HECKE GROUPS, LINEAR RECURRENCES, AND KEPLER LIMITS

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Abstract

We study the linear fractional transformations in the Hecke group $G(\Phi)$ where Φ is either root of $x^2 - x - 1$ (the larger root being the "golden ratio" $\phi = 2\cos\frac{\pi}{5}$.) Let $g \in G(\Phi)$ and let z be a generic element of the upper half-plane. Exploiting the fact that $\Phi^2 = \Phi + 1$, we find that $g(z)$ is a quotient of linear polynomials in z such that the coefficients of z^1 and z^0 in the numerator and denominator of $g(z)$ appear themselves to be linear polynomials in Φ with coefficients that are certain multiples of Fibonacci numbers. We make somewhat less detailed observations along similar lines about the functions in $G(2\cos\frac{\pi}{k})$ for k greater than or equal to 5.

1. Introduction

Let $G(\lambda)$ be the Hecke group generated by the linear fractional transformations $S\colon z \mapsto -1/z$ and $T_\lambda\colon z \mapsto z + \lambda$ and let $G_k = G(2\cos\frac{\pi}{k})$. This article describes numerical experiments carried out to study Hecke groups, mainly G_k for $k \geq 5$. In this article, an n-tuple of symbols

$$\overrightarrow{w} = \{w_1, w_2, w_3, ..., w_n\}$$

representing an ordered set of integers is called a *word* on $\mathbf{Z}$ and we write $|\overrightarrow{w}|$ for n. A typical element of $G(\lambda)$ takes the form

$$T_\lambda^{w_1} S T_\lambda^{w_2} S ... S T_\lambda^{w_n} = g_{\lambda;\overrightarrow{w}}.$$

This representation is not unique. For example, a function g in $G(\lambda)$ can be described by a word of length n for arbitrarily large n, because any word representing g can be padded with zeros and the resulting word will also represent g. Consequently, when studying all g represented by words $\overrightarrow{w}$ with $|\overrightarrow{w}|$ less than or equal to N, we can restrict attention to the words w such that $|\overrightarrow{w}|$ is equal to N.

Let ϕ, ϕ^*, represent the larger and smaller roots of $x^2 - x - 1$, respectively. The problem of expressing (for $g \in G_5$) $g = g_{\phi;\overrightarrow{w}}$ in terms of the w_i was raised by Leo in

[9] and discussed by his student Sherkat in [12]; the first purpose of this article is to write down conjectures addressing this question. Our calculations indicate that, for arbitrary $\lambda, z \in \mathbf{C}$, $g_{\lambda;\overrightarrow{w}}(z)$ is a rational function of z and λ in polynomials of λ-degree less than or equal to k. Here are the first few:

$$g_{\lambda;\{w_1\}}(z) = \frac{1 \cdot z + w_1\lambda}{0 \cdot z + 1},$$

$$g_{\lambda;\{w_1,w_2\}}(z) = \frac{w_1\lambda \cdot z + w_1w_2\lambda^2 - 1}{1 \cdot z + w_2\lambda},$$

and

$$g_{\lambda;\{w_1,w_2,w_3\}}(z) = \frac{(w_1w_2\lambda^2 - 1) \cdot z + w_1w_2w_3\lambda^3 - (w_1 + w_3)\lambda}{w_2\lambda \cdot z + w_2w_3\lambda^2 - 1}.$$

Following [12], we simplify the above expressions for $g_{\lambda;\overrightarrow{w}}$ when λ is ϕ or ϕ^* by repeatedly making the substitution $\Phi^2 = \Phi + 1$ ($\Phi = \phi$ or $\Phi = \phi^*$.) The coefficients in $g_{\Phi;\overrightarrow{w}}$ become linear polynomials in Φ:

$$g_{\Phi;\{w_1\}}(z) = \frac{1 \cdot z + w_1\Phi}{0 \cdot z + 1},$$

$$g_{\Phi;\{w_1,w_2\}}(z) = \frac{w_1\Phi \cdot z + w_1w_2\Phi + w_1w_2 - 1}{1 \cdot z + w_2\Phi},$$

and

$$g_{\Phi;\{w_1,w_2,w_3\}}(z) = \frac{(w_1w_2\Phi + w_1w_2 - 1) \cdot z + (2w_1w_2w_3 - w_1 - w_3)\Phi + w_1w_2w_3}{w_2\Phi \cdot z + w_2w_3\Phi + w_2w_3 - 1}.$$

