THE MEAN VALUE OF A HYBRID ARITHMETIC FUNCTION ASSOCIATED TO FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract
Let \(\lambda_f(n)\), \(\sigma(n)\), and \(\phi(n)\) denote the \(n\)th Fourier coefficient of holomorphic cusp form \(f\), the sum-of-divisors function, and the Euler totient function, respectively. We improve the error term of the asymptotic formula of \(\sum_{n \leq x} \lambda^j_f(n)\sigma^b(n)\phi^c(n)\) for \(j = 2, 4, 6\), and estimate the sum \(\sum_{n \leq x} \lambda^j_f(n)\sigma^b(n)\phi^c(n)\) for \(j = 7, 8\).

1. Introduction

In analytic number theory, one of the most basic goals is to establish the asymptotic formulae for the summation function

\[ S(x) = \sum_{n \leq x} a(n), \]

where \(a(n)\) is an arithmetic function. In this paper, we investigate a hybrid arithmetic function \(\lambda^j_f(n)\sigma^b(n)\phi^c(n)\), where \(\lambda_f(n)\) is the \(n\)-th Fourier coefficient of holomorphic cusp form \(f\), \(\sigma(n)\) is the sum-of-divisors function, and \(\phi(n)\) is the Euler totient function.

Assume that \(k\) is an even integer and \(H^*_k\) is the set of all normalized Hecke primitive eigencuspform of weight \(k\) for the full modular group \(SL_2(\mathbb{Z})\). The Fourier expansion of \(f \in H^*_k\) at \(z = \infty\) is

\[ f(z) = \sum_{n \geq 1} \lambda_f(n)n^{-\frac{k+1}{2}}e(z). \]
Here, $\lambda_f(n)$ is the eigenvalue of Hecke operator $T_n$, and $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

where $m \geq 1$ and $n \geq 1$ are arbitrary integers. Fourier coefficients of holomorphic cusp forms are extremely significant and are of interest to many mathematicians. In 1974, Deligne [4] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n),$$

where $d(n)$ is the Dirichlet divisor function.

Rankin [17] studied the sum of Fourier coefficients of cusp forms over natural numbers, and showed that

$$\sum_{n \leq x} \lambda_f(n) \ll x^{1/3}(\log x)^{-\delta},$$

where $0 < \delta < 0.06$. In 1990, Ivić [5] obtained the following result:

$$\sum_{n \leq x} \lambda_f^2(n) = cx + O_f(x^{3/4}).$$

Recently, Manski, Mayle and Zbacnik [15] investigated a hybrid arithmetic function and showed

$$\sum_{n \leq x} d^n(n)\sigma^h(n)\phi^c(n) = x^{b+c+1}P_{2a-1}(\log x) + O(x^{b+c+ra+\epsilon}),$$

where $a, b, c$ are real numbers, $\frac{1}{2} \leq r_a < 1$, and $P_n(t)$ is a polynomial of degree $n$.

Subsequently, the mean values of the hybrid arithmetic function $\lambda_f^i(n)\sigma^h(n)\phi^c(n)$ were estimated by Li[14] for $i = 1, 2, 3, 4$, and by Cui[3] for $i = 5, 6$.

Combining the classical analytic method with properties of some primitive automorphic L-functions, we improve the results of $\sum_{n \leq x} \lambda_f^i(n)\sigma^h(n)\phi^c(n)$ for $i = 2, 4, 6$, and further study the summation $\sum_{n \leq x} \lambda_f^i(n)\sigma^h(n)\phi^c(n)$ for $i = 7, 8$. In detail, we obtain the following results.

**Theorem 1.** Let $b, c \in \mathbb{R}$, $f \in H_0^\infty$, and let $\lambda_f(n)$ denote its $n$-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda_f^2(n)\sigma^h(n)\phi^c(n) = C_0x^{b+c+1} + O\left(x^{b+c+\frac{32}{15}+\epsilon}\right),$$
where the $O$-constant depends on $f$.

