,2,3) = \frac{17+7\sqrt{7}}{27} \approx 1.315565$. For brevity, let $K = -L(1,2,3)$.

To prove that $\lambda(3) = K$, we will partition $\mathbb{N}'_3$ into four subsets:

$$\begin{aligned}M_1 &= \{(a,b,c) \in \mathbb{N}'_3 \mid c = 2a\}, \\ M_2 &= \{(a,b,c) \in \mathbb{N}'_3 \mid c = 2b\}, \\ M_3 &= \{(a,b,c) \in \mathbb{N}'_3 \mid c = a + b\}, \\ M_0 &= \big\{(a,b,c) \in \mathbb{N}'_3 \mid c \notin \{2a, 2b, a+b\}\big\}.\end{aligned}$$

Speaking very informally, we can think of M_1, M_2, M_3 as subsets of $\mathbb{N}'_3$ that have only two 'degrees of freedom'. For each $j \in \{0,1,2,3\}$, we will show that with finitely many exceptions, if $(a,b,c) \in M_j$ then $\cos a\theta + \cos b\theta + \cos c\theta \leq -K$ for some θ.

Suppose $(a,b,c) \in M_0$. We then have $0 < a < b < c$, $\gcd(a,b,c) = 1$, and $c \notin \{2a, 2b, a+b\}$. Let $S = \{\theta \in \mathbb{T} \mid c\theta = \pi\}$, which is an equispaced set of order c. Note that $2 - \cos a\theta - \cos b\theta$ is a nonnegative weight function, and consider

$$\delta = \Big\langle 2 - \cos a\theta - \cos b\theta,\ \frac{1}{2} + \cos a\theta + \cos b\theta \Big\rangle_S.$$

Now observe that none of a, b, or $b-a$ is a multiple of c, and that any of $2a$, $2b$, or $a+b$ that is a multiple of c must equal c. It then follows from properties of the dot product that

$$\delta = \Big\langle 2, \frac{1}{2} \Big\rangle_S - \Big\langle \cos a\theta, \cos a\theta \Big\rangle_S - \Big\langle \cos b\theta, \cos b\theta \Big\rangle_S = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

and then by Lemma 3.4, we conclude that $\frac{1}{2} + \cos a\theta + \cos b\theta \leq 0$ for some $\theta \in S$. But that θ satisfies $\cos c\theta = -1$ and therefore $\cos a\theta + \cos b\theta + \cos c\theta \leq -3/2 < -K$.

Next, suppose $(a,b,c) \in M_1$. If $a \leq 2$ then (a,b,c) must be $(2,3,4)$, and we observe that $\cos 2\theta + \cos 3\theta + \cos 4\theta = -2 < -K$ if $\theta = \pi/3$. We therefore assume $a \geq 3$. Now define

$$S_1 = \Big\{ \theta \in \mathbb{T} \mid a\theta = \frac{2\pi}{3} \Big\}, \qquad S_2 = \{ \theta \in \mathbb{T} \mid b\theta = \pi \},$$

which are equispaced sets of order a and b respectively. Note that $\gcd(a,b) = 1$ since any common divisor would divide $2a = c$. By Lemma 3.1, there exist $\theta_1 \in S_1$ and $\theta_2 \in S_2$ such that $|\theta_1 - \theta_2| \leq \frac{\pi}{ab}$. Then $\cos a\theta_1 = \cos 2\pi/3 = -1/2$ and $\cos c\theta_1 = \cos 2a\theta_1 = \cos 4\pi/3 = -1/2$. Also, we have $|b\theta_1 - \pi| = |b\theta_1 - b\theta_2| \leq \frac{\pi}{a}$, so by Lemma 3.2, we have

$$\cos b\theta_1 \leq -1 + \frac{\pi^2}{2a^2} \leq -1 + \frac{\pi^2}{18} \approx -0.45,$$

implying $\cos a\theta_1 + \cos b\theta_1 + \cos c\theta_1 \leq -2 + \frac{\pi^2}{18} \approx -1.45 < -K$.

