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A NEW APPROACH TO THE FRAENKEL CONJECTURE FOR LOW N VALUES

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Abstract

We present a new approach to deal with Fraenkel's conjecture, which describes how the integers can be partitioned into sets of rational Beatty sequences, in the case where the numerators of the moduli are equal. We use this approach to give a new proof of the known $n = 4$ case when the numerators are equal.

1. Introduction

A set of arithmetic progressions which partitions the integers is called a *disjoint covering system* (DCS). A classic result from the 1950's (see [5]) shows that in any DCS there must be two arithmetic progressions with the same common modulus, that is, any DCS admits *multiplicity.* Since then there has been considerable study of the ways the integers can be partitioned into arithmetic progressions and there have been generalizations of this concept. One of these generalizations is a disjoint covering system of Beatty sequences. A *Beatty sequence* is a sequence $S(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor : n \in \mathbb{Z}\}$, where $\alpha, \beta \in \mathbb{R}$ ($\lfloor \alpha n + \beta \rfloor$ is the integer part of $\alpha n + \beta$). Here, α is called the *modulus* of the sequence $S(\alpha, \beta)$. Similarly to partitioning the integers into arithmetic progressions, a *disjoint covering system of Beatty sequences* is a set of Beatty sequences $\{S(\alpha_i, \beta_i)\}_{i=1}^{k}$ such that every integer belongs to exactly one Beatty sequence. Clearly, if $\alpha_i \in \mathbb{Z}$ for all i then the system $\{S(\alpha_i, \beta_i)\}_{i=1}^{k}$ is a DCS. By density arguments, for a system $\{S(\alpha_i, \beta_i)\}_{i=1}^{n}$,

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} = 1. \tag{1}$$

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The concept of a covering system of Beatty sequences is attributed to Samuel Beatty [3] who proved in 1926 that if x, y are irrational positive numbers such that

$$\frac{1}{x} + \frac{1}{y} = 1, \tag{2}$$

then the sequences $\{ix\}_{i=1}^{\infty}, \{iy\}_{i=1}^{\infty}$ contain one and only one number between each pair of consecutive natural numbers. By (1) any irrational numbers x, y satisfy $\frac{1}{x} + \frac{1}{y} = 1$ if and only if the $S(x, 0), S(y, 0)$ induce a partition on the natural numbers. In fact, with appropriate β_1, β_2, the sequences $S(x, \beta_1), S(y, \beta_2)$ partition the whole set of integers. Apparently John William Strutt (Lord Rayleigh) in his book, "The theory of sound" from 1877 [12] was the first to refer, indirectly, to such systems.

As with DCS, we say that a disjoint covering system of Beatty sequences $\{S(\alpha_i, \beta_i)\}_{i=1}^{k}$ admits *multiplicity* if there exist $1 \leq i < j \leq k$ such that $\alpha_i = \alpha_j$. It is natural to ask if, as is the case with DCS, any such system of Beatty sequences has multiplicity. It follows from Beatty's result that the answer is negative if the integers are partitioned by only two Beatty sequences. However, if $k > 2$, Graham [7] showed that any system $\{S(\alpha_i, \beta_i)\}_{i=1}^{k}$ with irrational moduli must admit multiplicity. Moreover, Graham showed that for a system $\{S(\alpha_i, \beta_i)\}_{i=1}^{k}$, if α_i is irrational for some $1 \leq i \leq k$ then α_j is irrational for any $1 \leq j \leq k$.

We are left with the case where all the moduli are rational, in which case we call the sequences *rational Beatty sequences*. In this case we will always assume that in a fraction p/q, p and q are co-prime. We will call a set of rational Beatty sequences which partition the integers a *disjoint covering system of rational Beatty sequences*, or a DCS of RBS or, when there is no danger of confusion, just a DCS.