**Theorem 2.** Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its $n$-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda_f^b(n) \phi^c(n) = x^{b+c+1} P_1(\log x) + O\left(x^{b+c+\frac{281}{208}+\varepsilon}\right),$$

where $P_1(t)$ is a polynomial in $t$ of degree 1, and the $O$-constant depends on $f$.

**Theorem 3.** Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its $n$-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda_f^b(n) \phi^c(n) = x^{b+c+1} P_2(\log x) + O\left(x^{b+c+\frac{281}{612}+\varepsilon}\right),$$

where $P_2(t)$ is a polynomial in $t$ of degree 4, and the $O$-constant depends on $f$.

**Remark.** We improve the results in [3, Theorem 4], [14, Theorem 2] and [14, Theorem 4] respectively. By comparison, $23/37 = 0.621\cdots < 38/59 = 0.644\cdots$ ([14, Theorem 2]), $257/299 = 0.8595\cdots < 139/160 = 0.86875$ ([14, Theorem 4]), $201/208 = 0.9663\cdots < 631/652 = 0.9677\cdots$ ([3, Theorem 4]).

**Theorem 4.** Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its $n$-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda_f^b(n) \phi^c(n) = O\left(x^{b+c+\frac{281}{612}+\varepsilon}\right),$$

where the $O$-constant depends on $f$.

**Theorem 5.** Let $b, c \in \mathbb{R}$, $f \in H_k^*$, and let $\lambda_f(n)$ denote its $n$-th normalized Fourier coefficient. Then for any $\varepsilon > 0$,

$$\sum_{n \leq x} \lambda_f^b(n) \phi^c(n) = x^{b+c+1} P_{13}(\log x) + O\left(x^{b+c+\frac{117}{124}+\varepsilon}\right),$$

where $P_{13}(t)$ is a polynomial in $t$ of degree 13, and the $O$-constant depends on $f$.

2. Preliminaries

In order to prove Theorems 1-5, some relevant results will be given in this section. First, we will introduce several primitive automorphic L-functions.
The definition of the Riemann zeta function is
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \quad \text{Re}(s) > 1. \tag{2.1}
\]

The Hecke \(L\)-function of \(f \in H_k^*\) is a Dirichlet series and has Euler product representation
\[
L(f, s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_p (1 - \alpha_f(p)p^{-s})^{-1}(1 - \beta_f(p)p^{-s})^{-1}, \tag{2.2}
\]
where \(\alpha_f(p)\) and \(\beta_f(p)\) are called local roots or local parameters of \(L(f, s)\) at \(p\).

According to Deligne [10], they satisfy
\[
\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2.3}
\]

The inequality (1.1) ensures that the Dirichlet series (2.2) is absolutely convergent for \(\text{Re}(s) > 1\).

The \(j\)th symmetric power \(L\)-function attached to \(f \in H_k^*\) is defined as
\[
L(\text{sym}^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s})^{-1}. \tag{2.4}
\]

The definition of Rankin-Selberg \(L\)-function shows that
\[
L(\text{sym}^i f \times \text{sym}^j f, s) := \prod_p \prod_{m=0}^{i} \prod_{u=0}^{j} (1 - \alpha_f(p)^{(i+j)-2(m+u)} p^{-s})^{-1}. \tag{2.5}
\]

The above products over primes give the Dirichlet series for \(L(\text{sym}^j f, s)\) and \(L(\text{sym}^i f \times \text{sym}^j f, s)\). In view of (2.3)-(2.5), it is easy to see that \(L(\text{sym}^j f, s)\) and \(L(\text{sym}^i f \times \text{sym}^j f, s)\) converge absolutely in half-plane \(\text{Re}(s) > 1\).

Now we give the current best subconvexity (or convexity) bounds of these primitive automorphic \(L\)-functions.

The proofs of the statements in the following lemma may be found in [1], [6, Chapter 5], and [16].