Next, suppose $(a,b,c) \in M_2$. This case is very similar to $(a,b,c) \in M_1$. If $b \leq 2$ then (a,b,c) must be $(1,2,4)$, and we observe that $\cos\theta + \cos 2\theta + \cos 4\theta = -3/2 < -K$ if $\theta = 2\pi/3$. We therefore assume $b \geq 3$. We define

$$S_1 = \{ \theta \in \mathbb{T} \mid a\theta = \pi \}, \qquad S_2 = \Big\{ \theta \in \mathbb{T} \mid b\theta = \frac{2\pi}{3} \Big\},$$

which are equispaced sets of order a and b, and we note that $\gcd(a,b) = 1$, so there exist $\theta_1 \in S_1$ and $\theta_2 \in S_2$ such that $|\theta_1 - \theta_2| \leq \frac{\pi}{ab}$. Then $\cos b\theta_2 = \cos 2\pi/3 = -1/2$ and $\cos c\theta_2 = \cos 2b\theta_2 = \cos 4\pi/3 = -1/2$. Also, we have $|a\theta_2 - \pi| = |a\theta_2 - a\theta_1| \leq \frac{\pi}{b}$, so Lemma 3.2 gives us

$$\cos a\theta_2 \leq -1 + \frac{\pi^2}{2b^2} \leq -1 + \frac{\pi^2}{18} \approx -0.45,$$

implying $\cos a\theta_2 + \cos b\theta_2 + \cos c\theta_2 \leq -2 + \frac{\pi^2}{18} \approx -1.45 < -K$.

Finally, suppose $(a,b,c) \in M_3$. We will show the existence of a finite subset $M' \subset M_3$ such that if $(a,b,c) \in M_3 \setminus M'$, then $\cos a\theta + \cos b\theta + \cos c\theta \leq -K$ for some θ. That will reduce this case to checking the finitely many elements of M'. It will suffice to choose

$$M' = \{(a,b,c) \in M_3 \mid b \leq 32\} = \{(a,b,a+b) \in M_3 \mid 1 \leq a < b \leq 32\}$$

which has at most $\binom{32}{2} = 496$ elements. At the end of this section, we will give an alternative argument that avoids such a large finite set.

So suppose $(a, b, c) \in M_3 \setminus M'$, so $c = a + b$ and $b \geq 33$. We choose

$$S_1 = \Big\{\theta \in \mathbb{T} \mid a\theta = \frac{2\pi}{3}\Big\}, \qquad S_2 = \Big\{\theta \in \mathbb{T} \mid b\theta = \frac{2\pi}{3}\Big\},$$

which are equispaced sets of order a and b respectively. Note that $\gcd(a, b) = 1$ since any common divisor would divide $a+b = c$. By Lemma 3.1, there exist $\theta_1 \in S_1$ and $\theta_2 \in S_2$ such that $|\theta_1 - \theta_2| \leq \frac{\pi}{ab}$. Then $\cos b\theta_2 = \cos 2\pi/3 = -1/2$. Also, we have $|a\theta_2 - 2\pi/3| = |a\theta_2 - a\theta_1| \leq \frac{\pi}{b}$ and

$$\left|c\theta_2 - \frac{4\pi}{3}\right| = \left|\Big(a\theta_2 - \frac{2\pi}{3}\Big) + \Big(b\theta_2 - \frac{2\pi}{3}\Big)\right| = \left|\Big(a\theta_2 - \frac{2\pi}{3}\Big) + 0\right| \leq \frac{\pi}{b}.$$

By Lemma 3.3, we then have

$$\cos a\theta_2 \leq -\frac{1}{2} + \frac{3}{\pi}\frac{\pi}{b} \qquad \text{and} \qquad \cos c\theta_2 \leq -\frac{1}{2} + \frac{3}{\pi}\frac{\pi}{b}$$

which implies

$$\cos a\theta_2 + \cos b\theta_2 + \cos c\theta_2 \leq -\frac{3}{2} + \frac{6}{b} \leq -\frac{3}{2} + \frac{6}{33} \approx -1.318 < -K.$$

We remark that the case $(a, b, c) \in M_3$ can be dealt with more easily if we take as known the fact that

$$\mu(3) = M(0, 1, 3) = \sqrt{\frac{47 - 14\sqrt{7}}{27}} \approx 0.607346,$$

which was shown in Section 3 of [2]. Note that this result says that given any degree 3 Newman polynomial $F(z) = 1 + z^k + z^\ell$, we have $|F(z)| \leq M(0, 1, 3)$ for some $|z| = 1$. We apply this to the case $F(z) = 1 + z^a + z^{a+b}$. The statement

$$\big|1 + z^a + z^{a+b}\big| \leq \sqrt{\frac{47 - 14\sqrt{7}}{27}} \qquad \text{for some } |z| = 1$$

