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AN ALTERNATIVE PROOF OF MELHAM'S IDENTITIES

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Abstract

Melham conjectures 21 identities, all of which are analogous to Jacobi's two-square theorem. A small number of these have already been proved in various ways by other authors, and in 2013 Toh proved all of them using similar, known identities. In this paper we offer an alternative and straightforward method to proving all of them using the theory of modular forms, in particular, Sturm's bound, which we explain in detail.

1. Introduction

Define

$$G_3(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \qquad G_5(q) := \sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}, \qquad G_7(q) := \sum_{n=-\infty}^{\infty} q^{\frac{n(5n-3)}{2}},$$

the generating functions of the triangular, pentagonal, and heptagonal numbers, respectively, with $q = e^{2\pi i\tau}$, and τ in the upper half plane, H. Melham's 21 identities (all analogous to Jacobi's two-square theorem) [9] are all of the form

$$G_k(q^\alpha)G_k(q^\beta) = \text{an explicit } q\text{-series}$$

where $k = 3, 5$ or 7, $\alpha, \beta \in \mathbb{N}$. For example, using the labeling in his paper, his Identity (6) is

$$G_3(q)G_3(q^5) = \sum_{n=0}^{\infty} \left[\frac{q^{3n} + q^{7n+1}}{1 - q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1 - q^{20n+15}} \right].$$

We will frequently refer to the $G_k(q^\alpha)G_k(q^\beta)$ part as the left-hand side of the identity, and the explicit q-series as the right-hand side.

The method we shall use relies on the theory of *modular forms*. We provide a brief description here, but for a more thorough reading, the reader should look to such sources as [2], [8], [10]. A *congruence subgroup*, Γ, of $SL_2(\mathbb{Z})$ is a subgroup of $SL_2(\mathbb{Z})$ that contains

$$\Gamma(N) := \mathrm{Ker}\Big(SL_2(\mathbb{Z}) \to SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)\Big), \tag{1}$$

where $\Gamma(N)$ is called the *principal congruence subgroup.* The smallest such N is the *level* of Γ. Three such particular congruence subgroups of interest are:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}, \tag{2}$$

$$\begin{aligned} \Gamma^0(N) :&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_0(N) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod N \right\}, \end{aligned} \tag{3}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}, \tag{4}$$

where $*$ represents any number. All three groups have level N. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and for a complex function f, define

$$f|_A(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right). \tag{5}$$

A function f is a modular form of weight k, with k a positive integer, over a congruence subgroup Γ if f is holomorphic on H, $f|_\gamma$ remains bounded as $\Im(z) \to \infty$ for any $\gamma \in SL_2(\mathbb{Z})$, and verifies, for all $\gamma \in \Gamma$, $f|_\gamma = f$.

Our method is to show that the left-hand side, or LHS, (raised to an even power) of each identity is a type of theta function, a modular form over a certain congruence subgroup. We then show that the right-hand side, or RHS, (raised to the same even power) is also a modular form for the same weight over this subgroup, by showing that it can be written in terms of the Weierstrass zeta function. It is then a simple case of checking a small number of coefficients to show that each identity satisfies Sturm's bound, and thus that both sides must be equivalent. Melham mentions in his paper that a small number of these have already been proved in various ways by Hirschhorn [5], Sun [12], and Dickson [3] (combined with work from Adiga, Cooper, and Han [1]). In 2013 Toh [13] proved all of them using similar, known identities. The method demonstrated in this paper is easily expandable to similar identities, and a textbook demonstration of how to apply Sturm's bound.

2. The LHS as Theta Functions

We begin this section by defining a *lattice* as in [7, p. 149]. Let V be an n-dimensional real vector space. A lattice in V is a subgroup of V of the form

$$\Gamma = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n, \tag{6}$$

where $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors in V. An *integral lattice* is a lattice where the inner product of any two elements in the lattice is integral. An *even lattice* is an integral lattice where the inner product of an element with itself (or norm) is always even. The *dual lattice*, denoted Λ^*, of a lattice Λ, is the lattice of vectors having integral inner products with all the elements of Λ. Define the *discriminant* of a lattice to be $\mathrm{disc}(\Lambda) = |\Lambda^*/\Lambda|$. Finally, the *level* of a lattice Λ is the minimum $N \in \mathbb{N}$ with Nx^2 even for all $x \in \Lambda^*$ [4, p. 91].

Ebeling [4, p. 86] defines a *generalized Theta function* for a lattice $\Lambda \subset \mathbb{R}^n$, a point $z \in \mathbb{R}^n$, the variable $\tau \in H$, and a spherical polynomial P of degree r, as

$$\vartheta_{z+\Lambda,P}(\tau) := \sum_{x \in z+\Lambda} P(x)e^{\pi i \tau x^2} = \sum_{x \in z+\Lambda} P(x)q^{\frac{x^2}{2}}. \tag{7}$$

We have no need to discuss spherical polynomials; for our matters it is sufficient to state that the constant polynomial $P(x) = 1$ is spherical, of degree 0. We take $P(x) = 1$ from here on, and will drop it from our notation.

Lemma 1. *Let $\Lambda \subset \mathbb{R}^n$ (n even) be an even lattice of level N, $\rho \in \Lambda^*$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then for odd $d > 0$, $c \neq 0$,*

$$\vartheta_{\rho+\Lambda}(A\tau) = (c\tau + d)^{\frac{n}{2}} \left(\frac{\Delta}{d}\right) e^{\pi i a b \rho^2} \vartheta_{a\rho+\Lambda}(\tau) \tag{8}$$

with $\Delta := (-1)^{\frac{n}{2}} \mathit{disc}(\Lambda)$, $\left(\frac{\Delta}{d}\right)$ the Jacobi symbol, and for $c = 0$,

$$\vartheta_{\rho+\Lambda}(A\tau) = e^{\pi i a b \rho^2} \vartheta_{a\rho+\Lambda}(\tau). \tag{9}$$

Proof. The proof of this is Corollary 3.1 combined with the remarks before Theorem 3.2 of [4, pp. 92-94]. $\square$

Now, as $\rho \in \Lambda^*$, $\rho^2 \in \mathbb{Q}$. Let $\rho^2 = \frac{u}{v}$, with $\gcd(u, v) = 1$. We have

$$\vartheta^{2v}_{\rho+\Lambda}|_A = e^{2\pi i a b u} \vartheta^{2v}_{a\rho+\Lambda} = \vartheta^{2v}_{a\rho+\Lambda}.$$

We now take $n = 2$, and consider lattices in $\mathbb{R}^2$.

2.1. Triangular Numbers, $G_3(q)$

Let $\Lambda = \left\langle \begin{pmatrix} \sqrt{2\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2\beta} \end{pmatrix} \right\rangle$, with α, β integers. Then Λ is even, and has dual

$$\Lambda^* = \left\langle \begin{pmatrix} \frac{1}{\sqrt{2\alpha}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2\beta}} \end{pmatrix} \right\rangle. \tag{10}$$

Calculating the level N of Λ is straightforward, and gives $N = 4 \cdot \mathrm{lcm}(\alpha, \beta)$. Let $\rho = \begin{pmatrix} \sqrt{\frac{\alpha}{2}} \\ \sqrt{\frac{\beta}{2}} \end{pmatrix} \in \Lambda^*$. We have, with $q = e^{2\pi i\tau}$,

$$\begin{aligned} \vartheta_{\rho+\Lambda}(\tau) &= \sum_{n,m\in\mathbb{Z}} q^{\alpha n^2+\alpha n+\frac{\alpha}{4}+\beta m^2+\beta m+\frac{\beta}{4}} \\ &= q^{\frac{\alpha+\beta}{4}} \sum_{n,m\in\mathbb{Z}} q^{2\alpha\frac{n(n+1)}{2}+2\beta\frac{m(m+1)}{2}}. \end{aligned} \tag{11}$$

Hence

$$\vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = q^{\frac{\alpha+\beta}{8}} \sum_{n\in\mathbb{Z}} q^{\alpha\frac{n(n+1)}{2}} \sum_{m\in\mathbb{Z}} q^{\beta\frac{m(m+1)}{2}}, \tag{12}$$

so

$$\frac{1}{4} q^{-\frac{\alpha+\beta}{8}} \vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = G_3(q^\alpha)G_3(q^\beta). \tag{13}$$

Now, it is easy to see that $\rho + \Lambda = a\rho + \Lambda$ whenever $a \equiv 1 \pmod 2$. Further, as N is clearly even, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we can see that a must be odd. Now, $\rho^2 = \frac{\alpha+\beta}{2}$, hence if $\alpha+\beta$ is even then $\vartheta_{\rho+\Lambda}(\tau)^2$ is a weight 2 modular form for $\Gamma_0(N)$, and if $\alpha + \beta$ is odd then $\vartheta_{\rho+\Lambda}(\tau)^4$ is a weight 4 modular form for $\Gamma_0(N)$. Therefore $\vartheta_{\rho+\Lambda}(\frac{\tau}{2})^{2v}$ is a weight $2v$ modular form for $\Gamma' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, where v is defined as

$$v = \begin{cases} 1 & \alpha+\beta \equiv 0 \pmod 2 \\ 2 & \alpha+\beta \equiv 1 \pmod 2. \end{cases} \tag{14}$$

In other words, depending on the values of α and β, we need to raise $\vartheta_{\rho+\Lambda}(\tau)$ to an appropriate power $(2v)$, in order to ensure modularity. This turns out to be exactly analogous in the next two cases.

We see that for $A \in \Gamma'$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod{\frac{N}{2}}$. In other words,

$$\Gamma' = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(\frac{N}{2}\right) \quad : \quad b \text{ even} \right\} = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2). \tag{15}$$

Hence we can conclude that

$$\left(q^{\frac{\alpha+\beta}{8}} G_3(q^\alpha) G_3(q^\beta)\right)^{2v} \tag{16}$$

is a modular form for Γ' as well.