It is easy to see that there are ways of partitioning the integers with rational Beatty systems without multiplicity. For example, a DCS with 3 sequences is: $S(7/4, 0), S(7/2, -1), S(7, -3)$. Fraenkel [6] showed that for every positive integer n, the system

$$\{S\left((2^n - 1)/2^{n-i}, -2^{i-1} + 1\right)\}_{i=1}^{n} \tag{3}$$

is a DCS with distinct moduli. Fraenkel conjectured that the systems (3) are essentially the only ones (up to translation) without multiplicity. More precisely, we state

Fraenkel's Conjecture. (Fraenkel, see [11]) If $\{S(\alpha_i, \beta_i)\}_{i=1}^{n}$ form a DCS, with $n > 2$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ then $\alpha_i = p/q_i$, where $p = 2^n - 1, q_i = 2^{n-i}$, for $i = 1, 2, \ldots, n$.

Fraenkel's conjecture has been proved for $n \leq 7$. This was done for $n = 3$ by Morikawa [9], for $n = 4$ by Altman, Gaujal and Hordijk [1], for $n = 5$, and 6 by Tijdeman [14] and for $n = 7$ by Barát and Varjú [2].

In this note we present a new approach to deal with Fraenkel's conjecture which will hopefully be easier to generalize to larger values of n under the assumption that all the numerators of the moduli are equal. This assumption will enable us to associate with each q_i a set B_i containing q_i p-th roots of unity. Moreover, for $i \neq j$ the sets B_i and B_j are disjoint and therefore the sets B_i, $1 \leq i \leq n$ partition the group of all p-th roots of unity. This will allow us to carry out some manipulations which solve the problem for $n = 3, 4$, and which might be adapted to prove, under the equal numerators assumption, the conjecture for larger values of n (see e.g. Lemma 1). We record the case $n = 4$ as

Theorem A. (see [1]) *Let $\{S(p/q_i, \beta_i)\}_{i=1}^4$ be a disjoint covering system of rational Beatty sequences. If the moduli are distinct then $p = 15$ and $\{q_1, q_2, q_3, q_4\} = \{8, 4, 2, 1\}$.*

We mention that a proof for the $n = 3$ case can be easily deduced from the proof of the case $n = 4$.

2. From Beatty Sequences to a Partition of $\{0, 1, \ldots, p-1\}$

We start by introducing a relation between disjoint covering systems of Beatty sequences $\{S(p/q_i, \beta_i)\}_{i=1}^n$ and partitions of the set $\{0, 1, \ldots, p-1\}$ which will be important in the proof of Theorem A.

It is shown in [11] that for disjoint covering system of Beatty sequences $\{S(\frac{p_i}{q_i}, \beta_i)\}_{i=1}^n$, there exists a polynomial $f(z)$ whose coefficients are all equal to 1 such that

$$f(z) + \sum_{i=1}^{n} \frac{z^{b_i}}{1 - z^{p_i}} \cdot \frac{1 - z^{\overline{q}_i q_i}}{1 - z^{\overline{q}_i}} = \frac{1}{1-z}, \tag{4}$$

where $\overline{q}_i$ is the smallest non-negative integer satisfying

$$\overline{q}_i q_i \equiv -1 \pmod{p_i}. \tag{5}$$

However, in our situation, $p_i = p_j = p$ for every $1 \leq i, j \leq n$. Therefore, if ξ is a p-th primitive root of unity, by taking the limit as z goes to ξ in (4) we get

$$\sum_{i=1}^{n} \frac{\xi^{b_i}}{1 - \xi^{\overline{q}_i}} \cdot (1 - \xi^{q_i \overline{q}_i}) = 0. \tag{6}$$

Consequently, by (5)

$$\sum_{i=1}^{n} \frac{\xi^{b_i}}{1 - \xi^{\overline{q}_i}} = 0, \tag{7}$$

and then

$$\sum_{i=1}^{n} \xi^{b_i}(1+\xi^{\overline{q}_i}+\xi^{2\overline{q}_i}+\cdots+\xi^{(q_i-1)\overline{q}_i})=0. \tag{8}$$

Equation (8) induces a disjoint cover of the *p-th* roots of unity with corresponding sets M_i for $1 \leq i \leq n$ where

$$M_i = \{\xi^{b_i+j\overline{q}_i}\}_{j=0}^{q_i-1}. \tag{9}$$

Multiplying the sets by ξ^{-b_1}, we may assume that $M_1 = \{\xi^{j\overline{q}_1}\}_{j=0}^{q_1-1}$.