is equivalent to

$$\big|1 + z^a + z^{a+b}\big|^2 \leq \frac{47 - 14\sqrt{7}}{27} \qquad \text{for some } |z| = 1$$

but we also have, if $z = e^{i\theta}$,

$$\begin{aligned}\big|1 + z^a + z^{a+b}\big|^2 &= (1 + z^a + z^{a+b})(1 + z^{-a} + z^{-a-b}) \\ &= 3 + (z^a + z^{-a}) + (z^b + z^{-b}) + (z^{a+b} + z^{-a-b}) \\ &= 3 + 2\big(\cos a\theta + \cos b\theta + \cos(a + b)\theta\big).\end{aligned}$$

The condition

$$3 + 2\big(\cos a\theta + \cos b\theta + \cos(a+b)\theta\big) \le \frac{47 - 14\sqrt{7}}{27}$$

is equivalent to

$$\cos a\theta + \cos b\theta + \cos(a+b)\theta \le \frac{1}{2}\Big(\frac{47-14\sqrt{7}}{27} - 3\Big) = -\frac{17+7\sqrt{7}}{27} = L(1,2,3).$$

In [2], the fact that $\mu(3) = M(0,1,3)$ is proved by reducing to a finite number of cases, but their number of cases is much less than 496. Since we used the value of $\mu(3)$ just in the case where $(a,b,c) \in M_3$, it could be said that our evaluation of $\lambda(3)$ is a generalization of the evaluation of $\mu(3)$.

5. A Possible Strategy for $\lambda(4)$

The current author is unaware of how to evaluate $\lambda(4)$, but includes in this section a possible outline of a strategy where we reduce that problem to a finite set of problems that, speaking informally, have fewer 'degrees of freedom'.

We conjecture that $\lambda(4) = -L(1,2,3,4) \approx 1.519558$. To prove this, we must show that if $(a,b,c,d) \in \mathbb{N}_4'$, then $\cos a\theta + \cos b\theta + \cos c\theta + \cos d\theta \le L(1,2,3,4)$ for some θ.

Let $(a,b,c,d) \in \mathbb{N}_4'$, and define $S = \{\theta \in \mathbb{T} \mid d\theta = \pi\}$, which is an equispaced set of order d. Define the nonnegative weight function

$$\begin{aligned} w(\theta) &= (1-\cos a\theta) + (1-\cos b\theta) + 2(1-\cos c\theta)^2 \\ &= 1 - \cos a\theta + 1 - \cos b\theta + 3 - 4\cos c\theta + \cos 2c\theta \\ &= 5 - \cos a\theta - \cos b\theta - 4\cos c\theta + \cos 2c\theta. \end{aligned}$$

If we can show that

$$\Big\langle w(\theta),\ \frac{3}{5} + \cos a\theta + \cos b\theta + \cos c\theta \Big\rangle_S \le 0,$$

then it will follow that $\cos a\theta + \cos b\theta + \cos c\theta \le -3/5$ for some θ satisfying $\cos d\theta = -1$, so then $\cos a\theta + \cos b\theta + \cos c\theta + \cos d\theta \le -8/5 = -1.6 < L(1,2,3,4)$.

Using previously stated properties of the dot product, we note that evaluating

$$\Big\langle 5 - \cos a\theta - \cos b\theta - 4\cos c\theta + \cos 2c\theta,\ \frac{3}{5} + \cos a\theta + \cos b\theta + \cos c\theta \Big\rangle_S$$

depends on properties of the set $\{0,a,b,c,2c\} \pm \{0,a,b,c\}$. This is illustrated in Table 1. If all the nonzero numbers in the body of Table 1 are nonmultiples of d,

	0	a	b	c	$2c$
0	0	a	b	c	$2c$
a	a	$2a$	$b \pm a$	$c \pm a$	$2c \pm a$
b	b	$b \pm a$	$2b$	$c \pm b$	$2c \pm b$
c	c	$c \pm a$	$c \pm b$	$2c$	$2c \pm c$

Table 1: Elements of $\{0, a, b, c, 2c\} \pm \{0, a, b, c\}$

then properties of the dot product give us

$$\left\langle 5 - \cos a\theta - \cos b\theta - 4\cos c\theta + \cos 2c\theta,\ \frac{3}{5} + \cos a\theta + \cos b\theta + \cos c\theta \right\rangle_S$$
$$= 5 \cdot \frac{3}{5} - \frac{1}{2} - \frac{1}{2} - 4 \cdot \frac{1}{2} = 0$$

and so in that case, the desired conclusion follows. We now observe:

- The positive numbers $a,\ b,\ c,\ b-a,\ c-a,\ c-b$ are all less than d, and are hence nonmultiples of d.
- The positive numbers $2a,\ 2b,\ 2c,\ b+a,\ c+a,\ c+b,\ 2c-a,\ 2c-b$ are all less than $2d$, and hence if any of them are multiples of d, they are equal to d.
- The positive numbers $2c+a,\ 2c+b,\ 2c+c$ are all less than $3d$, and hence if any of them are multiples of d, they are equal to d or $2d$.