2.2. Pentagonal Numbers, $G_5(q)$

Let $\Lambda = \left\langle \begin{pmatrix} \sqrt{6\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{6\beta} \end{pmatrix} \right\rangle$, with α, β integers. Then Λ is even and has dual

$$\Lambda^* = \left\langle \begin{pmatrix} \frac{1}{\sqrt{6\alpha}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6\beta}} \end{pmatrix} \right\rangle. \tag{17}$$

The level of Λ is $N = 12 \cdot \mathrm{lcm}(\alpha, \beta)$. Let $\rho = \begin{pmatrix} -\sqrt{\frac{\alpha}{6}} \\ -\sqrt{\frac{\beta}{6}} \end{pmatrix} \in \Lambda^*$. We find

$$q^{-\frac{\alpha+\beta}{24}} \vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = G_5(q^\alpha) G_5(q^\beta). \tag{18}$$

Again, it is easy to see that $\rho + \Lambda = a\rho + \Lambda$ whenever $a \equiv 1 \pmod 6$. When $a \equiv -1 \pmod 6$ we see $a\rho + \Lambda = -(\rho + \Lambda)$, but as we are taking the norm of each shifted lattice point this is also acceptable. Further, as N is clearly even, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we can see that a must be odd and coprime to 3.

Now, $\rho^2 = -\frac{\alpha+\beta}{6}$, hence $\vartheta_{\rho+\Lambda}(\tau)^{2v}$ is a weight $2v$ modular form for the congruence subgroup $\Gamma' = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad : \quad a \equiv \pm 1 \pmod 6\}$ where

$$v = \begin{cases} 1 & \alpha + \beta \equiv 0 \pmod 6 \\ 6 & \alpha + \beta \equiv \pm 1 \pmod 6 \\ 3 & \alpha + \beta \equiv \pm 2 \pmod 6 \\ 2 & \alpha + \beta \equiv 3 \pmod 6. \end{cases} \tag{19}$$

Therefore $\vartheta_{\rho+\Lambda}(\frac{\tau}{2})^{2v}$ is a weight $2v$ modular form for $\Gamma'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Gamma' \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We see for $A \in \Gamma''$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod{\frac{N}{2}}$. In other words,

$$\begin{aligned} \Gamma'' &= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(\frac{N}{2}\right) \quad : \quad b \text{ even}, a \equiv \pm 1 \pmod 6 \right\} \\ &= \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2) \end{aligned} \tag{20}$$

as $1 = ad - bc \equiv d \pmod 6$. Hence we can conclude that

$$\left(q^{\frac{\alpha+\beta}{24}} G_5(q^\alpha) G_5(q^\beta)\right)^{2v} \tag{21}$$

is a modular form for Γ'' as well.

2.3. Heptagonal Numbers, $G_7(q)$

Let $\Lambda = \left\langle \begin{pmatrix} \sqrt{10\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{10\beta} \end{pmatrix} \right\rangle$, with α, β integers. Then Λ is even, and has dual

$$\Lambda^* = \left\langle \begin{pmatrix} \frac{1}{\sqrt{10\alpha}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{10\beta}} \end{pmatrix} \right\rangle. \tag{22}$$

The level of Λ is $N = 20 \cdot \mathrm{lcm}(\alpha, \beta)$. Let $\rho = \begin{pmatrix} -3\sqrt{\frac{\alpha}{10}} \\ -3\sqrt{\frac{\beta}{10}} \end{pmatrix} \in \Lambda^*$. We find

$$q^{-\frac{9(\alpha+\beta)}{40}} \vartheta_{\rho+\Lambda}\left(\frac{\tau}{2}\right) = G_7(q^\alpha) G_7(q^\beta). \tag{23}$$

Once more, we have $\rho + \Lambda = a\rho + \Lambda$ whenever $a \equiv 1 \pmod{10}$. Further, as N is clearly even, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we can see that a must be odd.

Now, $\rho^2 = \frac{\alpha+\beta}{10}$, hence $\vartheta_{\rho+\Lambda}(\tau)^{2v}$ is a weight $2v$ modular form for the congruence subgroup $\Gamma' = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \quad : \quad a \equiv 1 \pmod{10}\}$ where

$$v = \begin{cases} 1 & \alpha + \beta \equiv 0 \pmod{10} \\ 10 & \alpha + \beta \equiv \pm 1 \pmod{10} \\ 5 & \alpha + \beta \equiv \pm 2 \pmod{10} \\ 10 & \alpha + \beta \equiv \pm 3 \pmod{10} \\ 5 & \alpha + \beta \equiv \pm 4 \pmod{10} \\ 2 & \alpha + \beta \equiv 5 \pmod{10}. \end{cases} \tag{24}$$

Therefore $\vartheta_{\rho+\Lambda}(\frac{\tau}{2})^{2v}$ is a weight $2v$ modular form for $\Gamma'' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Gamma' \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We see for $A \in \Gamma''$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod{\frac{N}{2}}$. In other words,

$$\begin{aligned} \Gamma' &= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0\left(\frac{N}{2}\right) \quad : \quad b \text{ even, } a \equiv \pm 1 \pmod{10} \right\} \\ &= \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2) \cap \Gamma_1(10). \end{aligned} \tag{25}$$

3. The RHS as the Weierstrass Zeta Function

The *Weierstrass zeta function* is a function that naturally leads to an Eisenstein series of weight 1 [2, p. 138]. We write it as Z to avoid confusion and define it as

$$Z_\Lambda(z) := \frac{1}{z} + \sum_{\substack{w\in\Lambda\\ w\neq 0\\ w\neq z}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right) \tag{26}$$

for a 2 dimensional lattice $\Lambda := \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, and $z \in \mathbb{C}$. This function is not quite periodic, but rather with each step along the lattice increases by a lattice constant, $\eta_1(\Lambda)$ or $\eta_2(\Lambda)$, defined by:

$$\eta_1(\Lambda) := Z_\Lambda(z+\omega_1) - Z_\Lambda(z) \quad \text{and} \quad \eta_2(\Lambda) := Z_\Lambda(z+\omega_2) - Z_\Lambda(z). \tag{27}$$

If we assume without loss of generality that $\Im\left(\frac{\omega_1}{\omega_2}\right) > 0$, then these lattice constants satisfy the *Legendre relation*

$$\eta_2(\Lambda)\omega_1 - \eta_1(\Lambda)\omega_2 = 2\pi i. \tag{28}$$

We can thus define a periodic function, $Z^*_{\Lambda_\tau}$, for $u, v \in \mathbb{R}$, and $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$, as follows,

$$Z^*_{\Lambda_\tau}(u\tau + v) := Z_{\Lambda_\tau}(u\tau+v) - u\eta_1(\Lambda_\tau) - v\eta_2(\Lambda_\tau). \tag{29}$$

For some scalar m, and writing $Z(z \mid \omega_1, \omega_2) := Z_\Lambda(z)$ as is common, we see

$$\begin{aligned} Z(mz \mid m\omega_1, m\omega_2) &= \frac{1}{mz} + \sum_{\substack{w\in m\Lambda\\ w\neq 0\\ w\neq mz}} \left(\frac{1}{mz-w} + \frac{1}{w} + \frac{mz}{w^2}\right) \\ &= \frac{1}{m}\left(\frac{1}{z} + \sum_{\substack{w\in\Lambda\\ w\neq 0\\ w\neq z}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right)\right) \\ &= \frac{1}{m} Z(z \mid \omega_1, \omega_2). \end{aligned} \tag{30}$$

Applying this to $Z(z \mid 1, \tau) = Z_{\Lambda_\tau}(z)$ transformed by a matrix in $SL_2(\mathbb{Z})$, we find

$$\begin{aligned} Z\left(z \mid 1, \frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)Z(z(c\tau+d) \mid c\tau+d, a\tau+b) \\ &= (c\tau+d)Z(z(c\tau+d) \mid 1, \tau) \end{aligned} \tag{31}$$

as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We define

$$f_{r,s}(\tau) := Z\left(\frac{r\tau+s}{N} \mid 1, \tau\right). \tag{32}$$

Hence

$$\begin{aligned} f_{r,s}\left(\frac{a\tau+b}{c\tau+d}\right) &= Z\left(\frac{1}{N}\left(s+r\left(\frac{a\tau+b}{c\tau+d}\right)\right) \mid 1, \frac{a\tau+b}{c\tau+d}\right) \\ &= (c\tau+d)Z\left(\frac{1}{N}\left(s(c\tau+d)+r(a\tau+b)\right) \mid 1, \tau\right) \\ &= (c\tau+d)f_{ar+cs,br+ds}(\tau). \end{aligned} \tag{33}$$

Diamond and Shurman [2, p. 138] further state that the Weierstrass zeta function can be expressed as

$$Z_{\Lambda_\tau}(z) = \eta_2(\Lambda_\tau)z - \pi i\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}} - 2\pi i\sum_{n=1}^{\infty}\left[\frac{e^{2\pi iz}q^n}{1-e^{2\pi iz}q^n} - \frac{e^{-2\pi iz}q^n}{1-e^{-2\pi iz}q^n}\right]. \tag{34}$$

We consider the holomorphic function of τ,

$$Z^*_{\Lambda_\tau}\left(\frac{a\tau+b}{N}\right) = Z_{\Lambda_\tau}\left(\frac{a\tau+b}{N}\right) - \frac{a\eta_1(\Lambda_\tau)+b\eta_2(\Lambda_\tau)}{N} \tag{35}$$

which we call an Eisenstein series of weight 1.

