AN EXTENSION OF A FORMULA OF JOVOVIC

Shane Chern
Dept. of Mathematics, Penn State University, University Park, Pennsylvania
shanechern@psu.edu

Received: 12/28/18, Revised: 4/29/19, Accepted: 8/3/19, Published: 9/30/19

Abstract

Let $u > v$ be two positive integers and c be an integer. In this note, we give a closed form of the number of subsets of $\{1, 2, \ldots, un-1\}$ of vn elements whose sum is congruent to c modulo n. This extends a formula of Vladeta Jovovic.

1. Introduction

A result of Erdős, Ginzburg and Ziv [3] states as follows.

Theorem. *Each set of $2n-1$ integers contains some subset of n elements the sum of which is a multiple of n.*

In particular, if the set of $2n-1$ integers is $\{1, 2, \ldots, 2n-1\}$, Vladeta Jovovic conjectured a closed form of the number of such n-member subsets, denoted by $s(n)$:

$$s(n) = \frac{(-1)^n}{2n} \sum_{d|n} (-1)^d \phi\left(\frac{n}{d}\right) \binom{2d}{d}, \tag{1}$$

where the summation runs over all positive divisors of n, $\phi(\cdot)$ is Euler's totient function, and $\binom{2d}{d}$ is the d-th central binomial coefficient. This formula was later confirmed by Max Alekseyev [1], whose proof relies on heavy computation on trigonometric functions. We remark that $s(n)$ is sequence A145855 in the OEIS [5].

In this note, we not only provide an alternative proof of (1) that avoids the computation involving trigonometric functions in Alekseyev's proof, but also give the following extension.

Theorem 1.1. *Let $u > v$ be two positive integers and c be an integer. Let n be a positive integer. If $s(u, v; c; n)$ counts the number of subsets of $\{1, 2, \ldots, un-1\}$ of* *vn elements whose sum is congruent to c modulo n, then*

$$s(u,v;c;n) = \begin{cases} \dfrac{u-v}{un} \displaystyle\sum_{d|n} \mu\left(\dfrac{n}{d \cdot \gcd\left(c, \frac{n}{d}\right)}\right) \dfrac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d \cdot \gcd\left(c, \frac{n}{d}\right)}\right)} \dbinom{ud}{vd} \\ \qquad\qquad \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \dfrac{u-v}{un} \displaystyle\sum_{d|n} (-1)^d \mu\left(\dfrac{n}{d \cdot \gcd\left(c, \frac{n}{d}\right)}\right) \dfrac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d \cdot \gcd\left(c, \frac{n}{d}\right)}\right)} \dbinom{ud}{vd} \\ \qquad\qquad \text{if } n \text{ is even and } v \text{ is odd,} \end{cases} \tag{2}$$

where $\mu(\cdot)$ is the Möbius function and $\phi(\cdot)$ is Euler's totient function. We also adopt the convention that $\gcd(0,n) = n$.

3-mem.	Σ mod 3	3-mem.	Σ mod 3
$\{1,2,3\}$	0	$\{1,4,5\}$	1
$\{1,2,4\}$	1	$\{2,3,4\}$	0
$\{1,2,5\}$	2	$\{2,3,5\}$	1
$\{1,3,4\}$	2	$\{2,4,5\}$	2
$\{1,3,5\}$	0	$\{3,4,5\}$	0

(a) $n = 3$, $u = 2$, $v = 1$

2-mem.	Σ mod 2	2-mem.	Σ mod 2
$\{1,2\}$	1	$\{2,4\}$	0
$\{1,3\}$	0	$\{2,5\}$	1
$\{1,4\}$	1	$\{3,4\}$	1
$\{1,5\}$	0	$\{3,5\}$	0
$\{2,3\}$	1	$\{4,5\}$	1

(b) $n = 2$, $u = 3$, $v = 1$

4-mem.	Σ mod 2
$\{1,2,3,4\}$	0
$\{1,2,3,5\}$	1
$\{1,2,4,5\}$	0
$\{1,3,4,5\}$	1
$\{2,3,4,5\}$	0

(c) $n = 2$, $u = 3$, $v = 2$

Table 1: 2-, 3- and 4-Member subsets of $\{1,2,3,4,5\}$

In Table 1, we give three examples to illustrate Theorem 1.1. Here "k-mem." means k-member subsets of $\{1,2,3,4,5\}$ and "Σ mod d" means the sum of elements in the subset modulo d.