It follows that if $(a, b, c, d) \in \mathbb{N}'_4$ has the property that d is *not* equal to any of these 14 linear combinations of a, b, c

$$2a, 2b, 2c, b+a, c+a, c+b, 2c-a, 2c-b, 2c+a, c+\frac{1}{2}a, 2c+b, c+\frac{1}{2}b, 3c, \frac{3}{2}c$$

then $\cos a\theta + \cos b\theta + \cos c\theta + \cos d\theta \leq -8/5 < L(1, 2, 3, 4)$ for some θ. It then remains to deal with those (a, b, c, d) where d is equal to one of the 14 combinations listed above.

Enumerate those 14 linear combinations as $\varphi_1(a, b, c), \ldots, \varphi_{14}(a, b, c)$, and then for $1 \leq j \leq 14$, define

$$M_j = \{(a, b, c, d) \in \mathbb{N}'_4 \mid d = \varphi_j(a, b, c)\}.$$

Then each M_j, speaking very informally, is a subset of $\mathbb{N}'_4$ having only 3 'degrees of freedom'. If we can prove for each j that if $(a, b, c, d) \in M_j$, then $\cos a\theta + \cos b\theta + \cos c\theta + \cos d\theta \leq L(1, 2, 3, 4)$ for some θ, then that would complete the evaluation of $\lambda(4)$. Perhaps it is possible to reduce each M_j to a finite collection of subsets of $\mathbb{N}'_4$ that have 2 degrees of freedom.

6. Bounds on $\mu(5)$

It was observed in [6] that

$$M(0,1,2,6,9) = \min_{|z|=1} \left|1 + z + z^2 + z^6 + z^9\right| = 1$$

and it is suspected that $\mu(5) = 1$, although this appears to be difficult to prove. In this section, we give a short elementary argument showing that $\mu(5) \le \sqrt{3} \approx 1.732$, and a longer case analysis showing that $\mu(5) \le 1+\pi/5 \approx 1.628$. More generally, we will show that for any positive integer m, the problem of showing $\mu(5) \le 1 + \pi/m$ can be reduced to checking a finite number of cases. In particular, we will sketch a proof that $\mu(5) \le 1 + \pi/6 \approx 1.524$.

To show $\mu(5) \le K$, we must show that for all of the infinitely many $(0, a, b, c, d)$ in $\mathbb{N}_5''$, we have $\left|1 + z^a + z^b + z^c + z^d\right| \le K$ for some $|z| = 1$.

Let $(0, a, b, c, d) \in \mathbb{N}_5''$. To show $\left|1 + z^a + z^b + z^c + z^d\right| \le \sqrt{3}$ for some $|z| = 1$, define $S = \{\theta \in \mathbb{T} \mid d\theta = \pi\}$, and let $z = e^{i\theta}$. Observe that we have

$$\begin{aligned} \left|z^a + z^b + z^c\right|^2 &= (z^a + z^b + z^c)(z^{-a} + z^{-b} + z^{-c}) \\ &= 3 + (z^{b-a} + z^{a-b}) + (z^{c-a} + z^{a-c}) + (z^{c-b} + z^{b-c}) \\ &= 3 + \cos(b-a)\theta + \cos(c-a)\theta + \cos(c-b)\theta. \end{aligned}$$

Now note that $b - a,\ c - a,\ c - b$ are positive integers less than d, so they are not multiples of d. It follows that the average of $\left|z^a + z^b + z^c\right|^2$ over S is 3. Therefore at least one $\theta \in S$ satisfies $\left|z^a + z^b + z^c\right|^2 \le 3$ and hence $\left|1 + z^a + z^b + z^c + z^d\right| = \left|z^a + z^b + z^c\right| \le \sqrt{3}$.