Further calculations suggest that the coefficients of Φ^1 and Φ^0 in these expressions are always linear combinations of first-degree monomials h in the w_i such that the numerical coefficient of h is ± 1 times a Fibonacci number determined by the total degree of h; details are in the next section.

It is well known, of course, that the consecutive ratios F_n/F_{n-1} of Fibonacci numbers converge to ϕ. More generally, the limit of the ratios of consecutive elements of a linear recurrence L, when it exists, is called by Fiorenza and Vincenzi the *Kepler limit* of L. Certain roots of polynomials other than $x^2 - x - 1$ are also Kepler limits [5, 6], so we are led to consider the possibility that the G_5 phenomenon generalizes; our observations tend to confirm this guess. Section 2 describes what we found out about G_5, Section 3 describes less detailed observations for G_k with k between 5 and 33 (inclusive), and the final section provides some detail about our numerical experiments; documentation in the form of *Mathematica* notebooks is on our ResearchGate site for this article [4].

We state merely empirical claims in this article. We make several conjectures, but they, too, are based on empirical evidence, not on theoretical reasoning. When we say we have observed convergence of a sequence s_n (say) to a limit S, we mean that our plots of 1000 values of $\log|S - s_n|$ are apparently linear, with negative slope. We rely on our eyesight in this matter: we have not fitted our data to curves with a statistical package. Interested readers are invited to inspect the plots on our ResearchGate pages.

In the following section our observations were made on words in W of length 20, and those in the last section tested words of length 25. This means that we have in fact tested the claims on all shorter words as well.

In our tests, we identified functions in the G_k with their matrix representations: a function

$$T_\lambda^{w_1} S T_\lambda^{w_2} S ... S T_\lambda^{w_n}$$

was identified with the corresponding matrix product.

More information about the Hecke groups is available, for example, in [2].

Remark 1. The book [7] by Khovanskii apparently describes another method for approximating roots of polynomials using convergent sequences of ratios of elements of numerical sequences; but these sequences are not linear recurrences. (We have not seen [7], but Khovanskii's's method is described in [10], where the book is cited.)

2. The Group G_5

We make the following definitions.

1. The Fibonacci numbers are defined with the convention that $F_0 = 0, F_1 = F_2 = 1$, *etc.* It will be convenient to write $F_{-1} = 1$ as well in contexts where (see below) $\overrightarrow{s} = \emptyset$.

2. χ is the following Dirichlet character:

$$\chi(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Alternatively, with $(a|b)$ representing the Kronecker symbol, $\chi(n) = (n|2)$ if $n \equiv 0, 1, 2, 3, 4 \text{ or } 6 \pmod 8$, and $\chi(n) = -(n|2)$ otherwise.

3. W is the set of words $\overrightarrow{w}$ on $\mathbf{Z}$. The empty word $\overrightarrow{\emptyset}$ verifies $\overrightarrow{\emptyset} \in W$ and $\overrightarrow{\emptyset} \subset \overrightarrow{w}$ for any $\overrightarrow{w} \in W$.

4. We write the cardinal number of a set σ as $|\sigma|$. We apply the same notation to words $\overrightarrow{w}$ in W. We write $|\overrightarrow{\emptyset}| = 0$.

5. (a) For $\overrightarrow{w} \in W, \overrightarrow{w} \neq \overrightarrow{\emptyset}$, let $\overrightarrow{s} = \{w_{j_1}, w_{j_2}, ..., w_{j_m}\} \subset \overrightarrow{w} = \{w_1, ..., w_n\}$. If all of the $j_m \equiv m \pmod 2$,

then we write

$$\overrightarrow{s} \ll_1 \overrightarrow{w}.$$

We also write $\overrightarrow{\emptyset} \ll_1 \overrightarrow{w}$.

