GENERALIZED COSECANT NUMBERS AND THE HURWITZ ZETA FUNCTION

Victor Kowalenko
School of Mathematics and Statistics, The University of Melbourne, Victoria, Australia
vkowa@unimelb.edu.au

Received: 8/15/18, Accepted: 9/6/19, Published: 9/30/19

Abstract

This paper presents recent developments concerning the generalized cosecant numbers $c_{\rho,k}$, which emerge as the coefficients of the power series expansion for the important fundamental function $z^\rho/\sin^\rho z$. These coefficients can be computed for all, including complex, values of ρ by using the relatively novel graphical method known as the partition method for a power series expansion or by using intrinsic routines in Mathematica. In fact, they represent polynomials in ρ of degree k, where k is the power of z. In addition, though related to the Bernoulli numbers, they possess more properties and do not diverge like the former. The partition method for a power series expansion has the advantage that it yields the k-behaviour of the highest order coefficients. Thus, general formulas for such coefficients are derived by considering the properties of the highest part partitions. It is then shown how the generalized cosecant numbers are related to the specific symmetric polynomials that arise from summing over quadratic powers of integers. Consequently, integral values of the Hurwitz zeta function for even powers are expressed for the first time ever in terms of ratios of the generalized cosecant numbers.

1. Introduction

The cosecant numbers, c_k, are defined in [1] as the rational coefficients of the power series expansion for cosecant, in particular by

$$\csc z = \frac{1}{\sin z} \equiv \sum_{k=0}^{\infty} c_k \, z^{2k-1}. \tag{1}$$

Via the discrete graphical method known as the partition method for a power series expansion, a general formula for them is derived in terms of all the integer partitions summing to k, which is

$$c_k = (-1)^k \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^k i\,\lambda_i=k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} (-1)^{N_k} N_k! \prod_{i=1}^{k} \Big(\frac{1}{(2i+1)!}\Big)^{\lambda_i} \frac{1}{\lambda_i!}. \tag{2}$$

In this result, λ_i represents the multiplicity or the number of occurrences of each part i in the partitions, while the sum of the multiplicities or length of the partition is represented by N_k, i.e., $N_k = \sum_{i=1}^k \lambda_i$. For the partitions summing to k, the multiplicity of a part i ranges from zero to $\lfloor k/i \rfloor$, where $\lfloor x \rfloor$ denotes the floor function or the greatest integer less than or equal to x. Generally, when k is large, most of the multiplicities for a partition vanish as we shall see shortly. Using (2), one finds that $c_0 = 1$, $c_1 = 1/6$, $c_2 = 7/360$, $c_4 = 31/3 \cdot 7!$ and so on.

The reader should observe that an equivalence symbol has been introduced into (1) because the power series expansion on the right-hand side is divergent for $|z| \geq \pi$, while the left-hand side is always defined. For these values the right-hand side must be regularized in the manner described in [4]-[9]. Nevertheless, since the power series is convergent for the other values of z, one can replace the equivalence symbol by an equals sign. Then one obtains the standard power series for $z \csc z$, which is given as No. 1.411(11) in [3]. Consequently, the cosecant numbers can be expressed in terms of the Bernoulli numbers as

$$c_k = \frac{(-1)^{k+1}}{(2k)!}(2^{2k} - 2)B_{2k}. \tag{3}$$

As discussed in [1], the cosecant numbers possess far more properties than the Bernoulli numbers. They have also the major advantage that they do not diverge like them. In fact, in most situations such as 3 and the Euler Maclaurin summation formula, the divergence of the Bernoulli numbers is tamed by the factor of $(2k)!$ in the denominator. Thus, one is often required to divide two very large numbers by each other, which is avoided when using the cosecant numbers. Hence the above equation can be regarded as awkward or even clumsy. Nevertheless, the interested reader should consult [8], which shows how the Bernoulli numbers and polynomials can be evaluated by using the partition method for a power series expansion.