3.1. $E_{a,b,N}$ When $a \not\equiv 0 \pmod N$

If we have $a \not\equiv 0(N)$ we can expand the denominators, so we take $0 < a < N$ and, using the fact that $e^{2\pi iz} = e^{2\pi i\left(\frac{a\tau+b}{N}\right)} = e^{2\pi i\frac{a\tau}{N}}e^{2\pi i\frac{b}{N}} = q^{\frac{a}{N}}\zeta_N^b$ (with $\zeta_N := e^{\frac{2\pi i}{N}}$ as always), simplify using the Legendre relation:

$$\begin{aligned} Z^*_{\Lambda_\tau}\left(\frac{a\tau+b}{N}\right) &= \eta_2(\Lambda_\tau)\left(\frac{a\tau+b}{N}\right) - \pi i\frac{1+q^{\frac{a}{N}}\zeta_N^b}{1-q^{\frac{a}{N}}\zeta_N^b} - \frac{a\eta_1(\Lambda_\tau)+b\eta_2(\Lambda_\tau)}{N} \\ &\quad - 2\pi i\sum_{n=1}^{\infty}\left[\frac{q^{\frac{a}{N}}\zeta_N^b q^n}{1-q^{\frac{a}{N}}\zeta_N^b q^n} - \frac{q^{-\frac{a}{N}}\zeta_N^{-b}q^n}{1-q^{-\frac{a}{N}}\zeta_N^{-b}q^n}\right] \\ &= \frac{a}{N}\left(\tau\eta_2(\Lambda_\tau)-\eta_1(\Lambda_\tau)\right) - \pi i\left(1+2\sum_{m=1}^{\infty}q^{\frac{am}{N}}\zeta_N^{bm}\right) \\ &\quad - 2\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[q^{\frac{m(a+nN)}{N}}\zeta_N^{bm} - q^{\frac{m(-a+nN)}{N}}\zeta_N^{-bm}\right] \\ &= \frac{2\pi ia}{N} - \pi i - 2\pi i\sum_{m=1}^{\infty}q^{\frac{am}{N}}\zeta_N^{bm} \\ &\quad - 2\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[q^{m(\frac{a}{N}+n)}\zeta_N^{bm} - q^{\frac{m(-a+nN)}{N}}\zeta_N^{-bm}\right] \end{aligned}$$

$$= 2\pi i\left(\frac{a}{N}-\frac{1}{2}-\sum_{m=1}^{\infty}\left[\sum_{n=0}^{\infty}q^{\frac{m}{N}(a+nN)}\zeta_N^{bm}-\sum_{n=1}^{\infty}q^{\frac{m}{N}(-a+nN)}\zeta_N^{-bm}\right]\right)$$

$$= 2\pi i\left(\frac{a}{N}-\frac{1}{2}-\sum_{r=1}^{\infty}q^{\frac{r}{N}}\left[\sum_{\substack{s\mid r\\ s\equiv a\ (N)\\ s>0}}\zeta_N^{\frac{br}{s}}-\sum_{\substack{s\mid r\\ s\equiv -a\ (N)\\ s>0}}\zeta_N^{-\frac{br}{s}}\right]\right)$$

$$:= E_{a,b,N}(\tau). \tag{36}$$

Hence we have replaced $Z^*_{\Lambda_\tau}$ with $E_{a,b,N}$, to make our choices of a, b, N explicit. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then by our earlier result involving the transformation of $f_{r,s}(\tau)$ (Equation (33)),

$$E_{\alpha,\beta,N}|_M = E_{a\alpha+c\beta,b\alpha+d\beta,N}. \tag{37}$$

The first two indices can be reduced modulo N, hence we see that $E_{a,b,N}$ remains invariant under transformation by $\Gamma(N)$.

3.2. $E_{a,b,N}$ When $a \equiv 0 \pmod N$

For the case $a \equiv 0 \pmod N$ we start with

$$Z^*_{\Lambda_\tau}\left(\frac{a\tau+b}{N}\right) = \eta_2(\Lambda)\left(\frac{a\tau+b}{N}\right)-\pi i\frac{1+q^{\frac{a}{N}}\zeta_N^b}{1-q^{\frac{a}{N}}\zeta_N^b}$$
$$-2\pi i\sum_{n=1}^{\infty}\left[\frac{q^{\frac{a}{N}}\zeta_N^b q^n}{1-q^{\frac{a}{N}}\zeta_N^b q^n}-\frac{q^{-\frac{a}{N}}\zeta_N^{-b}q^n}{1-q^{-\frac{a}{N}}\zeta_N^{-b}q^n}\right]-\frac{a\eta_1(\Lambda)+b\eta_2(\Lambda)}{N}, \tag{38}$$

with $a = tN$, with $t \in \mathbb{Z}$. We consider $t = 0$, i.e., $a = 0$, and notice that the result will hold for any $a \equiv 0 \pmod N$ due to periodicity:

$$Z^*_{\Lambda_\tau}\left(\frac{b}{N}\right) = \eta_2(\Lambda)\left(\frac{b-b}{N}\right)-\pi i\frac{1+\zeta_N^b}{1-\zeta_N^b}-2\pi i\sum_{n=1}^{\infty}\left[\frac{\zeta_N^b q^n}{1-\zeta_N^b q^n}-\frac{\zeta_N^{-b}q^n}{1-\zeta_N^{-b}q^n}\right]$$
$$= -\pi i\frac{1+\zeta_N^b}{1-\zeta_N^b}-2\pi i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\zeta_N^{mb}q^{mn}-\zeta_N^{-mb}q^{mn}\right]$$
$$= 2\pi i\left(-\frac{1}{2}\cdot\frac{1+\zeta_N^b}{1-\zeta_N^b}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q^{nm}\left[\zeta_N^{mb}-\zeta_N^{-mb}\right]\right)$$

$$= 2\pi i\Bigg(\frac{1}{2} + \frac{1}{\zeta_N^b - 1} - \sum_{t=1}^{\infty} \sum_{\substack{s|t \\ s>0}} q^t \left[\zeta_N^{\frac{bt}{s}} - \zeta_N^{-\frac{bt}{s}}\right]\Bigg)$$

$$= 2\pi i\Bigg(\frac{1}{2} + \frac{1}{\zeta_N^b - 1} - \sum_{r=1}^{\infty} \sum_{\substack{s|r \\ s\equiv 0\ (N) \\ s>0}} q^{\frac{r}{N}} \left[\zeta_N^{\frac{br}{s}} - \zeta_N^{-\frac{br}{s}}\right]\Bigg)$$

$$:= E_{0,b,N}(\tau). \tag{39}$$

3.3. Defining $F_{a,c,N}$ for $a, c \not\equiv 0 \pmod N$

Now we consider the following linear combination, for $0 < a < N$:

$$\begin{aligned}
\frac{1}{2\pi i}F_{a,c,N}(\tau) := {} & \frac{1}{2\pi i}\sum_{b=0}^{N-1} \zeta_N^{-bc} E_{a,b,N}(\tau) \\
= {} & \sum_{b=0}^{N-1} \zeta_N^{-bc}\left(\frac{a}{N} - \frac{1}{2}\right) \\
& - \sum_{b=0}^{N-1} \zeta_N^{-bc} \sum_{r=1}^{\infty} \Bigg[\sum_{\substack{s|r \\ s\equiv a\ (N) \\ s>0}} q^{\frac{r}{N}} \zeta_N^{\frac{br}{s}} - \sum_{\substack{s|r \\ s\equiv -a\ (N) \\ s>0}} q^{\frac{r}{N}} \zeta_N^{\frac{-br}{s}} \Bigg] \\
= {} & \left(\frac{a}{N} - \frac{1}{2}\right) \sum_{b=0}^{N-1} \zeta_N^{-bc} \\
& - \sum_{r=1}^{\infty} \Bigg[\sum_{\substack{s|r \\ s\equiv a\ (N) \\ s>0}} q^{\frac{r}{N}} \sum_{b=0}^{N-1} \zeta_N^{b\left(\frac{r}{s} - c\right)} - \sum_{\substack{s|r \\ s\equiv -a\ (N) \\ s>0}} q^{\frac{r}{N}} \sum_{b=0}^{N-1} \zeta_N^{b\left(-\frac{r}{s} - c\right)} \Bigg] \\
= {} & \left(\frac{a}{N} - \frac{1}{2}\right) \sum_{b=0}^{N-1} \zeta_N^{-bc} - \sum_{r=1}^{\infty} \Bigg[\sum_{\substack{s|r \\ s\equiv a\ (N) \\ \frac{r}{s}\equiv c\ (N) \\ s>0}} q^{\frac{r}{N}} N - \sum_{\substack{s|r \\ s\equiv -a\ (N) \\ \frac{r}{s}\equiv -c\ (N) \\ s>0}} q^{\frac{r}{N}} N \Bigg] \\
= {} & \left(\frac{a}{N} - \frac{1}{2}\right) \sum_{b=0}^{N-1} \zeta_N^{-bc} - N\Bigg(\sum_{\substack{s\equiv a\ (N) \\ t\equiv c\ (N) \\ s,t>0}} q^{\frac{st}{N}} - \sum_{\substack{s\equiv -a\ (N) \\ t\equiv -c\ (N) \\ s,t>0}} q^{\frac{st}{N}} \Bigg).
\end{aligned} \tag{40}$$

So if we pick c such that $N \nmid c$, the sum on the left disappears, and we have

$$F_{a,c,N}(\tau) = -2\pi i N\left(\sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}} - \sum_{\substack{s\equiv -a\ (N)\\ t\equiv -c\ (N)\\ s,t>0}} q^{\frac{st}{N}}\right). \tag{41}$$

For simplicity we define

$$F^*_{a,c,N}(\tau) := \sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}} - \sum_{\substack{s\equiv -a\ (N)\\ t\equiv -c\ (N)\\ s,t>0}} q^{\frac{st}{N}} = -\frac{1}{2\pi i N}F_{a,c,N}(\tau). \tag{42}$$

We write

$$K_{a,c,N} := \sum_{\substack{s\equiv a\ (N)\\ t\equiv c\ (N)\\ s,t>0}} q^{\frac{st}{N}} \tag{43}$$

so that $F^*_{a,c,N} = K_{a,c,N} - K_{-a,-c,N}$. Suppose $\gcd(a,N) = d > 1$. If we take the sum over $c \equiv\ r \pmod{\frac{N}{d}}$ of the $F^*_{a,c,N}$ for some r, we observe

$$\begin{aligned}\sum_{\substack{0<c<N\\ c\equiv r\ \left(\frac{N}{d}\right)}} F^*_{a,c,N} &= \sum_{\substack{0<c<N\\ c\equiv r\ \left(\frac{N}{d}\right)}} K_{a,c,N} - \sum_{\substack{0<c<N\\ c\equiv -r\ \left(\frac{N}{d}\right)}} K_{-a,c,N}\\ &= K_{\frac{a}{d},r,\frac{N}{d}} - K_{-\frac{a}{d},-r,\frac{N}{d}}\\ &= F^*_{\frac{a}{d},r,\frac{N}{d}}. \end{aligned}\tag{44}$$

Of course, if $\frac{a}{d} \equiv -r \pmod{\frac{N}{d}}$ then this is 0.