Let $\gamma : \xi^i \longmapsto \xi^{-q_1 \cdot i}$ be an automorphism of the group of roots of unity $\{1, \xi, \xi^2, \ldots \xi^{p-1}\}$. This is indeed an automorphism since $(q_1, p) = 1$. Let $\widetilde{q_i}$ be the smallest non-negative integer satisfying $\widetilde{q_i} \equiv -q_1 \cdot \overline{q}_i \pmod p$, $(1 \leq i \leq n)$. Then $\widetilde{q_1} = 1$ and since $\{\overline{q}_i\}_{i=1}^{n}$ are distinct then so are $\{\widetilde{q_i}\}_{i=1}^{n}$. We get a new cover of the *p-th* roots of unity with $\widetilde{q_1} = 1$ which in turn induces a partition

$$\{0, 1, \ldots, p-1\} = \dot{\bigcup_i} B_i,$$

where

$$B_i = \left\{\widetilde{b_i} + j\widetilde{q_i} \pmod p\right\}_{j=0}^{q_i-1}. \tag{10}$$

Here, $\widetilde{b_i}$ is the smallest non-negative integer satisfying $\widetilde{b_i} \equiv -q_1 \cdot b_i \pmod p$ for $2 \leq i \leq n$. This implies that $\widetilde{b_1} = 0$ and $\widetilde{q_1} = 1$ so that $B_1 = \{0, 1, \ldots, q_1 - 1\}$.

From now on we will use the correspondence between the disjoint covering system of Beatty sequences $\{S(p/q_i, \beta_i)\}_{i=1}^{n}$ and the sets B_i without saying explicitly what that correspondence is.

3. Beatty Sequences and TG-sequences

In this section we provide some preliminary results concerning disjoint covering system of Beatty sequences and introduce and study TG-sequences. In the next lemma we write $B_1 + \widetilde{q}_2$ for $\{b + \widetilde{q}_2 : b \in B_1\}$.

Lemma 1. *Let $S(p/q_1, 0)$, $S(p/q_2, b_2)$ be disjoint Beatty sequences and let*

$$B_1 = \{0, \ldots, q_1 - 1\} \quad \textit{and } B_2 = \{\widetilde{b}_2, \widetilde{b}_2 + \widetilde{q}_2, \ldots, \widetilde{b}_2 + (q_2 - 1)\widetilde{q}_2\}$$

be the corresponding sets defined in (10). *If*

$$B_1 \cap (B_1 + \widetilde{q}_2) \neq \emptyset \tag{11}$$

then B_2 is an arithmetic progression in $[q_1, p-1]$ with common modulus $\widetilde{q}_2$ or $p - \widetilde{q}_2$.

Proof. Condition (11) implies that either $\widetilde{q}_2 < q_1$ or $\widetilde{q}_2 > p - q_1$. For $i \in \{0, \ldots, q_1 - 2\}$ the minimum of $\widetilde{b}_2 + (i+1)\widetilde{q}_2 - (\widetilde{b}_2 + i\widetilde{q}_2) \mod p$ and $\widetilde{b}_2 + i\widetilde{q}_2 - (\widetilde{b}_2 + (i+1)\widetilde{q}_2) \mod p$ is the minimum of $\widetilde{q}_2 \mod p$ and $p - \widetilde{q}_2 \mod p$, which is less than q_1. So B_1 does not fit in this gap. Therefore we have

$$1 < q_1 < \widetilde{b}_2 < \widetilde{b}_2 + \widetilde{q}_2 < \cdots < \widetilde{b}_2 + (q_2 - 1)\widetilde{q}_2 < p$$

or

$$1 < q_1 < \widetilde{b}_2 + (q_2 - 1)\widetilde{q}_2 < \cdots < \widetilde{b}_2 + \widetilde{q}_2 < \widetilde{b}_2 < p.$$

The result follows. □

The following lemma will be useful in the proof of Theorem A.

Lemma 2. *If $S(p_1/q_1, b_1) \cup S(p_2/q_2, b_2) = S(p_3/q_3, b_3)$ then one of the following holds:*
(a) $\{p_1/q_1, p_2/q_2\}$ is any size 2 subset of $\{7/1, 7/2, 7/4\}$,
(b) $p_1/q_1 = p_2/q_2$,
(c) $\{p_1/q_1, p_2/q_2\} = \{p/q, p/(p-2q)\}$ for some p and q.

