THE MEAN VALUE OF A HYBRID ARITHMETIC FUNCTION ASSOCIATED TO FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract

Let $\lambda_f(n)$, $\sigma(n)$, and $\phi(n)$ denote the n^{th} Fourier coefficient of holomorphic cusp form f, the sum-of-divisors function, and the Euler totient function, respectively. We improve the error term of the asymptotic formula of $\sum\limits_{n\le x}\lambda_f^j(n)\sigma^b(n)\phi^c(n)$ for $j=2,4,6$, and estimate the sum $\sum\limits_{n\le x}\lambda_f^j(n)\sigma^b(n)\phi^c(n)$ for $j=7,8$.

1. Introduction

In analytic number theory, one of the most basic goals is to establish the asymptotic formulae for the summation function

$$S(x)=\sum_{n\le x}a(n),$$

where $a(n)$ is an arithmetic function. In this paper, we investigate a hybrid arithmetic function $\lambda_f^j(n)\sigma^b(n)\phi^c(n)$, where $\lambda_f(n)$ is the n-th Fourier coefficient of holomorphic cusp form f, $\sigma(n)$ is the sum-of-divisors function, and $\phi(n)$ is the Euler totient function.

Assume that k is an even integer and H_k^* is the set of all normalized Hecke primitive eigencuspform of weight k for the full modular group $SL_2(\mathbb{Z})$. The Fourier expansion of $f\in H_k^*$ at $z=\infty$ is

$$f(z)=\sum_{n\ge 1}\lambda_f(n)n^{\frac{k-1}{2}}e(z).$$

Here, $\lambda_f(n)$ is the eigenvalue of Hecke operator T_n, and $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geq 1$ and $n \geq 1$ are arbitrary integers. Fourier coefficients of holomorphic cusp forms are extremely significant and are of interest to many mathematicians. In 1974, Deligne [4] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n), \tag{1.1}$$

where $d(n)$ is the Dirichlet divisor function.

Rankin [17] studied the sum of Fourier coefficients of cusp forms over natural numbers, and showed that

$$\sum_{n\leq x} \lambda_f(n) \ll x^{1/3}(\log x)^{-\delta},$$

where $0 < \delta < 0.06$. In 1990, Ivić [5] obtained the following result:

$$\sum_{n\leq x} \lambda_f^2(n) = cx + O_f(x^{\frac{3}{8}}).$$

Recently, Manski, Mayle and Zbacnik [15] investigated a hybrid arithmetic function and showed

$$\sum_{n\leq x} d^a(n)\sigma^b(n)\phi^c(n) = x^{b+c+1}P_{2^\alpha-1}(\log x) + O(x^{b+c+r_a+\varepsilon}),$$

where a, b, c are real numbers, $\frac{1}{2} \leq r_a < 1$, and $P_n(t)$ is a polynomial of degree n.

Subsequently, the mean values of the hybrid arithmetic function $\lambda_f^i(n)\sigma^b(n)\phi^c(n)$ were estimated by Li[14] for $i = 1, 2, 3, 4$, and by Cui[3] for $i = 5, 6$.

Combining the classical analytic method with properties of some primitive automorphic L-functions, we improve the results of $\sum\limits_{n\leq x} \lambda_f^i(n)\sigma^b(n)\phi^c(n)$ for $i = 2, 4, 6$, and further study the summation $\sum\limits_{n\leq x} \lambda_f^i(n)\sigma^b(n)\phi^c(n)$ for $i = 7, 8$. In detail, we obtain the following results.

Theorem 1. *Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,*

$$\sum_{n\leq x} \lambda_f^2(n)\sigma^b(n)\phi^c(n) = C_0 x^{b+c+1} + O\left(x^{b+c+\frac{23}{37}+\varepsilon}\right),$$

where the O-constant depends on f.

Theorem 2. *Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,*

$$\sum_{n\leq x} \lambda_f^4(n)\sigma^b(n)\phi^c(n) = x^{b+c+1}P_1(\log x) + O\left(x^{b+c+\frac{257}{299}+\varepsilon}\right),$$

where $P_1(t)$ is a polynomial in t of degree 1, and the O-constant depends on f.