4. The Identities

In this section, we use what we have covered in the previous sections to rearrange the identities conjectured by Melham to find equivalent identities which can then be easily proven. We continue to use the labeling from Melham's paper to refer to each of the identities.

4.1. Triangular Numbers

4.1.1. Identity (6)

As mentioned before, Melham's Identity (6) is

$$G_3(q)G_3(q^5) = \sum_{n=0}^{\infty}\left[\frac{q^{3n}+q^{7n+1}}{1-q^{20n+5}} - \frac{q^{13n+9}+q^{17n+12}}{1-q^{20n+15}}\right].$$

We expand the denominators in the RHS to give

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[q^{3n+(20n+5)m}+q^{7n+1+(20n+5)m}-q^{13n+9+(20n+15)m}-q^{17n+12+(20n+15)m}\right]\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[q^{\frac{(20n+5)(20m+3)}{20}-\frac{3}{4}}+q^{\frac{(20n+5)(20m+7)}{20}-\frac{3}{4}}\right.\\
&\qquad\qquad\qquad\qquad\left.-q^{\frac{(20n+15)(20m+13)}{20}-\frac{3}{4}}-q^{\frac{(20n+15)(20m+17)}{20}-\frac{3}{4}}\right]\\
&=q^{-\frac{3}{4}}\left(\sum_{\substack{a,b>0\\ a\equiv 5\ (20)\\ b\equiv 3\ (20)}}q^{\frac{ab}{20}}-\sum_{\substack{a,b>0\\ a\equiv -5\ (20)\\ b\equiv -3\ (20)}}q^{\frac{ab}{20}}+\sum_{\substack{a,b>0\\ a\equiv 5\ (20)\\ b\equiv 7\ (20)}}q^{\frac{ab}{20}}-\sum_{\substack{a,b>0\\ a\equiv -5\ (20)\\ b\equiv -7\ (20)}}q^{\frac{ab}{20}}\right)\\
&=q^{-\frac{3}{4}}\left(F^*_{5,3,20}+F^*_{5,7,20}\right). \qquad (45)
\end{aligned}$$

There exists a similar result for all the identities, that is, each identity is equivalent to an identity of the form

$$G_k(q^{\alpha})G_k(q^{\beta})=q^{\frac{\alpha+\beta}{8(2-k)}}\sum_{(a_i,c_i)}F^*_{a_i,c_i,4(k-2)}, \qquad (46)$$

with the sum going over a number of pairs (a_i, c_i). We will use these pairs later to calculate coefficients. Continuing with Identity (6), we recall by our earlier result that

$$\sum_{\substack{0<c<20\\ c\equiv 3\ (4)}}F^*_{5,c,20}=F^*_{1,3,4}=K_{1,3,4}-K_{3,1,4}=0. \qquad (47)$$

Hence

$$F^*_{5,3,20}+F^*_{5,7,20}=-F^*_{5,11,20}-F^*_{5,19,20} \qquad (48)$$

as

$$F^*_{5,15,20}=0. \qquad (49)$$

Noting that $1, 4$ are quadratic residues modulo 5, where as $2, 3$ are not, we see the RHS is equal to, where $\left(\frac{c}{p}\right)$ denotes the Legendre symbol,

$$-\frac{q^{-\frac{3}{4}}}{2}\sum_{\substack{0<c<20\\ c\equiv 3\ (4)}}\left(\frac{c}{5}\right)F^*_{5,c,20}, \qquad (50)$$

or equivalently,

$$\frac{q^{-\frac{3}{4}}}{2}\sum_{\substack{0<c<20\\ c\equiv 1\ (4)}}\left(\frac{c}{5}\right)F^*_{15,c,20}=-\frac{q^{-\frac{3}{4}}}{80\pi i}\sum_{b=0}^{19}E_{15,b,20}\sum_{\substack{0<c<20\\ c\equiv 1\ (4)}}\left(\frac{c}{5}\right)\zeta_{20}^{-bc}. \qquad (51)$$

Here we have used the fact that

$$F^*_{a,c,N} = -\frac{1}{2\pi i N}\sum_{b=0}^{N-1}\zeta_N^{-bc}E_{a,b,N}. \tag{52}$$

We will encounter sums of the form

$$\sum_{\substack{0<c<N\\ c\equiv\alpha\ (\beta)}}\left(\frac{c}{p}\right)\zeta_N^{-bc}$$

frequently. To deal with these we use a small lemma.

Lemma 2. *Let $\alpha,\beta,p\in\mathbb{N}$, $\alpha<\beta$, p an odd prime, $(\beta,p)=1$. Define $N=\beta p$, and let $\gamma\equiv\beta^{-1}$ modulo p. We have*

$$\sum_{\substack{0<c<N\\ c\equiv\alpha\ (\beta)}}\left(\frac{c}{p}\right)\zeta_N^{-bc} = \begin{cases}\zeta_N^{\alpha b(\gamma\beta-1)}\left(\frac{b\gamma}{p}\right)\sqrt{p} & p\equiv 1\ (mod\ 4)\\ -\zeta_N^{\alpha b(\gamma\beta-1)}\left(\frac{b\gamma}{p}\right)i\sqrt{p} & p\equiv 3\ (mod\ 4).\end{cases} \tag{53}$$

Proof. We define S as follows:

$$\begin{aligned} S &:= \sum_{\substack{0<c<N\\ c\equiv\alpha\ (\beta)}}\left(\frac{c}{p}\right)\zeta_N^{-bc}\\ &= \sum_{k=0}^{\left\lfloor\frac{N-\alpha}{\beta}\right\rfloor}\left(\frac{\alpha+k\beta}{p}\right)\zeta_N^{-b(\alpha+k\beta)}\\ &= \zeta_N^{-b\alpha}\sum_{k=0}^{\left\lfloor\frac{N-\alpha}{\beta}\right\rfloor}\left(\frac{\alpha+k\beta}{p}\right)\zeta_p^{-bk}. \end{aligned} \tag{54}$$

As $\alpha<\beta$ we have $\left\lfloor\frac{N-\alpha}{\beta}\right\rfloor = p-1$. Letting $n=\alpha+k\beta$ (which is of course a bijection modulo p), we therefore have $k=(n-\alpha)\gamma$ modulo p, so

$$\begin{aligned} S &= \zeta_N^{-\alpha b}\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_p^{-b(n-\alpha)\gamma}\\ &= \zeta_N^{-\alpha b}\zeta_p^{b\alpha\gamma}\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_N^{-b\gamma n}\\ &= \zeta_N^{\alpha b(\gamma\beta-1)}\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_N^{-b\gamma n}. \end{aligned} \tag{55}$$

This is a Gauss sum. Gauss proved [6] that

$$\sum_{n=0}^{p-1}\left(\frac{n}{p}\right)\zeta_p^{-bn}=\begin{cases}\left(\frac{b}{p}\right)\sqrt{p} & p\equiv 1 \pmod 4\\ -\left(\frac{b}{p}\right)i\sqrt{p} & p\equiv 3 \pmod 4\end{cases}\tag{56}$$

for p an odd prime. We can therefore conclude with the statement of the lemma. $\square$

Now, returning to the RHS of Identity (6), we have

$$\sum_{\substack{0<c<20\\ c\equiv 1\ (4)}}\left(\frac{c}{5}\right)\zeta_{20}^{-bc}=\left(\frac{b}{5}\right)\sqrt{5}.\tag{57}$$

Hence we see the RHS is

$$-\frac{\sqrt{5}}{80\pi i}q^{-\frac{3}{4}}\sum_{b=0}^{19}\left(\frac{b}{5}\right)\zeta_{20}^{15b}E_{15,b,20}(\tau).\tag{58}$$

From before, we have

$$\frac{1}{4}q^{-\frac{\alpha+\beta}{8}}\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)=G_3(q^{\alpha})G_3(q^{\beta})\tag{59}$$

with $\Gamma=\left\langle\begin{pmatrix}\sqrt{2\alpha}\\0\end{pmatrix},\begin{pmatrix}0\\\sqrt{2\beta}\end{pmatrix}\right\rangle$, and $\rho=\begin{pmatrix}\sqrt{\frac{\alpha}{2}}\\\sqrt{\frac{\beta}{2}}\end{pmatrix}\in\Gamma^{*}$. Hence Identity (6) is equivalent to

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)=-\frac{\sqrt{5}}{20\pi i}\sum_{b=0}^{19}\left(\frac{b}{5}\right)\zeta_{20}^{15b}E_{15,b,20}(\tau).\tag{60}$$

Raising both sides to the power of 2, as $\alpha+\beta=1+5=6$ is even, we see the LHS is a modular form for the congruence subgroup $\Gamma_0(10)\cap\Gamma^0(2)$, as detailed in Section 2.1. Thus we aim to show that the RHS (ignoring the constant term),

$$H(\tau):=\left(\sum_{b=0}^{19}\left(\frac{b}{5}\right)\zeta_{20}^{15b}E_{15,b,20}(\tau)\right)^{2},\tag{61}$$

is also a modular form for this subgroup. Before we do this, we must transform the other identities into a similar form. Our method above works perfectly for the majority of the identities, but there are exceptions to this. The first such identity is Identity (13), where we must use a slight variation. We detail this method next.