**Lemma 2.1.** For any \(\varepsilon > 0\), \(\frac{1}{2} \leq \sigma \leq 1\), and \(|t| \geq 2\), we have
\[
\zeta(\sigma + it) \ll_{\varepsilon} (1 + |t|)^{\max\left\{ \frac{1}{2}(1-\sigma), 0 \right\} + \varepsilon},
\]
\[
L(\text{sym}^2 f, \sigma + it) \ll_{f, \varepsilon} (1 + |t|)^{\max\left\{ \frac{1}{2}(1-\sigma), 0 \right\} + \varepsilon},
\]
\[
L(\text{sym}^3 f, \sigma + it) \ll_{f, \varepsilon} (1 + |t|)^{\max\left\{ \frac{2}{3}(1-\sigma), 0 \right\} + \varepsilon}, \quad j = 3, 4,
\]
Define \( L(\text{sym}^i f \times \text{sym}^j f, \sigma + it) \ll_{f, \varepsilon} (1 + |t|)^{\max\left\{ \frac{(i+1)(j+1)}{2}, (1-\sigma), 0 \right\} + \varepsilon}, \ i, j = 1, 2, 3, 4. \)

The following lemma describes the mean-value of symmetric power L-functions and their Rankin-Selberg L-functions.

**Lemma 2.2.** For \( i, j = 1, 2, 3, 4, \) any \( \varepsilon > 0, \) \( T \geq T_0 \) (where \( T_0 \) is sufficiently large), we have the estimates

\[
\int_T^{2T} \left| L\left( \text{sym}^i f, \frac{1}{2} + \varepsilon + it \right) \right|^2 \, dt \ll_{f, \varepsilon} T^{\frac{i+1}{2} + \varepsilon},
\]

\[
\int_T^{2T} \left| L\left( \text{sym}^i f \times \text{sym}^j f, \frac{1}{2} + \varepsilon + it \right) \right|^2 \, dt \ll_{f, \varepsilon} T^{\frac{(i+1)(j+1)}{2} + \varepsilon}.
\]

**Proof.** The first assertion is in [10, Lemma 2.5], and the second is in [11, Lemma 2.4]. \( \square \)

Now we introduce the Perron’s formula, which is proved in [9, section 1.2.1].

**Lemma 2.3.** Let \( F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \) and \( a(n) \leq A(n) \), and the series of \( F(s) \) converges absolutely for \( \sigma > 1 \), where \( A(n) \) is monotonically increasing function and \( \sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma} = O \left( \frac{1}{(\sigma-1)^\alpha} \right) \) with \( \sigma \to 1^+ \). If \( b > 1 \) and \( x = N + \frac{1}{2} \) with \( N \in \mathbb{N} \). Then for \( T \geq 2 \),

\[
\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} \, ds + O \left( \frac{x^b}{T(b-1)^\alpha} + \frac{x A(2x) \log x}{T} \right).
\]

**Lemma 2.4.** Let \( f \in H_k^0 \) and \( \lambda_f(n) \) denote its \( n^{th} \) normalized Fourier coefficient. Define

\[
F_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n) \sigma^b(n) \phi(n)}{n^s}.
\]

Then \( F_1(s) \) can be factored as

\[
F_1(s) = L(\text{sym}^4 f \times \text{sym}^3 f, s - b - c) L^5(\text{sym}^3 f \times \text{sym}^2 f, s - b - c) \]

\[
\times L^8(\text{sym}^3 f, s - b - c) L^8(f, s - b - c) H_1(s),
\]

where \( H_1(s) \) is absolutely convergent in \( \text{Re}(s) > b + c + \frac{1}{2} \).
**Proof.** Because each of \( \lambda_f(n) \), \( \sigma(n) \), and \( \phi(n) \) are multiplicative, \( \lambda_f^2(n) \sigma^2(n) \phi^2(n) \) is multiplicative. Therefore, the Dirichlet series \( F_1(s) \) can be rewirted as a product over primes:

\[
F_1(s) = \prod_p f_{1,p}(s) = \prod_p \sum_{k=0}^{\infty} \frac{\lambda_f^2(p^k) \sigma^2(p^k) \phi^2(p^k)}{p^{ks}} \\
= \prod_p \left( 1 + \frac{\lambda_f^2(p) \sigma^2(p) \phi^2(p)}{p^s} + \frac{\lambda_f^2(p^2) \sigma^2(p^2) \phi^2(p^2)}{p^{2s}} + \cdots \right).
\]