To get better bounds on $\mu(5)$, we will use some straightforward lemmas similar to those in Section 3.

Lemma 6.1. *If θ satisfies $|\theta - \pi| \le \delta$, then $\left|1 + e^{i\theta}\right| \le \delta$.*

Lemma 6.2. *Let k, ℓ, m be distinct positive integers with $\ell < m$, and let $g = \gcd(k, m - \ell)$. We then have*

$$\left|1 + z^k + z^\ell + z^m\right| \le \frac{\pi g}{k}$$

for some z on the unit circle.

Proof. Define the sets

$$\begin{aligned} S_1 &= \{\theta \in \mathbb{T} \mid k\theta = \pi\}, \\ S_2 &= \{\theta \in \mathbb{T} \mid (m-\ell)\theta = \pi\}, \end{aligned}$$

which are equispaced sets of order k and $m-\ell$ respectively. By Lemma 3.1, there exist $\theta_1 \in S_1$ and $\theta_2 \in S_2$ such that $|\theta_1 - \theta_2| \le \frac{\pi g}{k(m-\ell)}$, and hence

$$|(m-\ell)\theta_1 - \pi| = |(m-\ell)\theta_1 - (m-\ell)\theta_2| \le \frac{\pi g}{k}.$$

Then, if $z = e^{i\theta_1}$, we have

$$\begin{aligned}\left|1 + z^k + z^\ell + z^m\right| &= \left|1 + z^k + z^\ell(1 + z^{m-\ell})\right| = \left|1 + e^{ik\theta_1} + z^\ell(1 + z^{m-\ell})\right| \\ &= \left|1 - 1 + z^\ell(1 + z^{m-\ell})\right| = \left|z^\ell(1 + z^{m-\ell})\right| = \left|1 + z^{m-\ell}\right|\end{aligned}$$

which, by Lemma 6.1, is at most $\pi g/k$. □

Now to prove $\mu(5) \le 1 + \pi/5$, let $(0, a, b, c, d) \in \mathbb{N}_5''$, and define $\alpha(z) = 1 + z^a + z^b + z^c + z^d$. We must show that

$$|\alpha(z)| \le 1 + \frac{\pi}{5}$$

for some z on the unit circle. From the definition of $\mathbb{N}_5''$, we have not only $0 < a < b < c < d$, but also $c \ge d - a$, which implies $c > d - b$. We thus have the following inequalities, where both sides are positive integers:

$$\begin{aligned} d &> b - a \\ d &> c - b \\ d &> c - a \\ c &> b - a \\ c &> d - b \\ c &\ge d - a \end{aligned}$$

We now define

$$\begin{aligned} g_1 &= \gcd(d, b-a) & &\text{so } g_1 \le b - a < d \\ g_2 &= \gcd(d, c-b) & &\text{so } g_2 \le c - b < d \\ g_3 &= \gcd(d, c-a) & &\text{so } g_3 \le c - a < d \\ g_4 &= \gcd(c, b-a) & &\text{so } g_4 \le b - a < c \\ g_5 &= \gcd(c, d-b) & &\text{so } g_5 \le d - b < c \\ g_6 &= \gcd(c, d-a) & &\text{so } g_6 \le d - a \le c. \end{aligned}$$

Applying Lemma 6.2 to the case $(k, \ell, m) = (d, a, b)$, we have

$$\begin{aligned}|\alpha(z)| &\le |z^c| + \left|1 + z^d + z^a + z^b\right| \\ &= 1 + \left|1 + z^d + z^a + z^b\right| \le 1 + \frac{\pi g_1}{d} \qquad \text{for some } |z| = 1.\end{aligned}$$

Similarly, applying Lemma 6.2 to each of the cases

$$(k,\ell,m) = (d,b,c)$$
$$(k,\ell,m) = (d,a,c)$$
$$(k,\ell,m) = (c,a,b)$$
$$(k,\ell,m) = (c,b,d)$$
$$(k,\ell,m) = (c,a,d),$$

we can conclude that we have

$$|\alpha(z)| \le 1 + \frac{\pi g_2}{d} \qquad \text{for some } |z| = 1$$
$$|\alpha(z)| \le 1 + \frac{\pi g_3}{d} \qquad \text{for some } |z| = 1$$
$$|\alpha(z)| \le 1 + \frac{\pi g_4}{c} \qquad \text{for some } |z| = 1$$
$$|\alpha(z)| \le 1 + \frac{\pi g_5}{c} \qquad \text{for some } |z| = 1$$
$$|\alpha(z)| \le 1 + \frac{\pi g_6}{c} \qquad \text{for some } |z| = 1.$$

If any of the six numbers

$$\frac{g_1}{d},\ \frac{g_2}{d},\ \frac{g_3}{d},\ \frac{g_4}{c},\ \frac{g_5}{c},\ \frac{g_6}{c} \tag{1}$$

is less than or equal to $1/5$, we are done. So for the remainder of the proof, we assume each of those six numbers is greater than $1/5$.