Table 1: Triangular and Pentagonal Identities

Id.	$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$	$\boldsymbol{N}$	$\boldsymbol{p}$	$\binom{\boldsymbol{k_1}}{\boldsymbol{a_1}}$	$\binom{\boldsymbol{k_2}}{\boldsymbol{a_2}}$	$\binom{\boldsymbol{k_3}}{\boldsymbol{a_3}}$	$\binom{\boldsymbol{k_4}}{\boldsymbol{a_4}}$	$2v$
6, T	$\binom{1}{5}$	20	5	$\binom{15}{15}$				2
11, T	$\binom{1}{13}$	52	13	$\binom{39}{39}$				2
14, T	$\binom{1}{37}$	148	37	$\binom{111}{148}$				2
7, T	$\binom{1}{6}$	24	3	$\binom{9}{3}$	$\binom{21}{15}$			4
8, T	$\binom{2}{3}$			$\binom{21}{9}$	$\binom{9}{21}$			4
9, T	$\binom{1}{10}$	40	5	$\binom{5}{5}$	$\binom{25}{25}$			4
10, T	$\binom{2}{5}$			$\binom{5}{25}$	$\binom{25}{5}$			4
12, T	$\binom{1}{22}$	88	11	$\binom{33}{11}$	$\binom{77}{55}$			4
13*, T	$\binom{2}{11}$			$\binom{33}{33}$	$\binom{77}{77}$			4
15*, T	$\binom{1}{58}$	232	29	$\binom{29}{29}$	$\binom{145}{145}$			4
16, T	$\binom{2}{29}$			$\binom{145}{29}$	$\binom{29}{145}$			4
17, P	$\binom{1}{5}$	60	5	$\binom{35}{15}$	$\binom{5}{45}$			2
20, P	$\binom{1}{13}$	156	13	$\binom{143}{91}$	$\binom{65}{13}$			6
23*, P	$\binom{1}{37}$	444	37	$\binom{407}{259}$	$\binom{185}{37}$			6
18*, P	$\binom{1}{10}$	120	5	$\binom{105}{25}$	$\binom{45}{85}$	$\binom{105}{65}$	$\binom{45}{5}$	12
19*, P	$\binom{2}{5}$			$\binom{105}{5}$	$\binom{45}{65}$	$\binom{105}{85}$	$\binom{45}{25}$	12
21, P	$\binom{1}{22}$	264	11	$\binom{121}{11}$	$\binom{253}{143}$	$\binom{209}{187}$	$\binom{187}{209}$	12
22, P	$\binom{2}{11}$			$\binom{121}{121}$	$\binom{187}{187}$	$\binom{209}{209}$	$\binom{253}{253}$	12
24*, P	$\binom{1}{58}$	696	29	$\binom{493}{29}$	$\binom{145}{377}$	$\binom{319}{551}$	$\binom{667}{203}$	12
25*, P	$\binom{2}{29}$			$\binom{319}{667}$	$\binom{667}{319}$	$\binom{493}{145}$	$\binom{145}{493}$	12

Table 2: Heptagonal Identity

Id.	$\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$	$\boldsymbol{N}$	$\boldsymbol{p}$	$\binom{\boldsymbol{k_1}}{\boldsymbol{a_1}}$	$\binom{\boldsymbol{k_2}}{\boldsymbol{a_2}}$	$\binom{\boldsymbol{k_3}}{\boldsymbol{a_3}}$	$\binom{\boldsymbol{k_4}}{\boldsymbol{a_4}}$	$\binom{\boldsymbol{k_5}}{\boldsymbol{a_5}}$	$\binom{\boldsymbol{k_6}}{\boldsymbol{a_6}}$	$\binom{\boldsymbol{k_7}}{\boldsymbol{a_7}}$	$\binom{\boldsymbol{k_8}}{\boldsymbol{a_8}}$	$2v$
26, H	$\binom{1}{6}$	120	3	$\binom{111}{21}$	$\binom{51}{81}$	$\binom{39}{69}$	$\binom{99}{9}$	$\binom{63}{117}$	$\binom{3}{57}$	$\binom{87}{93}$	$\binom{27}{33}$	20

4.1.2. Identity (13)

This identity is slightly resistant to our previous method, and requires a little more effort. Recalling that $K_{a,c,N} - K_{-a,-c,N} = -\frac{1}{2\pi i N} F_{a,c,N} = F^*_{a,c,N}$, we expand the denominators, and find the identity is equivalent to

$$\begin{aligned} G_3(q^2)G_3(q^{11}) = q^{-\frac{13}{8}}\Big(&- K_{11,5,88} - K_{11,37,88} - K_{11,45,88} - K_{11,53,88} - K_{11,69,88} \\ &- K_{77,3,88} - K_{77,11,88} - K_{77,27,88} - K_{77,59,88} - K_{77,67,88} - K_{77,75,88} \\ &+ K_{33,15,88} + K_{33,23,88} + K_{33,31,88} + K_{33,47,88} + K_{33,71,88} \\ &+ K_{55,1,88} + K_{55,9,88} + K_{55,25,88} + K_{55,33,88} + K_{55,49,88} + K_{55,81,88}\Big). \end{aligned} \tag{62}$$

Now, noting that we can write $-K_{-a,-c,N} = F^*_{a,c,N} - K_{a,c,N}$, and also $K_{a,c,N} = F^*_{a,c,N} + K_{-a,-c,N}$, we find this is equivalent to (multiplying by $q^{\frac{13}{8}}$ for simplicity)

$$\begin{aligned} q^{\frac{13}{8}} G_3(q^2)G_3(q^{11}) = &-K_{11,5,88} - K_{11,37,88} - K_{11,45,88} - K_{11,53,88} - K_{11,69,88} \\ &+ (F^*_{11,85,88} - K_{11,85,88}) + (F^*_{11,77,88} - K_{11,77,88}) + (F^*_{11,61,88} - K_{11,61,88}) \\ &+ (F^*_{11,29,88} - K_{11,29,88}) + (F^*_{11,21,88} - K_{11,21,88}) + (F^*_{11,13,88} - K_{11,13,88}) \\ &+ (F^*_{33,15,88} + K_{55,73,88}) + (F^*_{33,23,88} + K_{55,65,88}) + (F^*_{33,31,88} + K_{55,57,88}) \\ &+ (F^*_{33,47,88} + K_{55,41,88}) + (F^*_{33,71,88} + K_{55,17,88}) \\ &+ K_{55,1,88} + K_{55,9,88} + K_{55,25,88} + K_{55,33,88} + K_{55,49,88} + K_{55,81,88} \\ = &F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} \\ &+ F^*_{33,15,88} + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88} \\ &- K_{11,5,88} - K_{11,13,88} - K_{11,21,88} - K_{11,29,88} - K_{11,37,88} - K_{11,45,88} - K_{11,53,88} \\ &- K_{11,61,88} - K_{11,69,88} - K_{11,77,88} - K_{11,85,88} \\ &+ K_{55,1,88} + K_{55,9,88} + K_{55,17,88} + K_{55,25,88} + K_{55,33,88} + K_{55,41,88} + K_{55,49,88} \\ &+ K_{55,57,88} + K_{55,65,88} + K_{55,73,88} + K_{55,81,88} \\ = &F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} + F^*_{33,15,88} \\ &+ F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88} - \sum_{r=0}^{10} K_{11,5+8r,88} + \sum_{r=0}^{10} K_{55,1+8r,88} \\ = &F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} + F^*_{33,15,88} \\ &+ F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88} - K_{1,5,8} + K_{5,1,8} \\ = &F^*_{11,85,88} + F^*_{11,77,88} + F^*_{11,61,88} + F^*_{11,29,88} + F^*_{11,21,88} + F^*_{11,13,88} + F^*_{33,15,88} \\ &+ F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88}. \end{aligned} \tag{63}$$

Hence Identity (13) is equivalent to (note this is the form we mentioned during Identity (6), giving us the pairs (a_i, c_i), required for the calculation of coefficients later)

$$\begin{aligned}G_3(q^2)G_3(q^{11}) =& q^{-\frac{13}{8}}\Big(F^*_{11,13,88} + F^*_{11,21,88} + F^*_{11,29,88} + F^*_{11,61,88} + F^*_{11,85,88} \\ &+ F^*_{33,15,88} + F^*_{33,23,88} + F^*_{33,31,88} + F^*_{33,47,88} + F^*_{33,71,88}\Big) \\ =& \frac{1}{2}q^{-\frac{13}{8}}\left(\sum_{\substack{0<c<88 \\ c\equiv 7\ (8)}} \left(\frac{c}{11}\right) F^*_{33,c,88} - \sum_{\substack{0<c<88 \\ c\equiv 3\ (8)}} \left(\frac{c}{11}\right) F^*_{77,c,88}\right)\end{aligned} \tag{64}$$

as -1 is not a quadratic residue modulo 11. This is therefore the same as

$$G_3(q^2)G_3(q^{11}) = \frac{\sqrt{11}}{352\pi} q^{-\frac{13}{8}} \sum_{b=0}^{87} \left(\frac{b}{11}\right) \left(\zeta_{88}^{33b} E_{33,b,88} - \zeta_{88}^{77b} E_{77,b,88}\right), \tag{65}$$

which becomes

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right) = \frac{\sqrt{11}}{88\pi} \sum_{b=0}^{87} \left(\frac{b}{11}\right) \left(\zeta_{88}^{33b} E_{33,b,88} - \zeta_{88}^{77b} E_{77,b,88}\right). \tag{66}$$

Hence

$$H(\tau) := \left(\sum_{b=0}^{87} \left(\frac{b}{11}\right) \left(\zeta_{88}^{33b} E_{33,b,88} - \zeta_{88}^{77b} E_{77,b,88}\right)\right)^4. \tag{67}$$

4.2. Table of Identities

We find that all identities are equivalent to an identity of the form

$$\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right) = \frac{\sqrt{p}}{N\pi Q} \sum_{b=0}^{N-1} \left(\frac{b}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i b} E_{a_i,b,N}(\tau) \tag{68}$$

with $Q = 1$ if $p \equiv 1 \pmod 4$, and $Q = i$ if $p \equiv 3 \pmod 4$. The second sum runs over the pairs (k_i, a_i), $i = 1, 2, \ldots$. For our triangular identities, there are either 1 or 2 pairs, for our pentagonal, 2 or 4 pairs. For the single heptagonal identity, there are 8 pairs. Therefore each of Melham's identities (once transformed) is defined simply by α, β (recall from Sections 2.1 - 2.3 that N is defined by α, β), p, the pairs (k_i, a_i), and whether the identity is triangular, pentagonal, or heptagonal. We display this information in Table 1 and Table 2, with the understanding that these equivalent identities can be obtained by simply applying the method used for Identity (6) or, in some cases, Identity (13) (which we mark with an asterisk) to Melham's original identities in his paper.