A better method of expressing the cosecant numbers is in terms of the Riemann zeta function. By using No. 9.616 in [3], we find that

$$c_k = 2\big(1 - 2^{1-2k}\big)\frac{\zeta(2k)}{\pi^{2k}}, \tag{4}$$

where $\zeta(2k)$ represents the Riemann zeta function. Thus, we observe that $c_k \approx 2\pi^{-2k}$ or $\pi^{2k} \approx 2/c_k$ for $k \gg 1$, although this approximation gives the misleading impression that the c_k are irrational. Consequently, (2) becomes another method of determining even integer values of the Riemann zeta function.

Reference [1] not only presents numerous applications of cosecant numbers, but also demonstrates how they are related to other numbers such as the secant numbers and, more importantly, how the sets of the resulting numbers can be generalized or extended by introducing an arbitrary power ρ to their generating function. Specifically, the generalized cosecant numbers are given by

$$\csc^{\rho} z = \frac{1}{\sin^{\rho} z} \equiv \sum_{k=0}^{\infty} c_{\rho,k} z^{2k-\rho}, \tag{5}$$

where

$$c_{\rho,k} = (-1)^k \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^{k} i\lambda_i = k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} (-1)^{N_k} (\rho)_{N_k} \prod_{i=1}^{k} \Big(\frac{1}{(2i+1)!}\Big)^{\lambda_i} \frac{1}{\lambda_i!}. \tag{6}$$

In (6), $(\rho)_{N_k}$ denotes the Pochhammer notation for $\Gamma(\rho + N_k)/\Gamma(\rho)$, where $\Gamma(x)$ represents the gamma function.

2. Computation

To calculate the generalized cosecant numbers via (6), we need to determine the specific contribution made by each integer partition that sums to k. For example, if we wish to evaluate $c_{\rho,6}$, then we require all the contributions made from the partitions summing to 6, which appear in the first column of Table 1. Each part in a partition is assigned a specific value, which depends on the function being studied. In the case of $x^{\rho}/\sin^{\rho} x$, the part i is assigned a value of $(-1)^{i+1}/(2i+1)!$. In addition, since each part occurs λ_i times in a partition, we need to multiply λ_i values or calculate $(-1)^{(i+1)\lambda_i}/((2i+1)!)^{\lambda_i}$. The second column in Table 1 displays the multiplicities of the parts in all the partitions summing to 6, while the third column presents the length N_k for each partition. Thus we see that most of the multiplicities vanish as stated earlier.

Associated with each partition is a multinomial factor that is determined by taking the factorial of N_k and dividing by the factorials of all the multiplicities. For example, for the partition $\{2, 1, 1, 1, 1\}$ in Table 1, we have $\lambda_1 = 4$ and $\lambda_2 = 1$, while the other multiplicities vanish. Hence, the multinomial factor becomes $5!/(4!\,1!) = 5$. When the function is accompanied by an arbitrary power, say ρ, a further modification must be made. Each partition is then multiplied by the Pochhammer factor of $\Gamma(N_k + \rho)/\Gamma(\rho)$ divided by $N_k!$. That is, for each partition we must include the extra factor of $(\rho)_{N_k}/N_k!$. For $\rho = 1$, this simply yields unity and thus, the multinomial factor remains unaffected. Consequently, (6) reduces to (2) for $\rho = 1$. For $\rho = 2$ we obtain the cosecant-squared numbers as given in

Partition	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	N_k
$\{6\}$						1	1
$\{5,1\}$	1				1		2
$\{4,2\}$		1		1			2
$\{4,1,1\}$	2			1			3
$\{3,3\}$			2				2
$\{3,2,1\}$	1	1	1				3
$\{3,1,1,1\}$	3		1				4
$\{2,2,2\}$		3					3
$\{2,2,1,1\}$	2	2					4
$\{2,1,1,1,1\}$	4	1					5
$\{1,1,1,1,1,1\}$	6						6

Table 1: Multiplicities of the partitions summing to 6

Theorem 5 of [1], which are given by

$$\begin{aligned} c_{2,k} &= (-1)^k \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^k i\lambda_i = k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} (-1)^{N_k} (2)_{N_k} \prod_{i=1}^{k} \Big(\frac{1}{(2i+1)!}\Big)^{\lambda_i} \frac{1}{\lambda_i!} \\ &= \frac{(2k-1)}{\big(1-2^{1-2k}\big)} c_k. \end{aligned} \tag{7}$$