Examples. (i). Let $n = 3$, $u = 2$ and $v = 1$. Then we are considering 3-member subsets of $\{1,2,3,4,5\}$, which are listed in Table 1a. On the other hand, it follows from Theorem 1.1 that $s(2,1;0;3) = 4$, $s(2,1;1;3) = 3$ and $s(2,1;2;3) = 3$.

(ii). Let $n = 2$, $u = 3$ and $v = 1$. Then we are considering 2-member subsets of $\{1,2,3,4,5\}$. One may compute from Theorem 1.1 that $s(3,1;0;2) = 4$ and $s(3,1;1;2) = 6$. The two values match with Table 1b.

(iii). Let $n = 2$, $u = 3$ and $v = 2$. Then we are considering 4-member subsets of $\{1, 2, 3, 4, 5\}$. It can be computed by Theorem 1.1 that $s(3, 2; 0; 2) = 3$ and $s(3, 2; 1; 2) = 2$, which, again, match with Table 1c.

Further, if we take $c = 0$ and 1 respectively in (2), we get the following corollaries.

Corollary 1.2. *Let $u > v$ be two positive integers. The number of subsets of $\{1, 2, \ldots, un - 1\}$ of vn elements which sum to a multiple of n equals*

$$\begin{cases} \dfrac{u-v}{un} \displaystyle\sum_{d|n} \phi\left(\frac{n}{d}\right) \binom{ud}{vd} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \dfrac{u-v}{un} \displaystyle\sum_{d|n} (-1)^d \phi\left(\frac{n}{d}\right) \binom{ud}{vd} & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases} \tag{3}$$

Corollary 1.3. *Let $u > v$ be two positive integers. The number of subsets of $\{1, 2, \ldots, un-1\}$ of vn elements which sum to one more than a multiple of n equals*

$$\begin{cases} \dfrac{u-v}{un} \displaystyle\sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{ud}{vd} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \dfrac{u-v}{un} \displaystyle\sum_{d|n} (-1)^d \mu\left(\frac{n}{d}\right) \binom{ud}{vd} & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases} \tag{4}$$

Remark. One may further take $u = 2$ and $v = 1$ in (3) to deduce Jovovic's formula (1).

2. Proof of the Main Result

We begin with some combinatorial arguments. Let $\{x_1, x_2, \ldots, x_{vn}\}$ be a subset of $\{1, 2, \ldots, un - 1\}$ with $1 \le x_1 < x_2 < \cdots < x_{vn} \le un - 1$. For each $1 \le i \le vn$, we write $x_i = x_i' + i$. It follows that $0 \le x_1' \le x_2' \le \cdots \le x_{vn}' \le (u - v)n - 1$. Hence, the numbers $x_1', x_2', \ldots, x_{vn}'$ uniquely determine a partition with the largest part not exceeding $(u - v)n - 1$ and the number of parts not exceeding vn. Here, as usual, a partition of a nonnegative integer N is a weakly decreasing sequence of positive integers (which are called the parts) that sum to N.

It is a standard result (cf. Chapter 3 in [2]) in the theory of partitions that the generating function of partitions into at most M parts, each not exceeding L, is given by the conventional q-binomial coefficient

$$\begin{bmatrix} M+L \\ M \end{bmatrix}_q := \prod_{k=1}^{M} \frac{1-q^{L+k}}{1-q^k} = \prod_{k=1}^{L} \frac{1-q^{M+k}}{1-q^k}.$$

We remark that it is indeed a polynomial in q of degree LM. Hence, the generating function of partitions with the largest part not exceeding $(u-v)n-1$ and the number of parts not exceeding vn is

$$R(q) = \sum_{m=0}^{vn((u-v)n-1)} r(m)q^m := \begin{bmatrix} un-1 \\ vn \end{bmatrix}_q . \tag{5}$$

Let $G(q) = \sum_{n\geq 0} g(n)q^n$ be a formal power series. From the orthogonality of roots of unity, we have the following identity

$$\sum_{\substack{n\geq 0 \\ n\equiv h \pmod H}} g(n)q^n = \frac{1}{H}\sum_{\ell=1}^{H} e^{-\frac{2\pi ih\ell}{H}} G\left(e^{\frac{2\pi i\ell}{H}}q\right). \tag{6}$$

To determine the expression of $s(u,v;c;n)$, we notice that $\sum_{i=1}^{vn} x_i \equiv c \pmod n$ is equivalent to

$$\sum_{i=1}^{vn} x_i' \equiv \begin{cases} c \pmod n & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \frac{n}{2}+c \pmod n & \text{if } n \text{ is even and } v \text{ is odd,} \end{cases}$$

since

$$\begin{aligned} 1+2+\cdots+vn &= \frac{vn(vn+1)}{2} \\ &\equiv \begin{cases} 0 \pmod n & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \frac{n}{2} \pmod n & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases} \end{aligned}$$