Proof. By [10, Lemma 2], the complement of $S(p_3/q_3, b_3)$ is $S(p_3/(p_3 - q_3), b_3 - \overline{q}_3)$. Hence

$$S(p_1/q_1, b_1) \cup S(p_2/q_2, b_2) \cup S(p_3/(p_3 - q_3), b_3 - \overline{q}_3) = \mathbb{Z}. \tag{12}$$

If p_1/q_1, p_2/q_2 and $p_3/(p_3 - q_3)$ are distinct we have the (proven) $n = 3$ case of Fraenkel's conjecture which gives part (a) of the Lemma. Otherwise two of the moduli appearing in (12) are equal. If $p_1/q_1 = p_2/q_2$ we get case (b), and if $p_3/(p_3 - q_3)$ equals one of the other moduli we get case (c). □

The sets B_i in (10) are all particular cases of TG-sequences hereby explained. Let a and d be residues modulo p with $(p, d) = 1$, q a positive integer less than p and consider the set $\{a + id \mod p : i = 0, \ldots, q-1\}$. We sort this set into a sequence $a_1 \le a_2 \le \cdots \le a_q$ and call this sequence a $TG-$*sequence* with q points and modulus p. Call the pairs (a_i, a_{i+1}), $i = 1, \ldots, q-1$ and (a_q, a_1) the *gaps* of the sequence. We say the gap between a_i and a_{i+1} has size $a_{i+1} - a_i$ for $i < q$ and the gap between a_q and a_1 has size $p + a_1 - a_q$.

The Three Gap Theorem. ([15]) *The number of distinct gaps sizes in a TG-sequence is at most 3, and if it equals 3 then the largest gap size equals the sum of the other two. If there is only one gap size then the number of points is $q = 1$.*

Proof. The first part of this theorem is well known in a different setting [15]. Suppose we have only one gap size and it equals c. Then $p = cq$, so c divides p. Without loss of generality we assume that a, in the definition of a TG-sequence, equals 0. Thus

$$\{b + ic : i = 0, \ldots, q-1\} \equiv \{id : i = 0, \ldots, q-1\} \mod p$$

for some integer b. Therefore, if $q > 1$, there exist integers i_1 and i_2 such that $0 \equiv b + i_1 c \mod p$ and $d \equiv b + i_2 c \mod p$. Therefore $d \equiv (i_2 - i_1)c \mod p$. Since $(p, d) = 1$, we have $(p, c) = 1$. But c divides p, so $c = 1$ and $p = q$. This contradicts the definition of a TG-sequence and we conclude $q = 1$. □

The reader will appreciate that TG stands for Three Gap. We will refer to gaps with the smallest size as *small* gaps and the others as *larger* gaps.

Corollary 1. *Let B be a TG-sequence with one larger gap so that the points in B form an arithmetic progression. Then using the notation of the definition, the common modulus of this arithmetic progression is either d or $p - d$.*

Proof. Using the notation of the definition of a TG-sequence, we may assume, without loss of generality, that $a = 0$ so that $B \equiv \{id : i = 0, \ldots, q-1\}$ modulo p where $(p, d) = 1$. Suppose this is the same set as $\{ic + b, i = 0, \ldots, q - 1\}$. Multiplying each term by d^{-1} gives

$$\begin{aligned} B' &\equiv \{0, 1, \ldots, q-1\} \mod p \\ &\equiv \{d^{-1}b, d^{-1}b + d^{-1}c, \ldots, d^{-1}b + (q-1)d^{-1}c\} \mod p. \end{aligned}$$

From the right hand side we see that $|B' \cap (B' + d^{-1}c)| = q - 1$, so $B' + d^{-1}c$ is congruent to $\{1, \ldots, q\}$ or $\{p - 1, 0, \ldots, q - 2\}$ modulo p, which implies that $d^{-1}c \equiv \pm 1 \mod p$, and so $c \equiv \pm d \mod p$, and the result follows. □