In the above result one can replace $(2)_{N_k}$ by $(N_k+1)(1)_{N_k}$. Hence we obtain (2) again except $N_k!$ is now multiplied by N_k+1. On the other hand, if $\rho=-1$, then the coefficients are simply equal to the power series expansion for $\sin z$ divided by z. Therefore, we arrive at

$$\sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^k i\lambda_i = k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} (-1)^{N_k} (\rho)_{N_k} \prod_{i=1}^{k} \Big(\frac{1}{(2i+1)!}\Big)^{\lambda_i} \frac{1}{\lambda_i!} = \frac{1}{(2k+1)!}. \tag{8}$$

Hence, we have an expression for the reciprocal of $(2k+1)!$ in terms of a sum over partitions summing to k.

If we examine (6) more closely, then we see that the product deals with calculating the contribution made by each partition based on the values of the multiplicities, while the sum refers to all partitions summing to k. Hence the sum covers the range of values for each multiplicity. For example, λ_1 attains a maximum value of k, which corresponds to the partition with k ones, while λ_2 attains a maximum value of $[k/2]$, which corresponds to the partition with $[k/2]$ twos in it. For odd values of k, the partition with $[k/2]$ twos also possesses a one, i.e., $\lambda_1 = 1$. Thus, it can be seen that the maximum value of λ_i is always $[k/i]$, which becomes the upper limit for each multiplicity in both (2) and (6). Moreover, each partition in the sums must satisfy the constraint, $\sum_{i=1}^{k} i\lambda_i = k$.

As an example, let us calculate $c_{\rho,6}$. According to Table 1 there are eleven partitions summing to 6. Therefore, we need to determine eleven contributions in the sum over the partitions. By applying the steps mentioned above to (6), we find that the contributions from the partitions in the same order as the table are

$$\begin{aligned} c_{\rho,6} = &-(\rho)_1 \frac{1}{13!} + \frac{(\rho)_2}{2!}\frac{2!}{1!\cdot 1!}\frac{1}{3!\cdot 11!} + \frac{(\rho)_2}{2!}\frac{2!}{1!\cdot 1!}\frac{1}{5!\cdot 9!} - \frac{(\rho)_3}{3!}\frac{3!}{1!\cdot 2!}\frac{1}{3!^2\cdot 9!} \\ &+ \frac{(\rho)_2}{2!}\frac{2!}{2!}\frac{1}{(7!)^2} - \frac{(\rho)_3}{3!}\frac{3!}{1!\cdot 1!\cdot 1!}\frac{1}{3!\cdot 5!\cdot 7!} + \frac{(\rho)_4}{4!}\frac{4!}{1!\cdot 3!}\frac{1}{(3!)^3\cdot 7!} - \frac{(\rho)_3}{3!}\frac{3!}{3!}\frac{1}{(5!)^3} \\ &+ \frac{(\rho)_4}{4!}\frac{4!}{2!\cdot 2!}\frac{1}{(3!)^2\cdot (5!)^2} - \frac{(\rho)_5}{5!}\frac{5!}{4!\cdot 1!}\frac{1}{(3!)^4\cdot 5!} + \frac{(\rho)_6}{6!}\frac{6!}{6!}\frac{1}{(3!)^6}. \end{aligned} \tag{9}$$

The interesting property of the above result is that when the length N_k appearing in the Pochhammer terms is even, the contribution from the partition is positive while if it is odd, then the contribution is negative. This behaviour applies to all even values of k. On the other hand, if k is odd, then the contributions with an odd number of parts are positive, while those from an even number of parts are negative. Furthermore, by introducing (9) into Mathematica [10] and wrapping it entirely around the combination of the Simplify and Expand routines, one obtains

$$\begin{aligned} c_{\rho,6} = &\frac{1}{5884534656000}\Big(1061376\rho + 3327584\rho^2 + 4252248\rho^3 + 2862860\rho^4 \\ &+ 1051050\rho^5 + 175175\rho^6\Big). \end{aligned} \tag{10}$$

Since the denominator equals $2/(9 \cdot 15!)$, we arrive at the $k = 6$ result in Table 2.