Theorem 3. *Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,*

$$\sum_{n\leq x} \lambda_f^6(n)\sigma^b(n)\phi^c(n) = x^{b+c+1}P_4(\log x) + O\left(x^{b+c+\frac{201}{208}+\varepsilon}\right),$$

where $P_4(t)$ is a polynomial in t of degree 4, and the O-constant depends on f.

Remark. We improve the results in [3, Theorem 4], [14, Theorem 2] and [14, Theorem 4] respectively. By comparison, $23/37 = 0.621\cdots < 38/59 = 0.644\cdots$ ([14, Theorem 2]), $257/299 = 0.8595\cdots < 139/160 = 0.86875$ ([14, Theorem 4]), $201/208 = 0.9663\cdots < 631/652 = 0.9677\cdots$ ([3, Theorem 4]).

Theorem 4. *Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,*

$$\sum_{n\leq x} \lambda_f^7(n)\sigma^b(n)\phi^c(n) = O\left(x^{b+c+\frac{67}{68}+\varepsilon}\right),$$

where the O-constant depends on f.

Theorem 5. *Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,*

$$\sum_{n\leq x} \lambda_f^8(n)\sigma^b(n)\phi^c(n) = x^{b+c+1}P_{13}(\log x) + O\left(x^{b+c+\frac{117}{118}+\varepsilon}\right),$$

where $P_{13}(t)$ is a polynomial in t of degree 13, and the O-constant depends on f.

2. Preliminaries

In order to prove Theorems 1-5, some relevant results will be given in this section. First, we will introduce several primitive automorphic L-functions.

The definition of the Riemann zeta function is

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p(1-p^{-s})^{-1}, \qquad Re(s)>1. \tag{2.1}$$

The Hecke L-function of $f\in H_k^*$ is a Dirichlet series and has Euler product representation

$$L(f,s)=\sum_{n=1}^{\infty}\lambda_f(n)n^{-s}=\prod_p(1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1}, \tag{2.2}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are called local roots or local parameters of $L(f,s)$ at p. According to Deligne [10], they satisfy

$$\lambda_f(p)=\alpha_f(p)+\beta_f(p), |\alpha_f(p)|=|\beta_f(p)|=\alpha_f(p)\beta_f(p)=1. \tag{2.3}$$

The inequality (1.1) ensures that the Dirichlet series (2.2) is absolutely convergent for $Re(s)>1$.
The jth symmetric power L-function attached to $f\in H_k^*$ is defined as

$$L(sym^j f,s):=\prod_p\prod_{m=0}^{j}(1-\alpha_f(p)^{j-m}\beta_f(p)^m p^{-s})^{-1}. \tag{2.4}$$

The definition of Rankin-Selberg L-function shows that

$$L(sym^i f\times sym^j f,s):=\prod_p\prod_{m=0}^{i}\prod_{u=0}^{j}(1-\alpha_f(p)^{(i+j)-2(m+u)}p^{-s})^{-1}. \tag{2.5}$$

The above products over primes give the Dirichlet series for $L(sym^j f,s)$ and $L(sym^i f\times sym^j f,s)$. In view of (2.3)-(2.5), it is easy to see that $L(sym^j f,s)$ and $L(sym^i f\times sym^j f,s)$ converge absolutely in half-plane $Re(s)>1$.

Now we give the current best subconvexity (or convexity) bounds of these primitive automorphic L-functions.

The proofs of the statements in the following lemma may be found in [1], [6, Chapter 5], and [16].

Lemma 2.1. *For any $\varepsilon>0$, $\frac{1}{2}\le\sigma\le 1$, and $|t|\ge 2$, we have*

$$\zeta(\sigma+it)\ll_\varepsilon (1+\mid t\mid)^{max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon},$$

$$L(sym^2 f,\sigma+it)\ll_{f,\varepsilon} (1+\mid t\mid)^{max\{\frac{4}{3}(1-\sigma),0\}+\varepsilon},$$

$$L(sym^j f,\sigma+it)\ll_{f,\varepsilon} (1+\mid t\mid)^{max\{\frac{j+1}{2}(1-\sigma),0\}+\varepsilon},\ j=3,4,$$

$$L(sym^i f \times sym^j f, \sigma + it) \ll_{f,\varepsilon} (1+ \mid t \mid)^{max\{\frac{(i+1)(j+1)}{2}(1-\sigma),0\}+\varepsilon}, \; i,j = 1,2,3,4.$$

The following lemma describes the mean-value of symmetric power L-functions and their Rankin-Selberg L-functions.