Each identity has a corresponding $H(\tau)$, obtained by ignoring the constant term and raising to a certain even power, of the form

$$H(\tau) = \left(\sum_{b=0}^{N-1} \left(\frac{b}{p} \right) \sum_{(k_i, a_i)} (-1)^{i+1} \zeta_N^{k_i b} E_{a_i,b,N}(\tau) \right)^{2v} \tag{69}$$

for some integer v, as discussed in Sections 2.1 - 2.3. This is what we wish to show is a modular form for the relevant subgroup.

5. Identities Under Transformation by the Congruence Subgroup

For all our identities, we have already shown that the LHS (as a theta function) raised to some even power is modular for Γ', where

$$\Gamma' = \Gamma_0 \left(\frac{N}{2} \right) \cap \Gamma^0(2), \tag{70}$$

further intersected with $\Gamma_1(10)$ for the heptagonal case. As we have also seen, the RHS of each identity is an Eisenstein series, and is therefore modular with respect to $\Gamma(N)$, where thankfully the N of the LHS is equal to the N of the RHS. In order to reduce the amount of coefficients needed to be calculated, we would like to show that the RHS (raised to the same even power as the LHS) is also modular for Γ'. As the RHS is already modular for $\Gamma(N)$, we only need to consider elements in $\Gamma'' = \Gamma'/\Gamma(N)$. A set of generators for Γ'' is

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \frac{N}{2} & 1 \end{pmatrix},$$

$$C_0 = -I, \quad C_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix}, \quad C_2 = \begin{pmatrix} c_2 & 0 \\ 0 & c_2^{-1} \end{pmatrix}, \quad \ldots$$

where the C_j continue to include all c_j necessary to generate $(\mathbb{Z}/N\mathbb{Z})^*$ (noticing we have already included -1).

5.1. Invariance Under Transformation by Matrix A

We begin this section with a lemma we will require.

Lemma 3. *Let p be an odd prime, $p \mid N$. Let* $\gcd(d, N) = g$*, and* $kg = d$*,* $Mg = N$*,* *all integers. Then*

$$\sum_{b=0}^{N-1}\left(\frac{b}{p}\right)\zeta_N^{-db}=\sum_{b=0}^{N-1}\left(\frac{b}{p}\right)\zeta_M^{-kb}=\begin{cases}0 & p\neq M\\ \left(\frac{k}{p}\right)\frac{N}{p}\sqrt{p} & p=M\equiv 1\ (mod\ 4)\\ -\left(\frac{k}{p}\right)\frac{iN}{p}\sqrt{p} & p=M\equiv 3\ (mod\ 4).\end{cases} \tag{71}$$

Proof. Define

$$\begin{aligned}S:&=\sum_{b=0}^{p-1}\left(\frac{b}{p}\right)\sum_{r=0}^{\frac{N}{p}-1}\zeta_M^{-k(b+pr)}\\ &=\sum_{b=0}^{p-1}\left(\frac{b}{p}\right)\zeta_M^{-kb}\sum_{r=0}^{\frac{N}{p}-1}\zeta_M^{-kpr}\\ &=\sum_{b=0}^{p-1}\left(\frac{b}{p}\right)\zeta_M^{-kb}\sum_{r=0}^{\frac{N}{p}-1}\zeta_{\frac{N}{p}}^{-kgr}.\end{aligned} \tag{72}$$

Now,

$$\sum_{r=0}^{\frac{N}{p}-1}\zeta_{\frac{N}{p}}^{-kgr}=\begin{cases}0 & kg\not\equiv 0\ (\mathrm{mod}\ \frac{N}{p})\\ \frac{N}{p} & kg\equiv 0\ (\mathrm{mod}\ \frac{N}{p}).\end{cases} \tag{73}$$

For some $t\in\mathbb{Z}$,

$$kg\equiv 0\ (\mathrm{mod}\ \frac{N}{p}) \iff k\frac{N}{M}=\frac{N}{p}t \iff kp=Mt \iff M\mid p. \tag{74}$$

So $S=0$ when $p\not\equiv 0\ (\mathrm{mod}\ M)$, and as p an odd prime, $p\neq M$. When $p\equiv 0\ (\mathrm{mod}\ M)$, i.e., $p=M$, we have

$$S=\frac{N}{p}\sum_{b=0}^{p-1}\left(\frac{b}{p}\right)\zeta_p^{-kb}=\begin{cases}\left(\frac{k}{p}\right)\frac{N}{p}\sqrt{p} & p\equiv 1\ (\mathrm{mod}\ 4)\\ -\left(\frac{k}{p}\right)\frac{N}{p}i\sqrt{p} & p\equiv 3\ (\mathrm{mod}\ 4),\end{cases} \tag{75}$$

which completes the proof. □

To show that $H(\tau)$ is invariant under the matrix A, we recall that $E_{a,b,N}$ is an Eisenstein series, and make use of the transformation formula in Equation (37). We also write

$$\sum_{j\geq 0}s_j(\alpha,\beta,N)q^{\frac{j}{N}}:=E_{\alpha,\beta,N}(\tau) \tag{76}$$

so that $s_j(\alpha,\beta,N)$ is the $q^{\frac{j}{N}}$ coefficient of $E_{\alpha,\beta,N}(\tau)$. We spot

$$\begin{aligned}
E_{a,b+a,N}(\tau) &= E_{a,b,N}(\tau+1)\\
&= 2\pi i\left(\frac{a}{N}-\frac{1}{2}-\sum_{r=1}^{\infty} q^{\frac{r}{N}}\zeta_N^r\left[\sum_{\substack{s|r\\ s\equiv a\ (N)\\ s>0}}\zeta_N^{\frac{br}{s}}-\sum_{\substack{s|r\\ s\equiv -a\ (N)\\ s>0}}\zeta_N^{-\frac{br}{s}}\right]\right)\\
&= \sum_{j\geq 0} s_j(a,b,N)\zeta_N^j q^{\frac{j}{N}}. \qquad (77)
\end{aligned}$$

Thus

$$\begin{aligned}
H|_A &= \left(\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\sum_{(k_i,a_i)}(-1)^{i+1}\zeta_N^{k_i\beta}E_{a_i,\beta+2a_i,N}\right)^{2v}\\
&= \left(\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\sum_{(k_i,a_i)}(-1)^{i+1}\zeta_N^{k_i\beta}\sum_{j\geq 0} s_j(a_i,\beta,N)\zeta_N^{2j}q^{\frac{j}{N}}\right)^{2v}\\
&= \left(\sum_{j\geq 0}\zeta_N^{2j}q^{\frac{j}{N}}\sum_{(k_i,a_i)}(-1)^{i+1}\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\zeta_N^{k_i\beta}s_j(a_i,\beta,N)\right)^{2v}. \qquad (78)
\end{aligned}$$

By Lemma 3, we have

$$\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\zeta_N^{k_i\beta}s_j(a_i,\beta,N)=0 \qquad (79)$$

unless $s_j(a_i,\beta,N)$ has a $\zeta_N^{m\beta}$ term with $\gamma := (m+k_i,N)=\frac{N}{p}$, hence we need

$$m=-k_i+d_i\gamma \qquad (80)$$

with $d_i \not\equiv 0$ modulo p. For $j>0$,

$$s_j(a_i,\beta,N) = -2\pi i\left[\sum_{\substack{s|j\\ s\equiv a_i\ (N)\\ s>0}}\zeta_N^{\frac{\beta j}{s}}-\sum_{\substack{s|j\\ s\equiv -a_i\ (N)\\ s>0}}\zeta_N^{-\frac{\beta j}{s}}\right]. \qquad (81)$$

Therefore we have $j=\pm(a_i+lN)m$ for some $l\in\mathbb{Z}$, hence $j\equiv \pm a_i m \pmod N$. We note that for all our identities $p \mid a_i$, hence $j\equiv \pm a_i m \equiv \pm a_i(-k_i+d_i\gamma)\equiv \pm a_i k_i \pmod N$.

For triangular identities of the form that have just one pair of (a_i,k_i), we notice that $a_i k_i \equiv \frac{N}{4} \pmod N$. Hence $\zeta_N^{2j}=\pm 1$ for all j of nonvanishing terms. These identities were raised to the power of 2, hence $H|_A = H$.

For triangular identities that have two pairs of (a_i, k_i), we see that $a_i k_i \equiv \frac{N}{8}$ or $\pm\frac{3N}{8} \pmod N$. Hence $\zeta_N^{2j} = \pm i$ for all j of nonvanishing terms. These identities were raised to the power of 4, hence $H|_A = H$.

For pentagonal identities that have two pairs of (a_i, k_i), we see that $a_i k_i \equiv \frac{3N}{4}$ (Identity (17)) or $\frac{5N}{12}$ (Identities (20), (23)) modulo N. Hence for Identity (17) $\zeta_N^{2j} = -1$, and for the others, $\zeta_N^{2j} = \zeta_6^5$ for all j of nonvanishing terms. Identity (17) was raised to the power of 2, and Identities (20) and (23) were raised to the power of 6, hence in both cases $H|_A = H$.