From the theory of Hecke operators, we obtain the recursive relation

\[
\lambda_f(p^j) = \lambda_f(p^{j-1}) \lambda_f(p) - \lambda_f(p^{j-2}), \quad j \geq 2.
\]

By induction, we have

\[
\lambda_f(p^j) = \frac{\alpha_f(p^{j+1}) - \beta_f(p^{j+1})}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^{j} \alpha_f(p)^{j-m} \beta_f(p)^m.
\]

Therefore,

\[
f_{1,p}(s) = 1 + \frac{(\alpha_f(p) + \beta_f(p))^7 (p + 1)^b (p - 1)^c}{p^s} \\
\quad + \frac{\left( \frac{\alpha_f^3(p) - \beta_f^3(p)}{\alpha_f(p) - \beta_f(p)} \right)^7}{p^{2s}} (p^2 + p + 1)^b (p^2 - p)^c + \cdots \\
= 1 + \frac{(\alpha_f(p) + \beta_f(p))^7 (p + 1)^b (p - 1)^c}{p^s} \\
\quad + \frac{\left( \alpha_f^2(p) + \alpha_f(p) \beta_f(p) + \beta_f^2(p) \right)^7}{p^{2s}} (p^2 + p + 1)^b (p^2 - p)^c + \cdots \\
= 1 + \frac{(\alpha_f(p) + \beta_f(p))^7}{p^{s-b-c}} + O\left(p^{2(b+c-\sigma)} + p^{(b+c-\sigma-1)}\right).
\]
Then,

\[ F_1(s) = \prod_p f_{1,p}(s) \]

\[ = \prod_p \left( 1 + \frac{(\alpha_f(p) + \beta_f(p))^7}{p^{s+b+c}} + O\left( p^{2(b+c-a)} + p^{(b+c-a-1)} \right) \right) \]

\[ = \prod_p \left( 1 + \frac{\alpha_f^7(p) + 7\alpha_f^2(p) + 21\alpha_f^3(p) + 35\alpha_f(p) + 35\beta_f(p) + 21\beta_f^3(p) + 7\beta_f^2(p)}{p^{s+b+c}} \right. \]

\[ + O\left( p^{2(b+c-a)} + p^{(b+c-a-1)} \right) \]\n
\[ = L(\text{sym}^4 f \times \text{sym}^3 f, s - b - c) \]

\[ \times \prod_p \left( 1 + \frac{5\alpha_f^2(p) + 18\alpha_f^3(p) + 31\alpha_f(p) + 31\beta_f(p) + 18\beta_f^3(p) + 5\beta_f^2(p)}{p^{s+b+c}} \right. \]

\[ + O\left( p^{2(b+c-a)} + p^{(b+c-a-1)} \right) \]

\[ = L(\text{sym}^4 f \times \text{sym}^3 f, s - b - c) L^5(\text{sym}^3 f \times \text{sym}^2 f, s - b - c) \]

\[ \times \prod_p \left( 1 + \frac{8\alpha_f^2(p) + 16\alpha_f^3(p) + 16\beta_f(p) + 8\beta_f^3(p)}{p^{s+b+c}} + O\left( p^{2(b+c-a)} + p^{(b+c-a-1)} \right) \right) \]

\[ = L(\text{sym}^4 f \times \text{sym}^3 f, s - b - c) L^5(\text{sym}^3 f \times \text{sym}^2 f, s - b - c) L^8(\text{sym}^3 f, s - b - c) \]