Lemma 2.2. *For $i, j = 1, 2, 3, 4$, any $\varepsilon > 0$, $T \geq T_0$ (where T_0 is sufficiently large), we have the estimates*

$$\int_T^{2T} \left| L\left(sym^j f, \frac{1}{2} + \varepsilon + it\right)\right|^2 dt \ll_{f,\varepsilon} T^{\frac{j+1}{2}+\varepsilon},$$

$$\int_T^{2T} \left| L\left(sym^i f \times sym^j f, \frac{1}{2} + \varepsilon + it\right)\right|^2 dt \ll_{f,\varepsilon} T^{\frac{(i+1)(j+1)}{2}+\varepsilon}.$$

Proof. The first assertion is in [10, Lemma 2.5], and the second is in [11, Lemma 2.4]. □

Now we introduce the Perron's formula, which is proved in [9, section 1.2.1].

Lemma 2.3. *Let $F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ and $a(n) \leq A(n)$, and the series of $F(s)$ converges absolutely for $\sigma > 1$, where $A(n)$ is monotonically increasing function and $\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = O\left(\frac{1}{(\sigma-1)^{\alpha}}\right)$ with $\sigma \to 1^+$. If $b > 1$ and $x = N + \frac{1}{2}$ with $N \in \mathbb{N}$. Then for $T \geq 2$,*

$$\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} ds + O\left(\frac{x^b}{T(b-1)^{\alpha}} + \frac{xA(2x)\log x}{T}\right).$$

Lemma 2.4. *Let $f \in H_k^*$ and $\lambda_f(n)$ denote its n^{th} normalized Fourier coefficient. Define*

$$F_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^7(n)\sigma^b(n)\phi^c(n)}{n^s}.$$

Then $F_1(s)$ can be factored as

$$\begin{aligned} F_1(s) = &L(sym^4 f \times sym^3 f, s-b-c) L^5(sym^3 f \times sym^2 f, s-b-c) \\ &\times L^8(sym^3 f, s-b-c) L^8(f, s-b-c) H_1(s), \end{aligned}$$

where $H_1(s)$ is absolutely convergent in $Re(s) > b + c + \frac{1}{2}$.

Proof. Because each of $\lambda_f(n)$, $\sigma(n)$, and $\phi(n)$ are multiplicative, $\lambda_f^7(n)\sigma^b(n)\phi^c(n)$ is multiplicative. Therefore, the Dirichlet series $F_1(s)$ can be rewrited as a product over primes:

$$F_1(s) = \prod_p f_{1,p}(s) = \prod_p \sum_{k=0}^{\infty} \frac{\lambda_f^7(p^k)\sigma^b(p^k)\phi^c(p^k)}{p^{ks}}$$
$$= \prod_p \left(1 + \frac{\lambda_f^7(p)\sigma^b(p)\phi^c(p)}{p^s} + \frac{\lambda_f^7(p^2)\sigma^b(p^2)\phi^c(p^2)}{p^{2s}} + \cdots\right).$$

From the theory of Hecke operators, we obtain the recursive relation

$$\lambda_f(p^j) = \lambda_f(p^{j-1})\lambda_f(p) - \lambda_f(p^{j-2}),\ j \geq 2.$$

By induction, we have

$$\lambda_f(p^j) = \frac{\alpha_f(p)^{j+1} - \beta_f(p)^{j+1}}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m.$$