Note that d and $b-a$ are integer multiples of g_1, with $d > b-a$. The condition $g_1/d > 1/5$ is equivalent to $d < 5g_1$, which implies that $d \in \{2g_1, 3g_1, 4g_1\}$. Then since $b-a$ is an integer multiple of g_1 smaller than d, one of the following must be true:

$$d = 2g_1 \text{ and } b-a = g_1 \implies \frac{1}{2}d = b-a$$
$$d = 3g_1 \text{ and } b-a = g_1 \implies \frac{1}{3}d = b-a$$
$$d = 3g_1 \text{ and } b-a = 2g_1 \implies \frac{2}{3}d = b-a$$
$$d = 4g_1 \text{ and } b-a = 1g_1 \implies \frac{1}{4}d = b-a$$
$$d = 4g_1 \text{ and } b-a = 3g_1 \implies \frac{3}{4}d = b-a$$

So we have $rd = b - a$, where

$$r \in \left\{\frac{1}{4},\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4}\right\}.$$

We will now apply similar reasoning to the other five expressions in (1). Note that the following hold:

$$\begin{gathered}
d \text{ and } c-b \text{ are integer multiples of } g_2 \text{ with } d > c-b, \text{ and } d < 5g_2\\
d \text{ and } c-a \text{ are integer multiples of } g_3 \text{ with } d > c-a, \text{ and } d < 5g_3\\
c \text{ and } b-a \text{ are integer multiples of } g_4 \text{ with } c > b-a, \text{ and } c < 5g_4\\
c \text{ and } d-b \text{ are integer multiples of } g_5 \text{ with } c > d-b, \text{ and } c < 5g_5\\
c \text{ and } d-a \text{ are integer multiples of } g_6 \text{ with } c \geq d-a, \text{ and } c < 5g_6.
\end{gathered}$$

Notice the slight difference in the last line. This means that our list of possibilities for c and $d - a$ will include $c = d - a = g_6$.

Using similar reasoning as before, we can conclude:

$$\begin{gathered}
rd = b-a, \text{ where } r \in \{\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4}\}\\
sd = c-b, \text{ where } s \in \{\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4}\}\\
td = c-a, \text{ where } t \in \{\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4}\}\\
uc = b-a, \text{ where } u \in \{\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4}\}\\
vc = d-b, \text{ where } v \in \{\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4}\}\\
wc = d-a, \text{ where } w \in \{\tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ 1\}.
\end{gathered}$$

Thus there are finitely many possibilities for r, s, t, u, v, w. Notice that this remains true if we change our goal from showing $\mu(5) \leq 1 + \pi/5$ to showing $\mu(5) \leq 1 + \pi/m$. In that case, the possible values for r, s, t, u, v, w are fractions between 0 and 1 whose denominators are less than m.

We claim that for each of the finitely many possible triples r, s, u, there is at most one element of $\mathbb{N}_5''$ satisfying the three conditions

$$\begin{aligned}
rd &= b-a\\
sd &= c-b\\
uc &= b-a.
\end{aligned}$$

To verify this claim, we write these conditions as

$$\begin{aligned}
1a - 1b + 0c + rd &= 0\\
0a + 1b - 1c + sd &= 0\\
1a - 1b + uc + 0d &= 0,
\end{aligned}$$

which we temporarily regard as a system of equations in *rational* unknowns a, b, c, d. Writing as a matrix and row-reducing, we have

$$\begin{bmatrix} 1 & -1 & 0 & r \\ 0 & 1 & -1 & s \\ 1 & -1 & u & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & r \\ 0 & 1 & -1 & s \\ 0 & 0 & u & -r \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & r \\ 0 & 1 & -1 & s \\ 0 & 0 & 1 & -\frac{r}{u} \end{bmatrix}.$$