(b) If $\overrightarrow{s} \ll_1 \overrightarrow{w}$ and $|\overrightarrow{s}| > 1$, we write

$$\overrightarrow{s} \ll_2 \overrightarrow{w}.$$

(c) Let $\overrightarrow{s}, \overrightarrow{w}$ be as in definition 5a, except that all of the j_m satisfy $j_m \equiv m - 1 \pmod 2$. Then we write

$$\overrightarrow{s} \ll_3 \overrightarrow{w}.$$

We also write $\overrightarrow{\emptyset} \ll_3 \overrightarrow{w}$.

(d) If $\overrightarrow{s} \ll_3 \overrightarrow{w}$ and $|\overrightarrow{s}| > 1$, we write $\overrightarrow{s} \ll_4 \overrightarrow{w}$.

6. (a) For $\overrightarrow{w} \in W$, the formal product

$$m_{\overrightarrow{w}} = \prod_{w_i \in \overrightarrow{w}} w_i.$$

We also write

$$m_{\overrightarrow{\emptyset}} = 1.$$

(b) $M_{\overrightarrow{w}}$ is the set of all linear combinations with coefficients in the integers of monomials $m_{\overrightarrow{s}}$ such that $\overrightarrow{s} \subset \overrightarrow{w}$.

(c) M is the union of the $M_{\overrightarrow{w}}$ as $\overrightarrow{w}$ ranges over W.

Remark 2. In view of the identities $\Phi^2 = \Phi + 1$ for $\Phi = \phi$ or ϕ^*, it is clear that

(i) For each j between 1 and 8 (inclusive), there is a function $f_j : W \to M$ such that $f_j(\overrightarrow{w}) \in M_{\overrightarrow{w}}$ and, for all $g_{\Phi;\overrightarrow{w}} \in G(\Phi)$ and z with $\Im z$ positive,

$$g_{\Phi;\overrightarrow{w}}(z) = \frac{(f_1(\overrightarrow{w})\Phi + f_2(\overrightarrow{w}))z + f_3(\overrightarrow{w})\Phi + f_4(\overrightarrow{w})}{(f_5(\overrightarrow{w})\Phi + f_6(\overrightarrow{w}))z + f_7(\overrightarrow{w})\Phi + f_8(\overrightarrow{w})}.$$

Referring to the introduction, for example:

$$f_3(w_1, w_2, w_3) = 2w_1w_2w_3 - w_1 - w_3$$

and

$$f_6(w_1, w_2, w_3) = 0.$$

(ii) For each j between 1 and 8 (inclusive), there is a function $\nu_j : W \times W \mapsto \mathbf{Z}$ determined by the condition

$$f_j(\overrightarrow{w}) = \sum_{\overrightarrow{s} \subset \overrightarrow{w}} \nu_j(\overrightarrow{s}, \overrightarrow{w}) m_{\overrightarrow{s}}.$$

The following observations describe the functions represented by words of length less than or equal to 20 in G_k for k between 5 and 50 (inclusive.)

Observation 1. (a)

$$\nu_1(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|} & \text{if } \overrightarrow{s} \ll_1 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$\nu_2(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_1 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$\nu_3(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} -\chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|} & \text{if } \overrightarrow{s} \ll_1 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(d)

$$\nu_4(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} -\chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_2 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(e)

$$\nu_5(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_3 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(f)

$$\nu_6(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}| - 1)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_4 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(g)

$$\nu_7(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|} & \text{if } \overrightarrow{s} \ll_3 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

(h)

$$\nu_8(\overrightarrow{s}, \overrightarrow{w}) = \begin{cases} \chi(|\overrightarrow{w}| - |\overrightarrow{s}|)F_{|\overrightarrow{s}|-1} & \text{if } \overrightarrow{s} \ll_3 \overrightarrow{w}, \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 1. Observation 1 holds for words of arbitrary length and all k greater than or equal to 5.