Therefore,

$$f_{1,p}(s) = 1 + \frac{\left(\alpha_f(p) + \beta_f(p)\right)^7 (p+1)^b(p-1)^c}{p^s}$$
$$+ \frac{\left(\frac{\alpha_f^3(p) - \beta_f^3(p)}{\alpha_f(p) - \beta_f(p)}\right)^7 (p^2+p+1)^b(p^2-p)^c}{p^{2s}} + \cdots$$

$$= 1 + \frac{\left(\alpha_f(p) + \beta_f(p)\right)^7 (p+1)^b(p-1)^c}{p^s}$$
$$+ \frac{\left(\alpha_f^2(p) + \alpha_f(p)\beta_f(p) + \beta_f^2(p)\right)^7 (p^2+p+1)^b(p^2-p)^c}{p^{2s}} + \cdots$$

$$= 1 + \frac{\left(\alpha_f(p) + \beta_f(p)\right)^7}{p^{s-b-c}} + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right).$$

Then,

$$
\begin{aligned}
F_1(s) &= \prod_p f_{1,p}(s)\\
&= \prod_p \left(1 + \frac{(\alpha_f(p)+\beta_f(p))^7}{p^{s-b-c}} + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right)\right)\\
&= \prod_p \left(1 + \frac{\alpha_f^7(p) + 7\alpha_f^5(p) + 21\alpha_f^3(p) + 35\alpha_f(p) + 35\beta_f(p) + 21\beta_f^3(p) + 7\beta_f^5(p) + \beta_f^7(p)}{p^{s-b-c}}\right.\\
&\quad \left. + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right)\right)\\
&= L(sym^4 f \times sym^3 f, s-b-c)\\
&\quad \times \prod_p \left(1 + \frac{5\alpha_f^5(p) + 18\alpha_f^3(p) + 31\alpha_f(p) + 31\beta_f(p) + 18\beta_f^5(p) + 5\beta_f^5(p)}{p^{s-b-c}}\right.\\
&\quad \left. + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right)\right)\\
&= L(sym^4 f \times sym^3 f, s-b-c)L^5(sym^3 f \times sym^2 f, s-b-c)\\
&\quad \times \prod_p \left(1 + \frac{8\alpha_f^3(p) + 16\alpha_f(p) + 16\beta_f(p) + 8\beta_f^3(p)}{p^{s-b-c}} + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right)\right)\\
&= L(sym^4 f \times sym^3 f, s-b-c)L^5(sym^3 f \times sym^2 f, s-b-c)L^8(sym^3 f, s-b-c)\\
&\quad \times \prod_p \left(1 + \frac{8\alpha_f(p) + 8\beta_f(p)}{p^{s-b-c}} + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right)\right)\\
&= L(sym^4 f \times sym^3 f, s-b-c)L^5(sym^3 f \times sym^2 f, s-b-c)L^8(sym^3 f, s-b-c)\\
&\quad \times L^8(f, s-b-c)\prod_p \left(1 + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right)\right)\\
&= L(sym^4 f \times sym^3 f, s-b-c)L^5(sym^3 f \times sym^2 f, s-b-c)L^8(sym^3 f, s-b-c)\\
&\quad \times L^8(f, s-b-c)H_1(s),
\end{aligned}
$$

where $H_1(s)$ is absolutely convergent in $Re(s) > b + c + \frac{1}{2}$. □

The proof of the following lemma is similar to that of Lemma 2.4.

Lemma 2.5. *Let $f \in H_k^*$ and $\lambda_f(n)$ denote its n-th normalized Fourier coefficient. Define*

$$F_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^8(n)\sigma^b(n)\phi^c(n)}{n^s}.$$

Then $F_2(s)$ can be factored as

$$
\begin{aligned}
F_2(s) = &\, L(sym^4 f \times sym^4 f, s-b-c)L^6(sym^4 f \times sym^2 f, s-b-c)\\
&\times L^{13}(sym^4 f, s-b-c)L^{21}(sym^2 f, s-b-c)\zeta^{13}(s-b-c)H_2(s),
\end{aligned}
$$

where $H_2(s)$ is absolutely convergent in $Re(s) > b + c + \frac{1}{2}$.

3. Proofs of Theorems 1-5

The proofs of Theorems 1-5 are similar, so we just prove Theorem 5 in this section.