As a consequence, we have that

$$\begin{aligned} s(u,v;c;n) &= \begin{cases} \displaystyle\sum_{\substack{m=0 \\ m\equiv c \pmod n}}^{vn((u-v)n-1)} r(m) & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \displaystyle\sum_{\substack{m=0 \\ m\equiv \frac{n}{2}+c \pmod n}}^{vn((u-v)n-1)} r(m) & \text{if } n \text{ is even and } v \text{ is odd} \end{cases} \\ &= \begin{cases} \displaystyle\frac{1}{n}\sum_{\ell=1}^{n} e^{-\frac{2\pi ic\ell}{n}} R\left(e^{\frac{2\pi i\ell}{n}}\right) & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \displaystyle\frac{1}{n}\sum_{\ell=1}^{n} (-1)^\ell e^{-\frac{2\pi ic\ell}{n}} R\left(e^{\frac{2\pi i\ell}{n}}\right) & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases} \end{aligned} \tag{7}$$

In the next lemma, we evaluate $R(e^{2\pi i\ell/n})$.

Lemma 2.1. *Let ℓ be a positive integer. Then*

$$R\left(e^{\frac{2\pi i\ell}{n}}\right) = \frac{u-v}{u}\binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)}. \tag{8}$$

Proof. Recall that

$$R(q) = \begin{bmatrix} un-1 \\ vn \end{bmatrix}_q = \prod_{k=1}^{(u-v)n-1} \frac{1-q^{vn+k}}{1-q^k}.$$

Hence,

$$R\left(e^{\frac{2\pi i\ell}{n}}\right) = \prod_{k=1}^{(u-v)n-1} \lim_{q\to e^{\frac{2\pi i\ell}{n}}} \frac{1-q^{vn+k}}{1-q^k},$$

where

$$\lim_{q\to e^{\frac{2\pi i\ell}{n}}} \frac{1-q^{vn+k}}{1-q^k} = \begin{cases} 1 & \text{if } \left(e^{\frac{2\pi i\ell}{n}}\right)^k \neq 1, \\ \frac{vn+k}{k} & \text{if } \left(e^{\frac{2\pi i\ell}{n}}\right)^k = 1. \end{cases}$$

Furthermore, $\left(e^{\frac{2\pi i\ell}{n}}\right)^k = 1$ implies that n divides $k\ell$.

For convenience, we write $n = n'd$ and $\ell = \ell'd$ where $d = \gcd(\ell, n)$. Then $n \mid k\ell$ is equivalent to $n' \mid k$. Hence,

$$\begin{aligned} \prod_{k=1}^{(u-v)n-1} \lim_{q\to e^{\frac{2\pi i\ell}{n}}} \frac{1-q^{vn+k}}{1-q^k} &= \prod_{\substack{k=1 \\ k\equiv 0 \pmod{n'}}}^{(u-v)n-1} \frac{vn+k}{k} = \prod_{k'=1}^{(u-v)d-1} \frac{vn+k'n'}{k'n'} \\ &= \prod_{k'=1}^{(u-v)d-1} \frac{vd+k'}{k'} = \binom{ud-1}{(u-v)d-1} = \frac{u-v}{u}\binom{ud}{vd}, \end{aligned}$$

where in the second identity we set $k = k'n'$.

Consequently, we have that

$$R\left(e^{\frac{2\pi i\ell}{n}}\right) = \frac{u-v}{u}\binom{ud}{vd} = \frac{u-v}{u}\binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)},$$

which is our desired result. $\square$

If n is odd or n and v are both even, then it follows from (7) and (8) that

$$\begin{aligned}
s(u,v;c;n) &= \frac{1}{n}\sum_{\ell=1}^{n} e^{-\frac{2\pi i c\ell}{n}} R\left(e^{\frac{2\pi i \ell}{n}}\right)\\
&= \frac{u-v}{un}\sum_{\ell=1}^{n} e^{-\frac{2\pi i c\ell}{n}} \binom{u\cdot \gcd(\ell,n)}{v\cdot \gcd(\ell,n)}\\
&= \frac{u-v}{un}\sum_{d\mid n}\binom{ud}{vd}\sum_{\substack{\ell=1\\ \gcd(\ell,n)=d}}^{n} e^{-\frac{2\pi i c\ell}{n}}\\
&= \frac{u-v}{un}\sum_{d\mid n}\binom{ud}{vd}\sum_{\substack{\ell'=1\\ \gcd\left(\ell',\frac{n}{d}\right)=1}}^{\frac{n}{d}} \exp\left(-\frac{2\pi i c\ell'}{n/d}\right)\\
&= \frac{u-v}{un}\sum_{d\mid n}\mu\left(\frac{n}{d\cdot \gcd\left(c,\frac{n}{d}\right)}\right)\frac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)}\binom{ud}{vd},
\end{aligned}$$