For pentagonal identities that have four pairs of (a_i, k_i), we see that $a_i k_i \equiv \frac{7N}{8}$ or $\frac{3N}{8}$ (Identities (18), (19)) or $\frac{N}{24}$, $\frac{11N}{24}$, $\frac{13N}{24}$, $\frac{17N}{24}$ (Identities (21), (22), (24), (25)) modulo N. Hence for Identities (18), (19) $\zeta_N^{2j} = -i$, and for the others, $\zeta_N^{2j} = \zeta_{12}$, ζ_{12}^{11}, or ζ_{12}^5 for all j of nonvanishing terms. All of these identities were raised to the power of 12, hence in both cases $H|_A = H$.

For the one heptagonal identity we have $a_i k_i \equiv 51 \equiv \frac{17N}{40} \pmod N$ for all 8 pairs. Thus $\zeta_N^{2j} = \zeta_{20}^{17}$, and as the identity is raised to the power of 20, $H|_A = H$.

5.2. Invariance Under Transformation by Matrix B

To show that $H(\tau)$ is invariant under the matrix $B = \begin{pmatrix} 1 & 0 \\ \frac{N}{2} & 1 \end{pmatrix}$ we notice

$$H|_B = H|_{B_1^{-1}}|_{B_2}|_{B_1} \tag{82}$$

with

$$B_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & -\frac{N}{2} \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

because $B = B_1^{-1} B_2 B_1$. If $H|_{B_1}$ remains fixed under transformation by B_2, then H will be fixed by B. Therefore we just need to check that $H|_{B_1}$ is a sum of powers of $q^{\frac{2}{N}}$. We have

$$\begin{aligned} H|_{B_1} &= \left(\sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \zeta_N^{k_i\beta} E_{\beta,-a_i,N} \right)^{2v} \qquad (83) \\ &= \left(2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \left[\frac{\beta}{N} - \frac{1}{2} + \frac{1}{2\pi i} \sum_{r=1}^{\infty} q^{\frac{r}{N}} s_{-r}(\beta, a_i, N) \right] \right)^{2v} \\ &= \left(S + \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{\beta=0}^{N-1} \left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \sum_{r=1}^{\infty} q^{\frac{r}{N}} s_{-r}(\beta, a_i, N) \right)^{2v} \end{aligned}$$

with S some constant. Rearranging the RHS, we get

$$
\begin{aligned}
H|_{B_1} &= \Bigg(S - 2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{r=1}^{\infty} q^{\frac{r}{N}} \Bigg[\sum_{\beta=0}^{N-1} \sum_{\substack{s|r \\ s\equiv\beta\ (N) \\ s>0}} \left[\left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \zeta_N^{-\frac{a_i r}{s}}\right] \\
&\quad - \sum_{\beta=0}^{N-1} \sum_{\substack{s|r \\ s\equiv-\beta\ (N) \\ s>0}} \left[\left(\frac{\beta}{p}\right) \zeta_N^{k_i\beta} \zeta_N^{\frac{a_i r}{s}}\right]\Bigg]\Bigg)^{2v} \qquad (84) \\
&= \Bigg(S - 2\pi i \sum_{(k_i,a_i)} (-1)^{i+1} \sum_{r=1}^{\infty} q^{\frac{r}{N}} \Bigg[\sum_{\substack{s|r \\ s>0}} \left[\left(\frac{s}{p}\right) \zeta_N^{k_i s} \zeta_N^{-\frac{a_i r}{s}}\right] \\
&\quad - \sum_{\substack{s|r \\ s>0}} \left[\left(\frac{-s}{p}\right) \zeta_N^{-k_i s} \zeta_N^{\frac{a_i r}{s}}\right]\Bigg]\Bigg)^{2v} \\
&= \Bigg(S - 2\pi i \sum_{r=1}^{\infty} q^{\frac{r}{N}} \Bigg[\sum_{\substack{s|r \\ s>0}} \left(\frac{s}{p}\right) \sum_{(k_i,a_i)} (-1)^{i+1} \Bigg[\zeta_N^{k_i s - a_i \frac{r}{s}} \\
&\quad - \left(\frac{-1}{p}\right) \zeta_N^{-k_i s + a_i \frac{r}{s}}\Bigg]\Bigg]\Bigg)^{2v}.
\end{aligned}
$$

Define

$$
G := \sum_{(k_i,a_i)} (-1)^{i+1} \left[\zeta_N^{k_i s - a_i \frac{r}{s}} - \left(\frac{-1}{p}\right) \zeta_N^{-k_i s + a_i \frac{r}{s}}\right]. \qquad (85)
$$

We aim to show $G = 0$ whenever r is odd. Suppose first that $p \equiv 1 \pmod 4$, then, as we always have an even number of pairs of (a_i, k_i), using a simple trigonometric identity,

$$
\begin{aligned}
G &= \sum_{(k_i,a_i)} (-1)^{i+1} \left[\zeta_N^{k_i s - a_i \frac{r}{s}} - \zeta_N^{-k_i s + a_i \frac{r}{s}}\right] \\
&= 2i \sum_{(k_i,a_i)} (-1)^{i+1} \sin\left[\frac{2\pi}{N}\left(k_i s - a_i \frac{r}{s}\right)\right] \\
&= 2i \sum_{\text{odd } i} \left(\sin\left[\frac{2\pi}{N}\left(k_i s - a_i \frac{r}{s}\right)\right] - \sin\left[\frac{2\pi}{N}\left(k_{i+1} s - a_{i+1} \frac{r}{s}\right)\right]\right) \\
&= 4i \sum_{\text{odd } i} \cos[\theta_1] \sin[\theta_2], \qquad (86)
\end{aligned}
$$

with $\theta_1 = \frac{\pi}{N}\left(s(k_i + k_{i+1}) - \frac{r}{s}(a_i + a_{i+1})\right)$ and $\theta_2 = \frac{\pi}{N}\left(s(k_i - k_{i+1}) - \frac{r}{s}(a_i - a_{i+1})\right)$. Similarly, if $p \equiv 3 \pmod 4$ we have

$$\begin{aligned} G &= \sum_{(k_i,a_i)} (-1)^{i+1}\left[\zeta_N^{k_i s - a_i \frac{r}{s}} + \zeta_N^{-k_i s + a_i \frac{r}{s}}\right] \\ &= 2\sum_{(k_i,a_i)} (-1)^{i+1}\cos\left[\frac{2\pi}{N}\left(k_i s - a_i \frac{r}{s}\right)\right] \\ &= -4\sum_{\text{odd } i} \sin\left[\theta_1\right]\sin\left[\theta_2\right]. \end{aligned} \tag{87}$$

For identities that have $p \equiv 1 \pmod 4$, we note that for odd i, $k_i - k_{i+1} \equiv a_i - a_{i+1} \equiv \frac{N}{2}$ modulo N. Thus G vanishes when r is odd, as s and $\frac{r}{s}$ must share the same parity.

For identities that have $p \equiv 3 \pmod 4$, we are not as restricted, and we can have either (for odd i) $k_i - k_{i+1} \equiv a_i - a_{i+1} \equiv \frac{N}{2}$ or $k_i + k_{i+1} \equiv a_i + a_{i+1} \equiv \frac{N}{2}$ modulo N. For all of these identities we have this requirement, hence G vanishes for odd r, as required.

5.3. Invariance Under Transformation by Matrices C_j

To show these identities remain fixed under matrices of the form (disregarding the subscript for now) $C = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ modulo N, we apply the transformation formula again:

$$H|_C = \left(\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\sum_{(k_i,a_i)} (-1)^{i+1}\zeta_N^{k_i\beta} E_{ca_i, c^{-1}\beta, N}\right)^{2v}. \tag{88}$$

We let $\beta' = c^{-1}\beta$ and find, as we must have $(c, N) = 1$ and $p \mid N$,

$$\begin{aligned} H|_C &= \left(\sum_{\beta'=0}^{N-1}\left(\frac{c\beta'}{p}\right)\sum_{(k_i,a_i)} (-1)^{i+1}\zeta_N^{k_i c\beta'} E_{ca_i, c^{-1}\beta, N}\right)^{2v} \\ &= \left(\frac{c}{p}\right)^{2v}\left(\sum_{\beta'=0}^{N-1}\left(\frac{\beta'}{p}\right)\sum_{(k_i,a_i)} (-1)^{i+1}\zeta_N^{k_i c\beta'} E_{ca_i, \beta', N}\right)^{2v}. \end{aligned} \tag{89}$$

This will be equal to H if c acts on the pairs (k_i, a_i), by multiplication modulo N, by mapping each to $\pm(k_j, a_j)$, with $j = 1, 2, \ldots$ and so on. To see this we consider the effect of one mapping. We allow for each (k_i, a_i) to be mapped to either itself, its negative, a different (k_j, a_j), or the negative of that. If (k_i, a_i) is mapped to itself or (k_j, a_j), there are no issues. If it is mapped to the negative of one of these we notice, using $E_{-a,-b,N} = -E_{a,b,N}$,

$$\begin{aligned}\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\zeta_N^{-k_i\beta}E_{-a_i,\beta,N} &= -\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\zeta_N^{-k_i\beta}E_{a_i,-\beta,N}\\ &= -\left(\frac{-1}{p}\right)\sum_{\beta=0}^{N-1}\left(\frac{\beta}{p}\right)\zeta_N^{k_i\beta}E_{a_i,\beta,N}. \qquad (90)\end{aligned}$$

If $p \equiv 3 \pmod 4$ then the negatives cancel, and we are done. If however $p \equiv 1 \pmod 4$, we need to have either all the pairs (k_i, a_i) map to negatives, or for each pair that maps to a negative match to a pair (k_j, a_j) where j has a different parity than i, which will be accounted for by the $(-1)^{i+1}$ term.