\[ \times \prod_p \left( 1 + \frac{8\alpha_f(p) + 8\beta_f(p)}{p^{s+b+c}} + O\left( p^{2(b+c-a)} + p^{(b+c-a-1)} \right) \right) \]

\[ = L(\text{sym}^4 f \times \text{sym}^3 f, s - b - c) L^5(\text{sym}^3 f \times \text{sym}^2 f, s - b - c) L^8(\text{sym}^3 f, s - b - c) \]

\[ \times L^1(\text{sym}^3 f, s - b - c) H_1(s), \]

where \( H_1(s) \) is absolutely convergent in \( \text{Re}(s) > b + c + 1/2 \).

The proof of the following lemma is similar to that of Lemma 2.4.

**Lemma 2.5.** Let \( f \in H^*_k \) and \( \lambda_f(n) \) denote its \( n \)-th normalized Fourier coefficient. Define

\[ F_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n) \sigma^b(n) \phi^c(n)}{n^s}. \]

Then \( F_2(s) \) can be factored as

\[ F_2(s) = L(\text{sym}^4 f \times \text{sym}^3 f, s - b - c) L^6(\text{sym}^4 f \times \text{sym}^2 f, s - b - c) \]

\[ \times L^{13}(\text{sym}^4 f, s - b - c) L^{21}(\text{sym}^2 f, s - b - c) \zeta^{13}(s - b - c) H_2(s), \]

where \( H_2(s) \) is absolutely convergent in \( \text{Re}(s) > b + c + 1/2 \).
3. Proofs of Theorems 1-5

The proofs of Theorems 1-5 are similar, so we just prove Theorem 5 in this section.

Proof of Theorem 5. Denote $\lambda_f(n)\sigma^b(n)\phi^c(n)$ by $f_2(n)$. From (1.1), we observe that $f_2(n) \leq B n^{b+c+\varepsilon}$, where $B$ is a real constant on $\varepsilon$. Let

$$L_2(f, s - b - c) = L(sym^4 f \times sym^4 f, s - b - c)L^6(sym^4 f \times sym^2 f, s - b - c)$$

$$\times L^{13}(sym^4 f, s - b - c)L^{21}(sym^2 f, s - b - c)\zeta^{13}(s - b - c).$$

From Lemma 2.5, we learn that $F_2(s) = L_2(f, s - b - c)H_2(s)$.

From the works of Coghell and Mochel [2], Jacquet and Shalika [7] [8], Shahidi [19] [20], and the reformulation of Rudnick and Sarnak [18], symmetric power L-function $L(sym^4 f, s)$ $(1 \leq i \leq 4)$ and Rankin-Selberg L-function $L(sym^4 f \times sym^4 f, s)$ $(1 \leq i < j \leq 4)$ can be extended to be an entire function on the whole complex plane. Lau and Wu [12] showed that for $1 \leq j \leq 4$, $L(sym^4 f \times sym^4 f, s)$ is entire except for simple poles at $s = 0, 1$, and satisfies a functional equation. Thus, $F_2(s)$ can be analytically continued to the half-plane $Re(s) > b + c + \frac{1}{2}$. In this region, $F_2(s)$ only has a pole $s = b + c + 1$ of order 14.

By using Lemma 2.3 (Perron’s formula), we obtain

$$\sum_{n \leq x} \lambda_f(n)^\sigma(n)\phi^c(n) = \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} \sum_{n=1}^\infty f_2(n) \frac{x^s}{ns} ds + O \left( \frac{x^{1+\varepsilon}}{T^{s-b-c-1}} \right) + O \left( \frac{xB(2x)^{b+c+\varepsilon} \log x}{T} \right)$$

$$= \frac{1}{2\pi i} \int_{b+c+1+\varepsilon-iT}^{b+c+1+\varepsilon+iT} L_2(f, s - b - c)H_2(s) \frac{x^s}{s} ds + O \left( \frac{x^{b+c+1+\varepsilon}}{T} \right), \quad (3.1)$$

where $T$ with $1 \leq T \leq x$ is a parameter to be specified later.