Table 2 displays the generalized cosecant numbers up to $k = 15$, which have been obtained by introducing the multiplicities of all partitions summing to k into the sum in (2). For $k > 10$, the partition method for a power series expansion becomes laborious due to the exponential increase in the number of partitions. To circumvent this problem, a general computing methodology has been developed in [2] and [13], which is based on representing all the partitions summing to a specific order k by a partition tree and invoking the bivariate recursive central partition (BRCP) algorithm. From this computing methodology, general expressions for the coefficients of any power series expansion can be obtained. For example, the symbolic form in the case of the partitions summing to 6 is given by

$$\begin{aligned} DS[6] := &\, p[6]q[1]a + p[1]p[5]q[2]a\hat{}(2)2! + p[1]\hat{}(2)p[4]q[3]a\hat{}(3)3!/2! \\ &+ p[1]\hat{}(3)p[3]q[4]a\hat{}(4)4!/3! + p[1]\hat{}(4)p[2]q[5]a\hat{}(5)5!/4! + p[1]\hat{}(6)q[6] \\ &+ a\hat{}(6) + p[1]\hat{}(2)p[2]\hat{}(2)q[4]a\hat{}(4)4!/(2!2!) + p[1]p[2]p[3]q[3]a\hat{}(3)3! \\ &+ p[2]p[4]q[2]a\hat{}(2)2! + p[2]\hat{}(3)q[3]a\hat{}(3) + p[3]\hat{}(2)q[2]a\hat{}(2). \end{aligned} \tag{11}$$

k	$c_{\rho,k}$
0	1
1	$\frac{1}{3!}\rho$
2	$\frac{2}{6!}\left(2\rho+5\rho^2\right)$
3	$\frac{8}{9!}\left(16\rho+42\rho^2+35\rho^3\right)$
4	$\frac{2}{3\cdot 10!}\left(144\rho+404\rho^2+420\rho^3+175\rho^4\right)$
5	$\frac{4}{3\cdot 12!}\left(768\rho+2288\rho^2+2684\rho^3+1540\rho^4+385\rho^5\right)$
6	$\frac{2}{9\cdot 15!}\left(1061376\rho+3327594\rho^2+4252248\rho^3+2862860\rho^4+1051050\rho^5+175175\rho^6\right)$
7	$\frac{1}{27\cdot 15!}\left(552960\rho+1810176\rho^2+2471456\rho^3+1849848\rho^4+820820\rho^5+210210\rho^6 +25025\rho^7\right)$
8	$\frac{2}{45\cdot 18!}\left(200005632\rho+679395072\rho^2+978649472\rho^3+792548432\rho^4+397517120\rho^5 +125925800\rho^6+23823800\rho^7+2127125\rho^8\right)$
9	$\frac{4}{81\cdot 21!}\left(129369047040\rho+453757851648\rho^2+683526873856\rho^3+589153364352\rho^4 +323159810064\rho^5+117327450240\rho^6+27973905960\rho^7+4073869800\rho^8+282907625\rho^9\right)$
10	$\frac{2}{6075\cdot 22!}\left(38930128699392\rho+140441050828800\rho^2+219792161825280\rho^3 +199416835425280\rho^4+117302530691808\rho^5+47005085727600\rho^6+12995644662000\rho^7 +2422012593000\rho^8+280078548750\rho^9+15559919375\rho^{10}\right)$
11	$\frac{8}{243\cdot 25!}\left(494848416153600\rho+1830317979303936\rho^2+2961137042841600\rho^3 +2805729689044480\rho^4+1747214980192000\rho^5+755817391389984\rho^6 +232489541684400\rho^7+50749166067600\rho^8+7607466867000\rho^9+715756291250\rho^{10} +32534376875\rho^{11}\right)$
12	$\frac{2}{2835\cdot 27!}\left(1505662706987827200\rho+5695207005856038912\rho^2+9487372599204065280\rho^3 +9332354263294766080\rho^4+6096633539052376320\rho^5+2806128331871953088\rho^6 +937291839756592320\rho^7+229239926321406000\rho^8+40598842049766000\rho^9 +5005999501002500\rho^{10}+390802935022500\rho^{11}+14803141478125\rho^{12}\right)$
13	$\frac{232}{81\cdot 30!}\left(844922884529848320\rho+3261358271400247296\rho^2+5576528334428209152\rho^3 +5668465199488266240\rho^4+3858582205451484160\rho^5+1870620248833400064\rho^6 +667822651436228288\rho^7+178292330746770240\rho^8+35600276746834800\rho^9 +5225593531158000\rho^{10}+539680243602500\rho^{11}+35527539547500\rho^{12} +1138703190625\rho^{13}\right)$
14	$\frac{2}{1215\cdot 30!}\left(1383190150411557273600\rho+5438550955954777620480\rho^2 +9520277966410424647680\rho^3+9963522869920305561600\rho^4+7030409659600317952000\rho^5 +3563125373878394321920\rho^6+1344667951720621848320\rho^7+385269454103111177600\rho^8 +84369877134446904000\rho^9+14040489429586620000\rho^{10}+1737770384400050000\rho^{11} +152582323418525000\rho^{12}+8585822057312500\rho^{13}+235874232343750\rho^{14}\right)$
15	$\frac{1088}{729\cdot 35!}\left(56200973946476984008704\rho+224751194159631176407449\rho^2 +401910837930690543983001\rho^3+431774592520807259425996\rho^4 +314516377667793942941696\rho^5+165691720353903234153062\rho^6 +65564391936442058602342\rho^7+19922791941903925621747\rho^8 +4699575166447588018592\rho^9+861402610709293821168\rho^{10} +121477834916232394600\rho^{11}+12858745292219326500\rho^{12} +972018086752462750\rho^{13}+47294670578780625\rho^{14} +1126063585209062\rho^{15}\right)$