Corollary 2. *Let B be a TG-sequence with two larger gaps so that the points in B form two arithmetic progressions with common modulus c. Then, using the notation of the definition, c is congruent modulo p to either $2d$ or $-2d$.*

Proof. We may assume, without loss of generality, that $a = 0$ so that $B = \{id : i = 0, \ldots, q-1\}$ where $(p, d) = 1$. Let K_1 be one arithmetic progression and K_2 the other. Suppose $(q - 1)d$ does not belong to K_1 and consider $K_1 + d$. This is a subset of B. It cannot intersect both K_1 and K_2, neither can it be contained in K_1. Therefore $K_1 + d \subseteq K_2$. Similarly $K_2 \backslash \{(q - 1)d\} + d \subseteq K_1$. So if $0 \in K_1$ then $d \in K_2$, $2d \in K_1$, and so on. Thus $K_1 = \{0, 2d, \ldots\}$ and $K_2 = \{d, 3d, \ldots\}$, or vice versa. As in the proof of the last lemma we have $c \equiv \pm 2d \mod p$. □

No doubt this can be easily extended to more than 2 larger gaps. It also follows from the proof that $||K_1| - |K_2|| \leq 1$.

Corollary 3. *Let B be a TG-sequence with $q_2 > 1$ points and modulus p, with smallest gap size c and largest G, where $G > q_1$ for some integer q_1. Suppose that it has k larger gaps.*
(i) If the sequence has only two gap sizes, then

$$p - q_1 - q_2 \geq (k - 1)(G - 1) + (q_2 - k)(c - 1). \tag{13}$$

(ii) If the sequence has three gap sizes, then

$$p - q_1 - q_2 \geq (k-1)(G-c-1) + (q_2-k)(c-1). \tag{14}$$

The slightly awkward notation here will simplify the applications.

Proof. (i) Clearly p equals q_2 plus the number of points in the interiors of the gaps. The k larger gaps each contain $G-1$ points and the q_2-k small gaps each contain $c-1$ points. Thus, using the assumption $G > q_1$,

$$\begin{aligned} p &= q_2 + k(G-1) + (q_2-k)(c-1) \\ &\geq q_1 + q_2 + (k-1)(G-1) + (c-1)(q_2-k), \end{aligned}$$

giving the required result.

(ii) We have at least one gap of size G containing $G-1$ points, $k-1$ other larger gaps of size at least $G-c$ each containing at least $G-c-1$ points, and q_2-k small gaps each containing $c-1$ points. Thus,

$$p - q_2 \geq G - 1 + (k-1)(G-c-1) + (q_2-k)(c-1).$$

By assumption we have $G \geq q_1 + 1$, which establishes the inequality. $\square$

Example 1. Consider the TG sequence $\{7i : i = 0 \dots 3 \mod 13\} = \{0,1,7,8\}$. We have one larger gap of size 6, another of size 5 and 2 of size 1. In the notation of the corollary, $p = 13$, $q_1 = 5$, $q_2 = 4$, $G = 6$, $k = 2$ and $c = 1$. Then $p - q_1 - q_2$ and $(k-1)(G-c-1) + (q_2-k)(c-1)$ both equal 4.

4. Proof of Theorem A

By (1) we have $p = q_1 + q_2 + q_3 + q_4$. We may also assume that

$$q_1 \geq q_2 + 1 \geq q_3 + 2 \geq q_4 + 3 \geq 4 \tag{15}$$

and $(p, q_i) = 1$ for $1 \leq i \leq 4$. We cannot have equality in each of the first three inequalities in (15), for then $p = 4q_4 + 6$, which is not relatively prime to each q_i. Thus

$$q_1 \geq q_4 + 4. \tag{16}$$

With the notation of (10), $B_1 = \{0, 1, \dots q_1 - 1\}$ and B_2 is a TG-sequence with q_2 points and modulus p, which contains a gap of size at least $q_1 + 1$ (as B_1 is disjoint from B_2). Suppose that its small gaps have size c and that it has k larger gaps. Clearly the largest gap $G \geq q_1 + 1$ which is part of the hypothesis of Corollary 3.