Proof of Theorem 5. Denote $\lambda_f^8(n)\sigma^b(n)\phi^c(n)$ by $f_2(n)$. From (1.1), we observe that $f_2(n) \le Bn^{b+c+\varepsilon}$, where B is a real constant on ε. Let

$$\begin{aligned} L_2(f, s-b-c) = {} & L(sym^4 f \times sym^4 f, s-b-c) L^6(sym^4 f \times sym^2 f, s-b-c) \\ & \times L^{13}(sym^4 f, s-b-c) L^{21}(sym^2 f, s-b-c) \zeta^{13}(s-b-c). \end{aligned}$$

From Lemma 2.5, we learn that $F_2(s) = L_2(f, s-b-c)H_2(s)$.

From the works of Cogdell and Mochel [2], Jacquet and Shalika [7] [8], Shahidi [19] [20], and the reformulation of Rudnick and Sarnak [18], symmetric power L-function $L(sym^i f, s)$ $(1 \le i \le 4)$ and Rankin-Selberg L-function $L(sym^i f \times sym^j f, s)$ $(1 \le i < j \le 4)$ can be extended to be an entire function on the whole complex plane. Lau and Wu [12] showed that for $1 \le j \le 4$, $L(sym^j f \times sym^j f, s)$ is entire except possible for simple poles at $s = 0, 1$, and satisfies a functional equation. Thus, $F_2(s)$ can be analytically continued to the half-plane $Re(s) > b + c + \frac{1}{2}$. In this region, $F_2(s)$ only has a pole $s = b + c + 1$ of order 14.

By using Lemma 2.3 (Perron's formula), we obtain

$$\begin{aligned} & \sum_{n \le x} \lambda_f^8(n)\sigma^b(n)\phi^c(n) \\ & = \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} \sum_{n=1}^{\infty} \frac{f_2(n)}{n^s} \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T^{s-b-c-1}} \right) + O\left(\frac{xB(2x)^{b+c+\varepsilon} \log x}{T} \right) \\ & = \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} L_2(f, s-b-c)H_2(s) \frac{x^s}{s} ds + O\left(\frac{x^{b+c+1+\varepsilon}}{T} \right), \qquad (3.1) \end{aligned}$$

where T with $1 \le T \le x$ is a parameter to be specified later.

Since our goal is to estimate the integral in (3.1), we need to consider the closed contour Γ:

$$\begin{aligned} I &= [b+c+1+\varepsilon-iT, b+c+1+\varepsilon+iT], \\ II &= [b+c+1+\varepsilon+iT, b+c+\frac{1}{2}+\varepsilon+iT], \\ III &= [b+c+\frac{1}{2}+\varepsilon+iT, b+c+\frac{1}{2}+\varepsilon-iT], \\ IV &= [b+c+\frac{1}{2}+\varepsilon-iT, b+c+1+\varepsilon-iT]. \end{aligned}$$

Let

$$\begin{aligned} I_1 &= \int_I L_2(f, s-b-c)H_2(s)\frac{x^s}{s} ds, & I_2 &= \int_{II} L_2(f, s-b-c)H_2(s)\frac{x^s}{s} ds, \\ I_3 &= \int_{III} L_2(f, s-b-c)H_2(s)\frac{x^s}{s} ds, & I_4 &= \int_{IV} L_2(f, s-b-c)H_2(s)\frac{x^s}{s} ds. \end{aligned}$$

By Cauchy's Residue theorem, we obtain

$$\begin{aligned}\frac{1}{2\pi i}I_1 &= \frac{1}{2\pi i}\int_{\Gamma} L_2(f,s-b-c)H_2(s)\frac{x^s}{s}ds - \frac{1}{2\pi i}(I_2+I_3+I_4)\\ &= x^{b+c+1}P_{13}(\log x) - \frac{1}{2\pi i}\left(I_2+I_3+I_4\right),\end{aligned} \tag{3.2}$$

where $P_{13}(t)$ is the polynomial of degree 13 in t.