3. Higher-order Hecke Groups

Definition. Let $t(x)$ be a polynomial $\sum_{j=0}^{d} a_j x^j$ and $\gamma(t) := \gcd\{j \text{ s.t. } a_j \neq 0\}$. If $\gamma(t) = 1$, we say that t is *stable* . Whether or not t is stable, we associate to it the family of d^{th}-order linear recurrences Λ_t with kernel $\{-a_{d-1}, -a_{d-2}, ..., -a_0\}$. Let $\lambda = 2\cos\frac{\pi}{k}$ with minimal polynomial $p_\lambda = p$ (say.) Under certain conditions [5, 6], a root $x = \kappa_p$ of $p(x)$ is the Kepler limit of one of the $L_p \in \Lambda_p$. The elements of $G(\lambda) = G_k$ have the form

$$g_{\lambda;\overrightarrow{w}}(z) = \frac{(\sum_{j=0}^{d-1} f_{\lambda,1,j}(\overrightarrow{w})\lambda^j) \cdot z + \sum_{j=0}^{d-1} f_{\lambda,2,j}(\overrightarrow{w})\lambda^j}{(\sum_{j=0}^{d-1} f_{\lambda,3,j}(\overrightarrow{w})\lambda^j) \cdot z + \sum_{j=0}^{d-1} f_{\lambda,4,j}(\overrightarrow{w})\lambda^j}. \tag{1}$$

(Equation (1) is clear, as in the G_5 case, by substitution.)

For pragmatic reasons, we restrict our attention to $f = f_{\lambda,1,0}$ in the following observations.

Observation 2. For k between 5 and 500 (inclusive), $\gamma(p) = 1$ if k is odd and $\gamma(p) = 2$ if k is even.

Conjecture 2. For polynomials of the form $p = p_\lambda$, the statements in the above observation hold for all k greater than or equal to 5.

Observation 3. Let k lie between 5 and 33 (inclusive).

(a) There is a function $\nu^{(k)}\colon W \times W \mapsto \mathbf{Z}$ such that

$$f(\overrightarrow{w}) = \sum_{\overrightarrow{s} \subset \overrightarrow{w}} \nu^{(k)}(\overrightarrow{s}, \overrightarrow{w}) m_{\overrightarrow{s}}. \tag{2}$$

with

$$\overrightarrow{s_1}, \overrightarrow{s_2} \subset \overrightarrow{w}$$

and

$$|\overrightarrow{s_1}| = |\overrightarrow{s_2}| \Rightarrow |\nu^{(k)}(\overrightarrow{s_1}, \overrightarrow{w})| = |\nu^{(k)}(\overrightarrow{s_2}, \overrightarrow{w})| \tag{3}$$

for all $\overrightarrow{w} \in W$ s.t. $|\overrightarrow{w}| = 25$.

(b) If k is odd, then for some particular $L_p \in \Lambda_p$ and all $\overrightarrow{s} \subset \overrightarrow{w}$ s.t. $|\overrightarrow{w}| = 25$:

(b1) $|\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})| \in L_p$ and (b2) $\kappa_p = \lambda$.

(In our experiments the sum on the right-hand side of Equation (2) typically contains over 6×10^4 terms, but twelve or fewer distinct values of $|\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})|$.)

(c) Suppose k is an even number between 6 and 32 (inclusive.) Then

(c1) clause (b1) still holds, but (b2) does not; in this situation, we found no L_p for which κ_p exists. (By design, our searches stop with the first instance of L_p satisfying (a), so this is far from decisive.)

(c2) For $k = 8, 10, 14, 16, 18, 22, 26$, and 32, the ratios of consecutive elements of the L_p we found in the experiments form two convergent sub-sequences with different limits.

(c3) For $k = 6, 12, 20, 24, 28$, and 30, the L_p terminate in a sequence in which alternate members are zero, so that the requisite ratios are alternately zero or undefined.

(d) Suppose $k = 12, 14, 20, 22, 24, 28$, or 30. After a substitution $y = x^2$, $p(x)$ is transformed to a stable polynomial $q_{\lambda^2}(y) = q(y)$ (say), and then λ^2 is the Kepler limit of a linear recurrence $L_q \in \Lambda_q$ containing the $|\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})|$.