Table 2: Generalized cosecant numbers $c_{\rho,k}$ up to $k=15$

Such an expression can easily be imported into Mathematica [10]. Then the coefficients of the inner series p[k] are set equal to the assigned values of the parts. For the generalized cosecant numbers, this means we set

$$p[k\_] := (-1)\hat{}(k+1)/(2k+1)!,$$

while the q[k], which are referred to as the coefficients of the outer series, are set equal to the coefficients of the binomial series, namely,

$$q[k\_] := \text{Pochhammer}[\rho, k]/k!.$$

In addition, we set the parameter a equal to unity, thereby obtaining $c_{\rho,6}$. On the other hand, changing p[k] to

$$p[k\_] := (-1)\hat{}(k+1)/(2k)!,$$

will yield an entirely different set numbers known as the generalized secant numbers, $d_{\rho,k}$, [1, 2]. These numbers represent the coefficients in the power series expansion for $\sec^{\rho} z$. In terms of the partition method for a power series expansion they are given by

$$d_{\rho,k} = (-1)^k \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^{k} i\lambda_i = k}}^{k,[k/2],[k/3],\ldots,1} (-1)^N \, (\rho)_N \prod_{i=1}^{k} \left(\frac{1}{(2i)!} \right)^{\lambda_i} \frac{1}{\lambda_i!}. \tag{12}$$

Thus, the symbolic representation for DS[6] is not only general, but also powerful.

It should also be mentioned that Mathematica [10] via its SeriesCoefficient routine is able to determine the generalized cosecant numbers very quickly due to the fact that it has been optimized. For example, to obtain the first fifteen generalized cosecant numbers, one need only type

$$\text{In}[1] := \text{Table}[\text{SeriesCoefficient}[(z/\text{Sin}[z])\hat{}\rho, \{z, 0, k\}], \{k, 0, 30, 2\}]. \tag{13}$$