Since our goal is to estimate the integral in (3.1), we need to consider the closed contour $\Gamma$:

$$I = [b + c + 1 + \varepsilon - iT, b + c + 1 + \varepsilon + iT],$$

$$II = [b + c + 1 + \varepsilon + iT, b + c + \frac{1}{2} + \varepsilon + iT],$$

$$III = [b + c + \frac{1}{2} + \varepsilon + iT, b + c + \frac{1}{2} + \varepsilon - iT],$$

$$IV = [b + c + \frac{1}{2} + \varepsilon - iT, b + c + 1 + \varepsilon - iT].$$

Let

$$I_1 = \int_I L_2(f, s - b - c)H_2(s) \frac{x^s}{s} ds, \quad I_2 = \int_{II} L_2(f, s - b - c)H_2(s) \frac{x^s}{s} ds,$$

$$I_3 = \int_{III} L_2(f, s - b - c)H_2(s) \frac{x^s}{s} ds, \quad I_4 = \int_{IV} L_2(f, s - b - c)H_2(s) \frac{x^s}{s} ds.$$
By Cauchy’s Residue theorem, we obtain
\[
\frac{1}{2\pi i} I_1 = \frac{1}{2\pi i} \int_1 L_2(f, s - b - c) H_2(s) \frac{x^s}{s} ds - \frac{1}{2\pi i} (I_2 + I_3 + I_4)
\]
\[
= x^{b+c+1} P_{13}(\log x) - \frac{1}{2\pi i} (I_2 + I_3 + I_4),
\]
where \(P_{13}(t)\) is the polynomial of degree 13 in \(t\).

By Lemma 2.5, \(H_2(s)\) converges absolutely in the half-plane \(\text{Re}(s) > b + c + \frac{1}{2}\).
Thus, for the integrals over the horizontal segments, \(I_2\) and \(I_4\) can be estimated as
\[
\ll \int_{\frac{1}{2} + \varepsilon}^{1+\varepsilon} L_2(f, \sigma + iT) \frac{x^{b+c+\varepsilon}}{T^\sigma} d\sigma
\]
\[
\ll x^{b+c} \int_{\frac{1}{2} + \varepsilon}^{1+\varepsilon} L_2(f, \sigma + iT) \frac{x^{\sigma}}{T} d\sigma.
\]
Furthermore, Lemma 2.1 leads to
\[
I_2 + I_4 \ll x^{b+c} \int_{\frac{1}{2} + \varepsilon}^{1+\varepsilon} T^{\left(\frac{13}{2} + \varepsilon\right) \times 6 + \frac{5}{2} \times 13 + \frac{1}{2} \times 21 + \frac{11}{2} \times 13 (1-\sigma) + \varepsilon} \frac{x^{\sigma}}{T^\sigma} d\sigma
\]
\[
\ll x^{b+c} \max_{\frac{1}{2} + \varepsilon \leq \sigma \leq 1+\varepsilon} T^{117} \left(\frac{x}{T^{118}}\right)^{\sigma}
\]
\[
\ll x^{b+c+1+\varepsilon} + x^{b+c+\frac{1}{2} + \varepsilon} T^{58+\varepsilon}.
\]