Lemma 3. *With the above notation, $c, k \in \{1, 2\}$.*

Proof. Suppose that B_2 has only the two gap sizes G and c. Then by (i) of Corollary 3 and recalling that $p = q_1 + q_2 + q_3 + q_4$,

$$q_3 + q_4 \geq (k-1)(G-1) + (q_2 - k)(c-1) \geq (k-1)q_1 + (q_2 - k)(c-1).$$

If $k \geq 3$, the right hand side is at least $2q_1$, which is impossible by (15). So we have $k = 1$ or $k = 2$. If $c \geq 3$, the right hand side is at least $(k-1)q_1 + 2(q_2 - k)$, which is $2q_2 - 2$ when $k = 1$, and $q_1 + 2q_2 - 4$ when $k = 2$. In either case we get a contradiction with (15). Thus if B_2 has two gap sizes we have $1 \leq c \leq 2$ and $1 \leq k \leq 2$.

Now suppose we have three gap sizes. These are G, $G-c$ and c with $G > G-c > c$, which implies

$$G \geq 2c + 1 \tag{17}$$

and by (ii) of Corollary 3,

$$q_3 + q_4 \geq (k-1)(G-c-1) + (q_2 - k)(c-1). \tag{18}$$

We consider various combinations of c and k values.

If $c = 1$ and $k \geq 3$, (18) gives

$$q_3 + q_4 \geq (k-1)(q_1 - 1) \geq 2q_1 - 2,$$

which is impossible by (15).

If $c = 2$ and $k \geq 3$, (18) gives

$$\begin{aligned} q_3 + q_4 &\geq (k-1)(G-3) + q_2 - k \\ &= (k-1)(G-4) + q_2 - 1 \\ &\geq 2q_1 + q_2 - 7, \end{aligned}$$

which is again incompatible with (15).

If $c \geq 3$ then, using (17),

$$\begin{aligned} q_3 + q_4 &\geq (k-1)(G-c-1) + (q_2 - k)(c-1) \\ &= (k-1)(G-2c) + (c-1)(q_2 - 1) \\ &\geq 2q_2 - 2, \end{aligned}$$

which is again incompatible with (15). We have now eliminated all cases, for both two and three gaps, except those with $1 \leq c \leq 2$ and $1 \leq k \leq 2$. □

Next, by eliminating the cases $(c, k) = (2, 2), (1, 2)$ and $(1, 1)$, we will prove

Proposition 1. *With the above notation,* $q_1 = 2q_2$ *and* $\widetilde{q}_2 = 2$.

Proof. If $c = 2$ and $k = 2$ and we have three gaps (the two gap case is even simpler), we get

$$\begin{aligned} q_3 + q_4 &\geq G + q_2 - 5 \\ &\geq q_1 + q_2 - 4, \end{aligned}$$

so this case is also eliminated, leaving the cases $(c, k) = (1, 1)$, $(1, 2)$ or $(2, 1)$.

Suppose $c = 1$ and $k = 2$, then by (13) or (14) we have $p \geq 2q_1 + q_2 - 1$. By Corollary 2, $2\widetilde{q}_2 \equiv \pm 1 \mod p$, so that $2q_1 \equiv \pm q_2 \mod p$. If $2q_1 \equiv q_2 \mod p$ then $p = 2q_1 - q_2$, which contradicts the above inequality. If $2q_1 \equiv -q_2 \mod p$, we have $p = 2q_1 + q_2$. Since B_2 contains 2 larger gaps, and $c = 1$, each larger gap has size $q_1 + 1$ and contains q_1 points. One gap contains B_1 and the other $B_3 \cup B_4$. Therefore $B_3 \cup B_4$ is a translate of B_1. It follows that $S(p/q_3, b_3) \cup S(p/q_4, b_4) = S(p/q_1, b)$ for some b. Lemma 2 then says that either $p = 7$, which is impossible, or $q_3 = q_4$ which is impossible, or $q_1 > p/2$ which is also impossible. We conclude that we cannot have $c = 1$ and $k = 2$. We are left with the cases $(c, k) = (1, 1)$ or $(2, 1)$.

If $c = 1$ and $k = 1$, then, by Corollary 1, $\widetilde{q}_2 \equiv \pm 1 \mod p$, so $q_1 = q_2$, or $q_1 = p - q_2$, both of which are impossible.