The result for $k = 7$ generated by this command is:

$$\begin{aligned} \text{Out}[2] = {} & (8191\rho)/37362124800 + (15019(-1+\rho)\rho)/12454041600 + (517457(-2+\rho) \\ & \times (-1+\rho)\rho)/301771008000 + (169(-3+\rho)(-2+\rho)(-1+\rho)\rho)/183708000 \\ & + (83(-4+\rho)(-3+\rho)(-2+\rho)(-1+\rho)\rho)/391910400 + (7(-5+\rho)(-4+\rho) \\ & \times (-3+\rho)(-2+\rho)(-1+\rho)\rho)/335923200 + ((-6+\rho)(-5+\rho)(-4+\rho) \\ & \times (-3+\rho)(-2+\rho)(-1+\rho)\rho)/1410877440. \end{aligned} \tag{14}$$

Although there are far less contributions than by the partition method for a power series expansion, this form for the generalized cosecant numbers is still cumbersome because one must, once again, employ the Expand and Simplify routines in

Mathematica to obtain the forms displayed in Table 2. Moreover, by using this approach one cannot determine the k-dependence of the coefficients of the polynomials, whereas the partition method for a power series expansion is able to reveal this behaviour for the highest order coefficients as explained in the next section.

3. Coefficients of the Generalized Cosecant Numbers

Despite the fact that the contributions in (6) alternate in sign according to whether the length N_k of each partition is even or odd, the final forms for the generalized cosecant numbers only possess positive coefficients. Moreover, the highest order terms in the $c_{\rho,k}$ is $O(\rho^k)$. For example, from (10), we see that $c_{\rho,6}$ is a sixth order polynomial in ρ. The term that is responsible for the ρ^6 term in (6) emanates from the partition with six ones, represented by $\{1_6\}$, since its modified multinomial factor is $(\rho)_6/6!$. Consequently, we see that the generalized cosecant numbers are polynomials of degree k with fixed coefficients, which can be expressed as $c_{\rho,k} = \sum_{i=1}^{k} C_{k,i}\rho^i$. The aim of this section is to determine the highest order coefficients as functions of k.

Because we know that the highest order term in the generalized cosecant numbers is determined by the partition with the most number of ones, i.e., k ones or $\{1_k\}$, we can evaluate its contribution in (6) without much effort. This is simply $(\rho)_k/(3!)^k k!$. Therefore, the highest order term in ρ is the coefficient of ρ^k in the Pochhammer factor, which is given by

$$C_{k,k} = \frac{1}{(3!)^k k!}. \tag{15}$$

The coefficient of the next or second highest order term, namely, $C_{k,k-1}$, is the sum of the contributions due to two partitions. First, there is the ρ^{k-1} term from the partition with k ones and second, there is the highest order term from the partition with $k-2$ ones and a two, represented by $\{1_{k-2}, 2\}$. The first term follows from the coefficient of ρ^{k-1} in the Pochhammer factor in the previous calculation, while the second term represents the ρ^{k-1} power in $-(\rho)_{k-1}/(5!\cdot(3!)^{k-1}(k-2)!)$. Combining both contributions yields

$$C_{k,k-1} = \frac{1}{(3!)^k k!}\sum_{i=1}^{k-1} i - \frac{1}{5\cdot(3!)^{k-2}(k-2)!} = \frac{1}{5\cdot(3!)^k(k-2)!}. \tag{16}$$

The third highest order term in the $c_{\rho,k}$ or the term with the coefficient, $C_{k,k-2}$, is evaluated by the summing the contributions due to four partitions. These are: (1) the ρ^{k-2} power in the contribution from the partition with k ones, (2) the second leading order term in the contribution from the partition with $k-2$ ones and a two, and (3) the leading order terms in the contributions from the partitions with

$k-3$ ones and one three and $k-4$ ones and two twos, represented by $\{1_{k-3}, 3\}$ and $\{1_{k-4}, 2_2\}$, respectively.