Note that u is always nonzero. Continuing, we have

$$\begin{bmatrix} 1 & -1 & 0 & r \\ 0 & 1 & -1 & s \\ 0 & 0 & 1 & -\frac{r}{u} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & r \\ 0 & 1 & 0 & s-\frac{r}{u} \\ 0 & 0 & 1 & -\frac{r}{u} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & r+s-\frac{r}{u} \\ 0 & 1 & 0 & s-\frac{r}{u} \\ 0 & 0 & 1 & -\frac{r}{u} \end{bmatrix}.$$

It follows that the system has an infinite one-parameter family of *rational* solutions, given by

$$\begin{aligned} a &= \left(\frac{r}{u} - r - s\right)q \\ b &= \left(\frac{r}{u} - s\right)q \\ c &= \left(\frac{r}{u}\right)q \\ d &= q, \end{aligned}$$

where q can be any rational number. In other words, the rational solutions (a, b, c, d) are precisely the rational multiples of the 4-tuple

$$\left(\frac{r}{u} - r - s,\ \frac{r}{u} - s,\ \frac{r}{u},\ 1\right).$$

However, at most one rational multiple of a given rational 4-tuple can consist of relatively prime nonnegative integers. That is, there is at most one possible $(0, a, b, c, d) \in \mathbb{N}_5''$ for each choice of r, s, u.

For the specific case of showing $\mu(5) \le 1 + \pi/5$, it is possible to enumerate all the possibilities by hand. Recall that we have the restrictions

$$\begin{gathered} rd = b - a, \text{ where } r \in \{\tfrac14,\ \tfrac13,\ \tfrac12,\ \tfrac23,\ \tfrac34\} \\ sd = c - b, \text{ where } s \in \{\tfrac14,\ \tfrac13,\ \tfrac12,\ \tfrac23,\ \tfrac34\} \\ td = c - a, \text{ where } t \in \{\tfrac14,\ \tfrac13,\ \tfrac12,\ \tfrac23,\ \tfrac34\} \\ uc = b - a, \text{ where } u \in \{\tfrac14,\ \tfrac13,\ \tfrac12,\ \tfrac23,\ \tfrac34\} \\ vc = d - b, \text{ where } v \in \{\tfrac14,\ \tfrac13,\ \tfrac12,\ \tfrac23,\ \tfrac34\} \\ wc = d - a, \text{ where } w \in \{\tfrac14,\ \tfrac13,\ \tfrac12,\ \tfrac23,\ \tfrac34,\ 1\}. \end{gathered}$$

Notice that

$$(r + s)d = rd + sd = (b - a) + (c - b) = c - a = td$$

r	s	t
1/4	1/4	1/2
1/4	1/2	3/4
1/3	1/3	2/3
1/2	1/4	3/4

Table 2: Permissible values of r, s, t

u	v	w
1/4	1/4	1/2
1/4	1/2	3/4
1/4	3/4	1
1/3	1/3	2/3
1/3	2/3	1
1/2	1/4	3/4
1/2	1/2	1
2/3	1/3	1
3/4	1/4	1

Table 3: Permissible values of u, v, w

so $r + s = t$, and also notice that

$$(u + v)c = uc + vc = (b - a) + (d - b) = d - a = wc$$

so $u + v = w$. The only permissible values of r, s, t that satisfy this are shown in Table 2, and the only permissible values of u, v, w that satisfy this are shown in Table 3. Next, notice that we have $rd = b - a = uc$, which together with $c < d$ implies that $u > r$. Also notice that we have

$$sd = c - b \text{ implies } b = c - sd$$
$$vc = d - b \text{ implies } b = d - vc$$
$$c - sd = d - vc \text{ implies } c + vc = d + sd \text{ implies } (1 + v)c = (1 + s)d,$$

which together with $c < d$ implies $1 + v > 1 + s$, so $v > s$. The only permissible values of r, s, u, v satisfying both $u > r$ and $v > s$ are shown in Table 4. We now use the conditions $rd = uc$ and $(1 + v)c = (1 + s)d$ to conclude

$$\frac{r}{u} = \frac{c}{d} = \frac{1 + s}{1 + v}, \text{ implying } r(1 + v) = u(1 + s)$$

which only some of our possible values of r, s, u, v will satisfy. We see in Table 5 that only one of our possibilities satisfies $r(1 + v) = u(1 + s)$. We conclude that $(r, s, u, v) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3})$. We then have

$$\left(\frac{r}{u} - r - s,\ \frac{r}{u} - s,\ \frac{r}{u},\ 1\right) = \left(\frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1\right),$$

r	s	u	v
1/4	1/4	1/3	1/3
1/4	1/4	1/3	2/3
1/4	1/4	1/2	1/2
1/4	1/4	2/3	1/3
1/4	1/2	1/3	2/3
1/3	1/3	1/2	1/2
1/2	1/4	2/3	1/3