For the vertical segment, by using Lemma 2.2 and Cauchy’s inequality, \(I_3\) can be estimated as
\[
\ll \int_1^T \left( L \left( \text{sym}^4 f \times \text{sym}^4 f, \frac{1}{2} + \varepsilon + it \right) L^6 \left( \text{sym}^4 f \times \text{sym}^2 f, \frac{1}{2} + \varepsilon + it \right) \right.
\]
\[
\times L^{13} \left( \text{sym}^4 f, \frac{1}{2} + \varepsilon + it \right) L^{21} \left( \text{sym}^2 f, \frac{1}{2} + \varepsilon + it \right) \zeta^{13} \left( \frac{1}{2} + \varepsilon + it \right)
\]
\[
\times H_2(b + c + \frac{1}{2} + \varepsilon + it) \left| x^{b+c+\frac{1}{2} + \varepsilon} \right|_{|b+c+\frac{1}{2} + \varepsilon + it|} dt
\]
\[
\ll x^{b+c+\frac{1}{2} + \varepsilon} \int_1^T \left( L \left( \text{sym}^4 f \times \text{sym}^4 f, \frac{1}{2} + \varepsilon + it \right) L^6 \left( \text{sym}^4 f \times \text{sym}^2 f, \frac{1}{2} + \varepsilon + it \right) \right.
\]
\[
L^{13} \left( \text{sym}^4 f, \frac{1}{2} + \varepsilon + it \right) L^{21} \left( \text{sym}^2 f, \frac{1}{2} + \varepsilon + it \right) \zeta^{13} \left( \frac{1}{2} + \varepsilon + it \right) \left| \frac{1}{t} dt + x^{b+c+\frac{1}{2} + \varepsilon}.
\]

For convenience, we write
\[
L_{2,1}(f, s - b - c) = L(\text{sym}^4 f \times \text{sym}^4 f, s - b - c) L^4(\text{sym}^4 f \times \text{sym}^2 f, s - b - c)
\]
\[
\times L^6(\text{sym}^4 f, s - b - c),
\]
\[
L_{2,2}(f, s - b - c) = L^2(\text{sym}^4 f \times \text{sym}^2 f, s - b - c) L^5(\text{sym}^4 f, s - b - c).
\]
By a dyadic subdivision, we obtain

\[ I_3 \ll x^{b+c+1+\varepsilon} + x^{b+c+\frac{1}{2}+\varepsilon} \log T \max_{1 \leq T_1 \leq T} \frac{1}{T_1} \times \left( \prod_{\frac{1}{2} \leq T_1 \leq T} \left| \zeta^{13} \left( \frac{1}{2} + \varepsilon + it \right) L \left( \text{sym}^4 f, \frac{1}{2} + \varepsilon + it \right) \right| \right) \times \left( \frac{\int_{\frac{T_1}{2}}^{T_1} |L_{2,1}(f, \sigma + iT)|^2 \, dt}{\frac{T_1}{2}} \right)^{\frac{1}{2}} \times \left( \int_{\frac{T_1}{2}}^{T_1} |L_{2,2}(f, \sigma + iT)|^2 \, dt \right)^{\frac{1}{2}} \ll x^{b+c+1+\varepsilon} + x^{b+c+1+\varepsilon} T^{\frac{58}{5}} \ll x^{b+c+\frac{1}{2}+\varepsilon} T^{\frac{58}{5}}. \] (3.6)

Inserting (3.4) and (3.6) into (3.2), we have

\[ \frac{1}{2\pi i} \int_{\frac{b+c+1+\varepsilon + iT}{2}}^{b+c+1+\varepsilon + iT} L_1(f, s - b - c) H_2(s) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \int_{\Gamma} L_1(f, s - b - c) H_2(s) \frac{x^s}{s} \, ds - \frac{1}{2\pi i} (I_2 + I_3 + I_4) = x^{b+c+1} P_{13}(\log x) + O \left( x^{b+c+1+\frac{1}{2}+\varepsilon} T^{\frac{58}{5}} \right) + O \left( x^{b+c+1+\varepsilon} T \right). \]

Setting \( T = x^{\frac{1}{\varepsilon}} \), we obtain

\[ S_2(x) = x^{b+c+1} P_{13}(\log x) + O \left( x^{b+c+1+\frac{1}{10}} + \varepsilon \right). \]

This completes the proof of Theorem 5. \( \square \)

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References