By Lemma 2.5, $H_2(s)$ converges absolutely in the half-plane $Re(s) > b+c+\frac{1}{2}$. Thus, for the integrals over the horizontal segments, I_2 and I_4 can be estimated as

$$\begin{aligned}&\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} L_2(f,\sigma+iT)\frac{x^{b+c+\sigma}}{T}d\sigma\\ &\ll x^{b+c}\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} L_2(f,\sigma+iT)\frac{x^{\sigma}}{T}d\sigma.\end{aligned} \tag{3.3}$$

Furthermore, Lemma 2.1 leads to

$$\begin{aligned}I_2+I_4 &\ll x^{b+c}\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} T^{(\frac{25}{2}+\frac{15}{2}\times 6+\frac{5}{2}\times 13+\frac{4}{3}\times 21+\frac{13}{42}\times 13)(1-\sigma)+\varepsilon}\frac{x^{\sigma}}{T}d\sigma\\ &\ll x^{b+c}\max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} T^{117}\left(\frac{x}{T^{118}}\right)^{\sigma}\\ &\ll \frac{x^{b+c+1+\varepsilon}}{T} + x^{b+c+\frac{1}{2}+\varepsilon}T^{58+\varepsilon}.\end{aligned} \tag{3.4}$$

For the vertical segment, by using Lemma 2.2 and Cauchy's inequality, I_3 can be estimated as

$$\begin{aligned}&\ll \int_1^T \Bigg(\Bigg| L\left(sym^4 f\times sym^4 f,\frac{1}{2}+\varepsilon+it\right)L^6\left(sym^4 f\times sym^2 f,\frac{1}{2}+\varepsilon+it\right)\\ &\qquad\times L^{13}\left(sym^4 f,\frac{1}{2}+\varepsilon+it\right)L^{21}\left(sym^2 f,\frac{1}{2}+\varepsilon+it\right)\zeta^{13}\left(\frac{1}{2}+\varepsilon+it\right)\\ &\qquad\times H_2(b+c+\frac{1}{2}+\varepsilon+it)\Bigg|\frac{x^{b+c+\frac{1}{2}+\varepsilon}}{|b+c+\frac{1}{2}+\varepsilon+it|}\Bigg)dt\\ &\quad + x^{b+c+\frac{1}{2}+\varepsilon}\\ &\ll x^{b+c+\frac{1}{2}+\varepsilon}\int_1^T \Bigg| L\left(sym^4 f\times sym^4 f,\frac{1}{2}+\varepsilon+it\right)L^6\left(sym^4 f\times sym^2 f,\frac{1}{2}+\varepsilon+it\right)\\ &\quad L^{13}\left(sym^4 f,\frac{1}{2}+\varepsilon+it\right)L^{21}\left(sym^2 f,\frac{1}{2}+\varepsilon+it\right)\zeta^{13}\left(\frac{1}{2}+\varepsilon+it\right)\Bigg|\frac{1}{t}dt + x^{b+c+\frac{1}{2}+\varepsilon}.\end{aligned} \tag{3.5}$$

For convenience, we write

$$\begin{aligned}L_{2,1}(f,s-b-c) &= L(sym^4 f\times sym^4 f,s-b-c)L^4(sym^4 f\times sym^2 f,s-b-c)\\ &\qquad\times L^6(sym^4 f,s-b-c),\\ L_{2,2}(f,s-b-c) &= L^2(sym^4 f\times sym^2 f,s-b-c)L^6(sym^4 f,s-b-c).\end{aligned}$$

By a dyadic subdivision, we obtain

$$\begin{aligned}
I_3 &\ll x^{b+c+\frac{1}{2}+\varepsilon} + x^{b+c+\frac{1}{2}+\varepsilon}\log T \max_{1\le T_1\le T}\frac{1}{T_1} \\
&\quad\times\left(\max_{\frac{T_1}{2}\le t\le T_1}\left|\zeta^{13}\left(\frac{1}{2}+\varepsilon+it\right)L\left(sym^4 f,\frac{1}{2}+\varepsilon+it\right)L^{21}\left(sym^2 f,\frac{1}{2}+\varepsilon+it\right)\right|\right) \\
&\quad\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L_{2,1}(f,\sigma+iT)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_{\frac{T_1}{2}}^{T_1}\left|L_{2,2}(f,\sigma+iT)\right|^2 dt\right)^{\frac{1}{2}} \\
&\ll x^{b+c+\frac{1}{2}+\varepsilon} + x^{b+c+\frac{1}{2}+\varepsilon}T^{58} \\
&\ll x^{b+c+\frac{1}{2}+\varepsilon}T^{58}.
\end{aligned} \tag{3.6}$$