For example, Identity (7) has RHS

$$H = \left(\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{9b}E_{3,b,24} - \zeta_{24}^{21b}E_{15,b,24}\right)\right)^4. \qquad (91)$$

A set of generators for $(\mathbb{Z}/24\mathbb{Z})^*$ is 5, 7, 23. Of course $23 \equiv -1$ is trivial, so we focus first on 5. We get $5\cdot(k_1, a_1) = 5\cdot(9,3) = (21,15) \equiv (k_2, a_2)$ modulo 24, and $5\cdot(k_2,a_2) = 5\cdot(21,15) \equiv (9,3) \equiv (k_1,a_1)$ modulo 24. Thus H transforms to

$$\left(\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{21b}E_{15,b,24} - \zeta_{24}^{9b}E_{3,b,24}\right)\right)^4 = H. \qquad (92)$$

Similarly, for 7 we find $7\cdot(k_1,a_1) \equiv -(k_1,a_1)$, and $7\cdot(k_2,a_2) \equiv -(k_2,a_2)$ modulo 24. So H transforms to

$$\begin{aligned}&\left(\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{-9b}E_{-3,b,24} - \zeta_{24}^{-21b}E_{-15,b,24}\right)\right)^4\\ =&\left(-\left(\frac{-1}{3}\right)\sum_{b=0}^{23}\left(\frac{b}{3}\right)\left(\zeta_{24}^{9b}E_{3,b,24} - \zeta_{24}^{21b}E_{15,b,24}\right)\right)^4\\ =&H. \qquad (93)\end{aligned}$$

This is shown in Tables 3 and 4 below. Listed is the identity number, the value of N, a list of the c_i required, and the values of the pairs (k_i, a_i). The table then shows how the c_i permute the pairs. All permutations are in fact involutions, and we denote the fact that c_i swaps (k_i, a_i) to (k_j, a_j) as simply (i, j), whereas if c_i takes (k_i, a_i) to $-(k_j, a_j)$ and (k_j, a_j) to $-(k_i, a_i)$, we write $(i, -j)$. If (k_i, a_i) maps to itself, or to $-(k_i, a_i)$, we write (i), or $(-i)$, respectively. Of course, -1 just takes the pair (k_i, a_i) to $-(k_i, a_i)$.

Table 3: Identities with 2 pairs of (k_i, a_i)

Id.	$\boldsymbol{N}$	$\boldsymbol{c_1, c_2, c_3}$	$\boldsymbol{\binom{k_1}{a_1}}$	$\boldsymbol{\binom{k_2}{a_2}}$	$\boldsymbol{c_1}$ **acts**	$\boldsymbol{c_2}$ **acts**
7	24	$5, 7, 23$	$\binom{9}{3}$	$\binom{21}{15}$	$(1, 2)$	$(1, -2)$
8			$\binom{21}{9}$	$\binom{9}{21}$	$(1, 2)$	$(1, -2)$
9	40	$3, 11, 39$	$\binom{5}{5}$	$\binom{25}{25}$	$(1, -2)$	$(1, -2)$
10			$\binom{5}{25}$	$\binom{25}{5}$	$(1, -2)$	$(1, -2)$
12	88	$3, 5, 87$	$\binom{33}{11}$	$\binom{77}{55}$	$(1, -2)$	$(1, 2)$
13			$\binom{33}{33}$	$\binom{77}{77}$	$(1, -2)$	$(1, 2)$
15	232	$3, 5, 231$	$\binom{29}{29}$	$\binom{145}{145}$	$(1, -2)$	$(1, 2)$
16			$\binom{145}{29}$	$\binom{29}{145}$	$(1, -2)$	$(1, 2)$
17	60	7,13,59	$\binom{35}{15}$	$\binom{5}{45}$	$(1, -2)$	$(1), (2)$
20	156	7,11,155	$\binom{143}{91}$	$\binom{65}{13}$	$(1, 2)$	$(-1), (-2)$
23	444	5,7,443	$\binom{407}{259}$	$\binom{185}{37}$	$(1, -2)$	$(1, 2)$

For the three Identities (6), (11), (14), that have just a single pair, it is easy to find a set of generators that, for example, maps the pair to its negative. For the single heptagonal identity, we need to avoid matrices with first entry congruent to ± 3 modulo 10. So, as $N = 120$ we take $c_0 = 119$, $c_1 = 11$, $c_2 = 19$, and $c_3 = 29$. As before, c_0 is immediate. Next, c_1 takes (k_1, a_1) to $-(k_4, a_4)$ and vice versa, (k_2, a_2) to $-(k_3, a_3)$ and vice versa, (k_5, a_5) to $-(k_8, a_8)$ and vice versa, and (k_6, a_6) to $-(k_7, a_7)$ and vice versa. The value 19 swaps the pair with index 1 with the negative of the pair with index 2, 3 with -4, 5 with -6, 7 with -8. Finally, 29 swaps 1 and 4, 2 and 3, 5 and 8, 6 and 7.

6. Equivalence Using Sturm's Bound

Let $f = \sum_{k \in \mathbb{Z}} \alpha_k q^k$. We define $ord(f)$ to be the smallest such n that $\alpha_n \neq 0$. We first state a simplified version of Sturm's bound [11].

Theorem 1. *Define $\Gamma := SL_2(\mathbb{Z})$, and let f, g be modular forms on $\Gamma' \supseteq \Gamma(N)$ of weight k, a positive integer. Suppose $ord(f - g) > \frac{k[\Gamma:\Gamma']}{12}$. Then $f = g$.*

In other words, Sturm's bound says that given two modular forms over the same congruence subgroup and of the same weight, then they are equivalent if their q-expansions agree up to the $q^{\frac{kM}{12}}$ coefficient, where $k = 2v$ is the weight, and M is the index of the congruence subgroup in Γ.

Recall that we have defined for the triangular and pentagonal identities

$$\Gamma' = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2), \tag{94}$$

and for the heptagonal identity,

$$\Gamma' = \Gamma_0\left(\frac{N}{2}\right) \cap \Gamma^0(2) \cap \Gamma_1(10). \tag{95}$$

We have now shown that each side of our transformed identities are modular of weight $2v$ for $\Gamma'' = \Gamma'/\Gamma(N)$. We can now apply Sturm's bound. We have [2, p. 14], for the triangular and pentagonal cases,

$$M = [\Gamma : \Gamma'] = [\Gamma : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right) \tag{96}$$

as, despite the fact that $\Gamma' \neq \Gamma_0(N)$, they have the same index in Γ. To see this, we

Table 4: Identities with 4 pairs of (k_i, a_i)

Id.	N	c_1, c_2, c_3, c_4	$\binom{k_1}{a_1}$	$\binom{k_2}{a_2}$	$\binom{k_3}{a_3}$	$\binom{k_4}{a_4}$	c_1 **acts**	c_2 **acts**	c_3 **acts**
18	120	$7, 11, 19, 119$	$\binom{105}{25}$	$\binom{45}{85}$	$\binom{105}{65}$	$\binom{45}{5}$	$(1,-3),(2,-4)$	$(1,-2),(3,-4)$	$(1,-4),(2,-3)$
19			$\binom{105}{5}$	$\binom{45}{65}$	$\binom{105}{85}$	$\binom{45}{25}$	$(1,-3),(2,-4)$	$(1,-2),(3,-4)$	$(1,-4),(2,-3)$
21	264	$5, 7, 13, 263$	$\binom{121}{11}$	$\binom{253}{143}$	$\binom{209}{187}$	$\binom{187}{209}$	$(1,-4),(2,3)$	$(1,-3),(2,4)$	$(1,2),(3,-4)$
22			$\binom{121}{121}$	$\binom{187}{187}$	$\binom{209}{209}$	$\binom{253}{253}$	$(1,-2),(3,4)$	$(1,-3),(2,4)$	$(1,4),(2,-3)$
24	696	$5, 7, 13, 695$	$\binom{493}{29}$	$\binom{145}{377}$	$\binom{319}{551}$	$\binom{667}{203}$	$(1,-3),(2,-4)$	$(1,4),(2,3)$	$(1,2),(3,4)$
25			$\binom{319}{667}$	$\binom{667}{319}$	$\binom{493}{145}$	$\binom{145}{493}$	$(1,-3),(2,-4)$	$(1,4),(2,3)$	$(1,2),(3,4)$

simply notice that Γ' and $\Gamma_0(N)$ both share a common subgroup, $\Gamma_0(N) \cap \Gamma^0(2)$, and this common subgroup clearly has index 2 in both. For the heptagonal case, we simply have to double M. We need to check $\frac{kM}{12} + 1$ coefficients. Starting with Identity (6), we have $M = 36$, and we need to check $\frac{kM}{12} + 1 = 7$ coefficients. We use the form of

$$q^{\frac{\alpha+\beta}{8(k-2)}} G_k(q^{\alpha}) G_k(q^{\beta}) = \sum_{(a_i, c_i)} F^*_{a_i, c_i, 4(k-2)} \tag{97}$$

for ease, which as we mentioned before each identity can be written as. For Identity (6), this is

$$q^{\frac{3}{4}} G_3(q) G_3(q^5) = F^*_{5,3,20} + F^*_{5,7,20} \tag{98}$$

and find both sides start with

$$1 + 1 \cdot q + 0 \cdot q^2 + 1 \cdot q^3 + 0 \cdot q^4 + 1 \cdot q^5 + 2 \cdot q^6 + 0 \cdot q^7.$$

As the coefficients of this form agree, the coefficients of $H(\tau)$ and $\vartheta_{\rho+\Gamma}\left(\frac{\tau}{2}\right)^{2v}$ must also agree, and we are finished.

Our computations show that both the LHS and RHS have the same coefficients for all triangular, pentagonal, and heptagonal identities. Notice that while we always knew both sides were modular forms for $\Gamma(N)$, using that would have made the number of coefficients required for Sturm's Bound much larger, hence the efforts to show both sides were modular forms for the larger congruence subgroups. As an example, Identity (15) has $N = 232$, and $2v = 4$. With our efforts we only need to check just 121 coefficients, a relatively easy task. If we had used $\Gamma(N)$, we would have utilized the well known formula [2, p. 13],

$$M = [\Gamma : \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \tag{99}$$

and found that we had to check 1,559,041 coefficients, which is somewhat more daunting. Worse still, when we finally got round to checking Identities (24) and (25), with $N = 696$ and $2v = 12$, we would have had to check 224,501,761 coefficients. With what we now know we of course only have to check 1,441.

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