Conjecture 3. In the above observation, clause (a) holds for all k greater than or equal to 5, clause (b) holds for all odd k greater than or equal to 5, and clause (c1) holds for all even k greater than or equal to 6. One of clauses (c2) or (c3) holds for any even $k \geq 6$. Clause (d) holds for an unbounded set of even k greater than or equal to 6.

The conditions on polynomials under which linear recurrences with Kepler limits that are killed by them were established in [11] (cited in [5]).

A procedure (which can be invoked from computer algebra systems) for computing $p = p_\lambda$ for $\lambda = 2\cos\frac{\pi}{k}$ one at a time for individual k appeared in [2]; some information about the constant terms, in [1]; and, about the degree, in [8].

4. Data on the Linear Recurrences

4.1. Coefficients

This is a list of distinct coefficents of the $m_{\overrightarrow{s}}$ appearing in our calculations for Equation (2), (k between 5 and 33, inclusive):

5: 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657

6: 1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441

7: 1, 4, 14, 47, 155, 507, 1652, 5373, 17460, 56714, 184183

8: 1, 2, 8, 28, 96, 328, 1120, 3824, 13056, 44576, 152192, 519616

9: 1, 6, 27, 109, 417, 1548, 5644, 20349, 72846, 259579

10: 1, 5, 25, 100, 375, 1375, 5000, 18125, 65625, 237500, 859375, 3109375

11: 1, 6, 27, 110, 429, 1637, 6172, 23104, 86090, 319792

12: 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 564719

13: 1, 6, 27,110, 429, 1638, 6188, 23255, 87190, 326646

14: 1, 7, 49, 245, 1078, 4459, 17836, 69972, 271313, 1044435, 4002467

15: 1, 5, 20, 74, 265, 936, 3290, 11560, 40699, 143755, 509771

16: 1, 2, 16, 88, 416, 1820, 7616, 31008, 124032, 490312

17: 1, 8, 44, 208, 910, 3808, 15504, 62016, 245157

18: 1, 3, 18, 81, 333, 1323, 5184, 20196, 78489, 304722, 1182519

19: 1, 10, 65, 350, 1700, 7752, 33915, 144210

20: 1, 8, 45, 220, 1000, 4352, 18411, 76380, 312455

21: 1, 7, 35, 154, 636, 2534, 9877, 37962, 144571, 547239

22: 1, 11, 121, 847, 4840, 24684, 117249, 531069, 2326588

23: 1, 12, 90, 544, 2907, 14364, 67298

24: 1, 8, 44, 208, 911, 3824, 15656, 63136, 252241

25: 1, 10, 65, 350, 1700, 7752, 33915, 144210

26: 1, 13, 169, 1352, 8619, 48165, 247247, 1197196

27: 1, 18, 189, 1518

28: 1, 12, 91, 560, 3059, 15484, 74382

29: 1, 14, 119, 798, 4655, 24794

30: 1, 7, 35, 155, 650, 2653, 10676, 42635, 169555

31: 1, 16,152, 1120, 7084

32: 1 , 2, 32, 304, 2240, 14168

33: 1, 11, 77, 440, 2244, 10659, 48278, 211486

4.2. Initial Segments for Observations 3a - 3c

This section describes the results of a search for initial segments I of linear recurrences $L_p(p = p_\lambda, \lambda = 2\cos\frac{\pi}{k})$ with length equal to that of the kernel of L_p (so that I determines L_p) such that a sufficiently long initial segment of L_p contains the elements listed above for corresponding k.