where in the last identity we use the following evaluation of Ramanujan's sum (cf. Theorem 272 in [4])

$$c_q(m) := \sum_{\substack{\ell=1\\ \gcd(\ell,q)=1}}^{q} \exp\left(\frac{2\pi i m\ell}{q}\right) = \mu\left(\frac{q}{\gcd(m,q)}\right)\frac{\phi(q)}{\phi\left(\frac{q}{\gcd(m,q)}\right)}.$$

To show the second case of the main result, we have that if n is even and v is odd, then

$$s(u,v;c;n) = \frac{1}{n}\sum_{\ell=1}^{n}(-1)^{\ell} e^{-\frac{2\pi i c\ell}{n}} R\left(e^{\frac{2\pi i \ell}{n}}\right) = \frac{u-v}{un}\sum_{\ell=1}^{n}(-1)^{\ell} e^{-\frac{2\pi i c\ell}{n}}\binom{u\cdot \gcd(\ell,n)}{v\cdot \gcd(\ell,n)}.$$

Since n is even, we see that ℓ and $\gcd(\ell,n)$ have the same parity. Hence, $(-1)^{\ell} = (-1)^{\gcd(\ell,n)}$. Following the same argument as above, we conclude that

$$\begin{aligned}
s(u,v;c;n) &= \frac{u-v}{un}\sum_{\ell=1}^{n}(-1)^{\gcd(\ell,n)} e^{-\frac{2\pi i c\ell}{n}}\binom{u\cdot \gcd(\ell,n)}{v\cdot \gcd(\ell,n)}\\
&= \frac{u-v}{un}\sum_{d\mid n}(-1)^{d}\binom{ud}{vd}\sum_{\substack{\ell=1\\ \gcd(\ell,n)=d}}^{n} e^{-\frac{2\pi i c\ell}{n}}\\
&= \frac{u-v}{un}\sum_{d\mid n}(-1)^{d}\mu\left(\frac{n}{d\cdot \gcd\left(c,\frac{n}{d}\right)}\right)\frac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)}\binom{ud}{vd}.
\end{aligned}$$

We therefore arrive at the desired result.

3. Closing Remarks

If we replace the set $\{1, 2, \ldots, un-1\}$ in Theorem 1.1 by an arbitrary set of $un-1$ consecutive integers, say $\{a+1, a+2, \ldots, a+un-1\}$, the formula in (2) still holds. In fact, if $\{x_1, x_2, \ldots, x_{vn}\}$ (with $x_1 < x_2 < \cdots < x_{vn}$) is a subset, we may put $x_i = x_i' + a + i$ for each $1 \leq i \leq vn$ and then carry out the same procedure as in Section 2.

For a general set $\mathcal{S}$ of $2n-1$ integers in the theorem of Erdős, Ginzburg and Ziv stated at the beginning of this note, there seems to be no closed form that enumerates the number of n-member subsets whose elements sum to a multiple of n. Apparently, our approach would fail for general $\mathcal{S}$ as the initial combinatorial argument in Section 2 now does not make sense. On the other hand, following Alekseyev [1], we know that the number of such subsets is

$$-1 + \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \prod_{s \in \mathcal{S}} \left(1 + \exp\left(\frac{2\pi i(j+ks)}{n}\right)\right).$$

However, there is no obvious approach that can simplify this expression.

It would be an intriguing problem to find other examples of $\mathcal{S}$ that give nice closed forms of the number of the desired n-member subsets, or even may lead to parallel extensions like our Theorem 1.1.

Acknowledgements. I would like to thank the referee for helpful suggestions.

References

[1] M. Alekseyev, Proof of Jovovic's formula, unpublished manuscript (2008). Available at `http://oeis.org/A145855/a145855.txt`.

[2] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley Publishing Co., Reading, 1976.

[3] P. Erdős, A. Ginzburg, and A. Ziv, Theorem in the additive number theory, *Bull. Res. Counc. Israel Sect. F Math. Phys.* **10F** (1961), no. 1, 41–43.

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers. Sixth Edition*, Oxford University Press, Oxford, 2008.

[5] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.