In order to evaluate each of these contributions in general form, we now require the formula that gives the coefficient of ρ to an arbitrary power in each Pochhammer factor. According to [11, Chapter 24] and [12, Chapter 18], the Pochhammer polynomials can be expressed as

$$(y)_k = \frac{\Gamma(y+k)}{\Gamma(y)} = (-1)^k \sum_{j=0}^{k} (-1)^j s_k^{(j)} y^j, \tag{17}$$

where the integers $s_k^{(j)}$ are known as the (signed) Stirling numbers of the first kind with $s_k^{(0)} = s_0^{(k)} = 0$ for $k \geq 1$ and $s_0^0 = 1$. It should be mentioned that the Stirling numbers of the first kind are often represented as $s(k, j)$. They also satisfy the following recurrence relation:

$$s_{k+1}^{(j)} = s_k^{(j-1)} - k s_k^{(j)}. \tag{18}$$

In the appendix of [8], a general formula for these numbers is derived, which is given as

$$s_k^{(k-j)} = (-1)^j \sum_{i_j=j}^{k-1} i_j \sum_{i_{j-1}=j-1}^{i_j-1} \sum_{i_{j-2}=j-2}^{i_{j-1}-1} i_{j-2} \cdots \sum_{i_1=1}^{i_2-1} i_1. \tag{19}$$

By using this result, we can calculate general formulas for specific values of j ranging from 0 to 4. These are

$$s_k^{(k)} = 1, \quad s_k^{(k-1)} = -\binom{k}{2}, \quad s_k^{(k-2)} = \frac{(3k-1)}{4}\binom{k}{3},$$
$$s_k^{(k-3)} = -\binom{k}{2}\binom{k}{4}, \quad s_k^{(k-4)} = \frac{1}{48}\left(15k^3 - 30k^2 + 5k + 2\right)\binom{k}{5}. \tag{20}$$

Note that the denominator of the last result has been corrected here compared with [8], where 336 appears instead of 48. These results have been obtained by developing the nested sum formula given by (19) as a module in Mathematica [10]. In general, one obtains a polynomial in k of degree $2j$ with one of the terms in the polynomials being equal to $(-1)^j \binom{k}{j+1}$. In [2, Chapter 6], the partition method for a power series expansion is used to derive an alternative formula for the Stirling numbers of the first kind, which enables general formulas for $s_k^{(k-\ell)}$ to be determined more easily than using (19). Then it is found that the Stirling numbers of the first kind can be represented as $s_k^{(k-\ell)} = (-1)^\ell \binom{k}{\ell+1} r_\ell(k)$, where the $r_\ell(k)$ are polynomials of degree $\ell - 1$ and are displayed in Table 3. For odd values of ℓ the polynomials possess a common external factor of $k(k-1)$. Note also that some of the coefficients in the $\ell = 8$ and 9 results have been incorrectly transcribed in [2], but are corrected here.

ℓ	$r_\ell(k)$
1	1
2	$\frac{1}{4}(3k-1)$
3	$\frac{1}{2}k(k-1)$
4	$\frac{1}{48}\big(15k^3-30k^2+5k+2\big)$
5	$\frac{1}{16}k(k-1)\big(3k^2-7k-2\big)$
6	$\frac{1}{576}\big(63k^5-315k^4+315k^3+91k^2-42k-16\big)$
7	$\frac{1}{144}k(k-1)\big(9k^4-54k^3+51k^2+58k+16\big)$
8	$\frac{1}{3840}\big(135k^7-1260k^6+3150k^5-840k^4-2345k^3-540k^2+404k$ $+144\big)$
9	$\frac{1}{768}k(k-1)\big(15k^6-165k^5+465k^4+17k^3-648k^2-548k-144\big)$
10	$\frac{1}{9216}\big(99k^9-1485k^8+6930k^7-8778k^6-8085k^5+8195k^4+11792k^3$ $+2068k^2-2288k-768\big)$

Table 3: The polynomials $r_\ell(k)$ in the Stirling numbers of the first kind

As an aside, it should be mentioned that the corresponding results for the Stirling numbers of the second kind are also derived in [2].