Table 4: Permissible values of r, s, u, v satisfying $u > r$ and $v > s$

r	s	u	v	$1+s$	$1+v$	$r(1+v)$	$u(1+s)$
1/4	1/4	1/3	1/3	5/4	4/3	1/3	5/12
1/4	1/4	1/3	2/3	5/4	5/3	5/12	5/12
1/4	1/4	1/2	1/2	5/4	3/2	3/8	5/8
1/4	1/4	2/3	1/3	5/4	4/3	1/3	5/6
1/4	1/2	1/3	2/3	3/2	5/3	5/12	1/2
1/3	1/3	1/2	1/2	4/3	3/2	1/2	2/3
1/2	1/4	2/3	1/3	5/4	4/3	2/3	5/6

Table 5: Checking whether $r(1+v) = u(1+s)$

implying $(a, b, c, d) = (1, 2, 3, 4)$. But the polynomial $f(z) = 1 + z + z^2 + z^3 + z^4$ certainly satisfies $|f(z)| \leq 1 + \pi/5$ for some $|z| = 1$, because it has zeros at the nontrivial fifth roots of unity. This completes the proof that $\mu(5) \leq 1 + \pi/5$.

The above argument can be modified to show that $\mu(5) \leq 1+\pi/m$ for some other positive integers m, but doing so by hand is cumbersome and computer assistance is helpful. We will give a rough outline of an argument that $\mu(5) \leq 1 + \pi/6$.

This time, the argument involves finding r, s, t, u, v, w that satisfy

$$\begin{aligned} rd &= b - a \\ sd &= c - b \\ td &= c - a \\ uc &= b - a \\ vc &= d - b \\ wc &= d - a, \end{aligned}$$

where $0 < r, s, t, u, v < 1$, $0 < w \leq 1$, and r, s, t, u, v, w are fractions with denominators strictly less than 6. They must further satisfy

$$r + s = t, \qquad u + v = w, \qquad r < u, \qquad s < v, \qquad r(1 + v) = u(1 + s).$$

A finite search (aided by computer) reveals that the only eligible values of r, s, u, v

r	s	u	v	$1+s$	$1+v$
1/3	1/3	2/5	3/5	4/3	8/5
1/4	1/4	1/3	2/3	5/4	5/3
1/5	1/5	1/4	1/2	6/5	3/2
1/5	2/5	1/4	3/4	7/5	7/4
2/5	1/5	1/2	1/2	6/5	3/2
3/5	1/5	2/3	1/3	6/5	4/3

Table 6: Eligible r, s, u, v with denominators less than 6

$\frac{r}{u}-r-s$	$\frac{r}{u}-s$	$\frac{r}{u}$	(a,b,c,d)
1/6	3/6	5/6	(1, 3, 5, 6)
1/4	2/4	3/4	(1, 2, 3, 4)
2/5	3/5	4/5	(2, 3, 4, 5)
1/5	2/5	4/5	(1, 2, 4, 5)
1/5	3/5	4/5	(1, 3, 4, 5)
1/10	7/10	9/10	(1, 7, 9, 10)

Table 7: Eligible 4-tuples for verifying $\mu(5) \leq 1 + \pi/6$

are those shown in Table 6. (The values of $1+s$ and $1+v$ are included for convenience.) As before, each eligible (r, s, u, v) gives us a 4-tuple

$$\left(\frac{r}{u} - r - s,\ \frac{r}{u} - s,\ \frac{r}{u},\ 1\right)$$

of which precisely one integer multiple is an eligible (a, b, c, d). This gives us six possibilities, shown in Table 7. One can verify that for each of these six possibilities for (a, b, c, d), we have

$$\left|1 + z^a + z^b + z^c + z^d\right| \leq 1 + \pi/6$$

for some $|z| = 1$. This completes the sketch of the proof that $\mu(5) \leq 1 + \pi/6$.

In conclusion, note that the contributions of Campbell et al. and Goddard appeared gradually. The results in this paper can be regarded as extensions of that work. It appears that evaluating $\lambda(n)$ or $\mu(n)$ in finitely many steps is a genuinely subtle problem.

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