We conclude that $c = 2$ and $k = 1$, so that $\widetilde{q}_2 \equiv \pm 2 \mod p$. If $\widetilde{q}_2 = -2$ we have $q_1 + 2q_2 = p$. Then $B_2 = \{q_1, q_1+2, \ldots, p-2\}$ and $B_3 \cup B_4 = \{q_1+1, q_1+3, \ldots, p-1\}$, or vice versa. Either way B_2 is a translate of $B_3 \cup B_4$ so that $S(p/q_3, b_3) \cup S(p/q_4, b_4) = S(p/q_2, b)$ for some b. This is impossible by Lemma 2. So $\widetilde{q}_2 = 2$ and $q_1 = 2q_2$. $\square$

Proposition 2. *With the notation of Theorem A, we have $q_2 = 2q_3$.*

Proof. B_2 contains $q_2 - 1$ gaps of size 2, which must be filled by members of B_3 and B_4. We thus have

$$q_3 + q_4 \geq q_2 - 1. \tag{19}$$

Suppose $B_1 \cap (B_1 + \widetilde{q}_3) = \emptyset$. Then $\widetilde{q}_3 \geq q_1$, so $B_1 + \widetilde{q}_3$ can contain at most one member of B_3 and we therefore have $p \geq 2q_1 - 1 + q_3 = q_1 + 2q_2 + q_3 - 1$, which is impossible. Thus B_1 and $B_1 + \widetilde{q}_3$ have non-empty intersection. By Lemma 1, B_3 is an arithmetic progression with common modulus c equal to $\widetilde{q}_3$ or $p - \widetilde{q}_3$. The points in the gaps between the members of B_3 belong to B_2 or B_4. Since there are $q_3 - 1$ such gaps there is at least one gap with at most one member of B_4. Since $\widetilde{q}_2 \equiv 2 \mod p$, this allows at most 2 elements of B_2, so $c \leq 4$. If $c = 1$, we have $\widetilde{q}_1 \equiv \widetilde{q}_3$ or $\widetilde{q}_1 \equiv -\widetilde{q}_3$. The first implies that $q_1 = q_3$, and the second that $q_1 + q_3 = p$, both of which are impossible. If $c = 2$, we have $\widetilde{q}_3 \equiv \pm\widetilde{q}_2$, which leads to a similar contradiction. If $c = 3$, we can have members of B_2 in at most two the gaps in B_3. By considering possibilities, as before, we find this is impossible. So $c = 4$ which means $\widetilde{q}_3 \equiv \pm 4 \mod p$. If $\widetilde{q}_3 \equiv -4 \mod p$, we get $4q_3 + q_1 \equiv p \mod p$, so $3q_3 \equiv q_2 + q_4 \mod p$. This means $3q_3 = q_2 + q_4$. Then (19) implies that $3q_3 \leq q_3 + 2q_4 + 1$, which is impossible. So $\widetilde{q}_3 \equiv 4 \mod p$, which implies $q_1 = 4q_3$ and $q_2 = 2q_3$. $\square$

We are now ready to complete the proof of Theorem A.

Since $q_2 = 2q_3$, (19) gives $q_4 \geq q_3 - 1$, so, by (15), $q_4 = q_3 - 1$.

By considering $B_1 \cap (B_1 + \widetilde{q}_3)$ and using Lemma 1, as in the B_3 case we find that B_4 is an arithmetic progression, possibly with a single term. Now B_2 is an arithmetic progression with common modulus 2 and $2q_4 + 2$ terms. The $2q_4 + 1$ gaps between the terms must be filled with members of B_3 and B_4. If $q_4 > 1$, the common modulus of B_4 will be 4, leading to $\widetilde{q}_4 \equiv \pm\widetilde{q}_3 \mod p$, leading to $q_3 = q_4$ or $q_3 + q_4 = p$, both of which are impossible. We conclude that $q_4 = 1$, $q_3 = q_4 + 1 = 2$, $q_2 = 2q_3 = 4$ and $q_1 = 2q_2 = 8$, which completes the proof of Theorem A. $\square$

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