5: {0, 1}

6: {0, 1}

7: {0, 0, 1}

8: {0, 0, 1, 2}

9: {0, 0, 1}

10: {0, 0, 1, 5}

11: {0, 0, 0, 0, 1}

12: {0, 0, 0, 1}

13: {0, 0, 0, 0, 0, 1}

14: { -3, 1, -3, 0, -3, 0 }

15: {0, 0, 0, 1}

16: {0, 0, 0, 0, 0, 0, 1, 2}

17: {0, 0, 0 ,0, 0, 0, 0, 1}

18: {0, 0, 0, 0, 1, 3}

19: {0, 0, 0, 0, 0, 0, 0, 0, 1}

21: {0, 0, 0, 0, 0, 1}

22: {-1, 1, -1, 0, -1, 0, -1, 0, -1, 0}

23: {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}

24: {0, 0, 0, 0, 0, 0, 0, 1}

25: {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}

26: {-1, -1, -1, 0 , -1, 0, -1, 0, -1, 0, 1, 0}

27: {0, 0, 0, 0, 0, 0, 0, 0, 1}

28: {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}

29: {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}

30: {0, 0, 0, 0, 0, 0, 0, 1}

31: {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}

32: {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2}

33: {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}

4.3. Initial Segments for Observation 3d

This is a list of initial segments for $L_q \in \Lambda_q, q = q_{\lambda^2}, \lambda = 2\cos\frac{\pi}{k}, k = 12, 14, 20, 22, 24, 28,$ and 30, satisfying the conditions of observation 3d.

12: {0, 1}

14: {1, 0, 0}

20: {0, 0, 1}

22: {1, 0, 0, 0, 0}

24: {0, 0, 0, 1 }

28: {0, 0, 0, 0, 0, 1 }

30: {0, 0, 0, 1}

4.4. Kernels for the Linear Recurrences

We list these for the convenience of the reader. Below is the list of kernels for the $L_p \in \Lambda_p, (p = p_\lambda, \lambda = 2\cos\frac{\pi}{k}$ for k between 5 and 33 (inclusive.)

5: {1 ,1}

6: {0, 3}

7: {1, 2, -1}

8: {0, 4, 0, -2}

9: {0, 3, 1}

10: {0, 5, 0, -5}

11: {1, 4, -3, -3, 1}

12: {0, 4, 0, -1}

13: {1, 5, -4, -6, 3, 1}

14: {0, 7, 0, -14, 0, 7}

15: {-1, 4, 4, -1}

16: {0, 8, 0, -20, 0, 16, 0, -2}

17: {1, 7, -6, -15, 10, 10, -4, -1}

18: {0, 6, 0, -9, 0, 3}

19: {1, 8, -7, -21, 15, 20, -10, -5, 1}

20: {0, 8, 0, -19, 0, 12, 0, -1}

21: {-1, 6, 6, -8, -8, -1}

22: {0, 11, 0, -44, 0, 77, 0, -55, 0, 11}

23: {1, 10, -9, -36, 28, 56, -35, -35, 15, 6, -1}

24: {0, 8, 0, -20, 0, 16, 0, -1}

25: {0, 10, 0, -35, 1, 50, -5, -25, 5, 1}

26: {0, 13, 0, -65, 0, 156, 0, -182, 0, 91, 0, -13}

27: {0, 9, 0, -27, 0, 30, 0, -9, 1}

28: {0, 12, 0, -53, 0, 104, 0, -86, 0, 24, 0, -1}

29: {1, 13, -12, -66, 55, 165, -120, -210, 126, 126, -56, -28, 7, 1}

30: {0, 7, 0, -14, 0, 8, 0, -1}

31: {1, 14, -13, -78, 66, 220, -165, -330, 210, 252, -126, -84, 28, 8, -1 }

32: {0, 16, 0, -104, 0, 352, 0, -660, 0, 672, 0, -336, 0, 64, 0, -2}

33: {-1, 10, 10, -34, -34, 43, 43, -12, -12, -1}

Below is the list of kernels for the $L_q \in \Lambda_q, q = q_{\lambda^2}, \lambda = 2\cos\frac{\pi}{k}, k = 12, 14, 20, 22, 24, 28,$ and 30, satisfying the conditions of observation 3d.

12: {4, -1}

14: {7, -14, 7}

20: {8, -19, 12, -1}

22: {11, -44, 77, -55, 11}

24: {8, -20, 16, -1}

28: {12, -53, 104, -86, 24, -1}

30: {7, -14, 8, -1}

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