Inserting (3.4) and (3.6) into (3.2), we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} L_1(f,s-b-c)H_2(s)\frac{x^s}{s}ds \\
&= \frac{1}{2\pi i}\int_{\Gamma} L_1(f,s-b-c)H_2(s)\cdot\frac{x^s}{s}ds - \frac{1}{2\pi i}(I_2+I_3+I_4) \\
&= x^{b+c+1}P_{13}(\log x) + O\left(x^{b+c+\frac{1}{2}+\varepsilon}T^{58}\right) + O\left(\frac{x^{b+c+1+\varepsilon}}{T}\right).
\end{aligned}$$

Setting $T = x^{\frac{1}{118}}$, we obtain

$$S_2(x) = x^{b+c+1}P_{13}(\log x) + O\left(x^{b+c+\frac{117}{118}+\varepsilon}\right).$$

This completes the proof of Theorem 5. □

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References

[1] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Amer. Math. Soc.*, **30**(1) (2017), 205-224.

[2] J. Cogdell and P. Michel, On the complex moments of symmetric power L-functions at s = 1, *Int. Math. Res. Not.*, **31** (2004), 1561-1617.

[3] S. Cui, The average estimates for Fourier coefficients of holomorphic cusp forms, Master Thesis, *Shandong Normal University*, 2018.

[4] P. Deligne, La Conjecture de Weil. I, *Inst. Hautes Études Sci. Publ. Math.*, **43** (1974), 273-307.

[5] A. Ivić, Large values of certain number-theoretic error term, *Acta Arith.*, **56** (1990), 135-159.

[6] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc., 2004.

[7] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic representations I, *Amer. J. Math.*, **103**(3) (1981), 499-558.

[8] H. Jacquet and J. A. Shalika, On Euler products and the classification of automorphic forms II, *Amer. J. Math.*, **103**(4) (1981), 777-815.

[9] A. Karatsuba, *Complex Analysis in Number Theory*, CRC Press, Boca Raton, Florida, 1995.

[10] H. X. Lao, The cancellation of Fourier coefficient of cusp forms over different sparse sequences, *Acta Math. Sin. (Engl. Ser.)*, **29** (2013), 1963-1972.

[11] H. X. Lao and A. Sankaranarayanan, The average behavior of Fourier coefficients of cusp forms over sparse sequences, *Proc. Amer. Math. Soc.,* **137**(8) (2009), 2557-2565.

[12] Y. K. Lau and J. Wu, A density theorem on automorphic L-functions and some applications, *Trans. Amer. Math. Soc.*, **359** (2006), 441-472.

[13] X. Q. Li, Bounds for GL(3)GL(2) L-functons and GL(3) L-functions, *Ann. of Math.*, **173**(2) (2011), 301-336.

[14] T. T. Li, The asymptotic distrubution of the hybrid arithmetic functions, Master Thesis, *Shandong Normal University*, 2017.

[15] S. Manski, J. Mayle and N. Zbacnik, The asymptotic distribution of a hybird arithmetic function, *Integers*, **15** (2015), 1-16.

[16] M. McKee, H. W. Sun and Y. B. Ye, Improved subconvexity bounds for GL(3)GL(2) and GL(3) L-functions by weighted stationary phase, *Trans. Amer. Math. Soc.*, **370**(5) (2018), 3745-3769.

[17] R. A. Rankin, Sums of cusp form coefficients Automorphic forms and analytic number theory, *Univ. Montreal.*, 1990, 115-121.

[18] Z. Rudnick and P. Sarnak, Zeros of principal L-functions and random matrix theory, *Duke Math. J.*, **81**(2) (1996), 269-232.

[19] F. Shahidi, On certain L-functions, *Amer. J. Math.*, **103**(2) (1981), 297-355.

[20] F. Shahidi, Third symmetric power L-functions for GL(2), *Compos. Math.*, **70** (1989), 245-273.