With the above results, $C_{k,k-2}$ reduces to

$$C_{k,k-2} = \frac{s_k^{(k-2)}}{(3!)^k k!} + \frac{s_{k-1}^{(k-2)}}{5!\cdot(3!)^{k-2}(k-2)!} + \frac{s_{k-2}^{(k-2)}}{7!\cdot(3!)^{k-3}(k-3)!} + \frac{s_{k-2}^{(k-2)}}{2!\cdot(5!)^2(3!)^{k-4}(k-4)!}. \tag{21}$$

Introducing the results in (20), we find that the coefficients reduce to

$$C_{k,k-2} = \frac{21k+17}{175\,(3!)^{k+1}\,(k-3)!}. \tag{22}$$

The expressions for $C_{k,k-\ell}$ become more difficult to evaluate as ℓ increases. However, from the above results, we see that a pattern is developing. First, the power of 3! appears to be increasing as the power of ρ decreases. Next, the factorial in the denominator decrements by unity. That is, $C_{k,k-1}$ goes as $1/(3!)^k(k-2)!$, while $C_{k,k-2}$ goes as $1/(3!)^{k+1}(k-3)!$. Furthermore, the numerator for $C_{k,k-1}$ is constant, whereas it is linear in k for $C_{k,k-2}$. So we conjecture that the next coefficient is given by

$$C_{k,k-3} = \frac{ak^2+bk+c}{(3!)^{k+2}(k-4)!}. \tag{23}$$

The reason for choosing $1/(k-\ell-1)!$ in the denominator for $C_{k,k-\ell}$ is that the first value of these coefficients only begins when $k=\ell+1$. Thus, we can put $k=4$ in the above result and equate it to $C_{4,1}$ in Table 2, which equals 144/5443200. Similarly, we put $k = 5$ and $k = 6$ in (23) and equate the values respectively to $C_{5,2}$ and $C_{6,3}$ in the same table. Then we arrive at the following set of equations:

$$16a + 4b + c = \frac{216}{175}, \tag{24}$$

$$25a + 5b + c = \frac{312}{175}, \tag{25}$$

$$36a + 6b + c = \frac{2124}{875}. \tag{26}$$

The solution to the above set of equations is $a = 6/125$, $b = 102/875$ and $c = 0$, which means in turn that $C_{k,k-3}$ is given by

$$C_{k,k-3} = \frac{k^2 + 17k/7}{125(3!)^{k+1}(k-4)!}. \tag{27}$$

Putting $k = 8$ in this formula yields 73/26453952000, which agrees with the coefficient of 397517120 in Table 2 for $k = 8$ when it is multiplied by the external factor of $2/45 \cdot 18!$. In using this approach, it did not matter whether the power of 3! in the denominator was absolutely correct initially provided that the dependence upon k was correct, i.e., linear as opposed to quadratic or any other power. On the other hand, it is crucial that the factorial in the denominator is correct, although, as mentioned above, this is surmised from the fact that $C_{k,k-\ell}$ vanishes for $\ell > k$.

In the case of $C_{k,k-4}$ the conjecture becomes

$$C_{k,k-4} = \frac{ak^3 + bk^2 + ck + d}{(3!)^{k+4}(k-5)!}. \tag{28}$$

Then a set of four linear equations is required with k ranging from 5 to 8. Solving the equations using the LinearSolve routine in Mathematica [10] yields

$$C_{k,k-4} = \frac{3\big(3k^3 + 102k^2/7 + 289k/49 - 11170/539\big)}{625\,(3!)^{k+3}\,(k-5)!}. \tag{29}$$

Finally, putting $k = 9$ into the above result yields a value of 229051/733303549440000, which agrees with the coefficient of ρ^5 for $k = 9$ in Table 2.

Determining general formulas for the coefficients of the lowest order terms in the generalized cosecant numbers is a much more formidable problem. If we consider the preceding methods, then to obtain the equivalent of (16) for the lowest order coefficient, i.e., $C_{k,1}$, we need to evaluate the contributions from all partitions whereas previously we only needed a fixed number, e.g., four for determining $C_{k,k-2}$. This is clearly not possible since the number of partitions is not fixed, but increases exponentially. In addition, the conjectural approach breaks down when k appears in the second subscript of the coefficients. For example, to determine $C_{k,2}$ via this approach, ℓ would now be equal to $k-2$.

ρ	$k=6$	$k=8$	$k=10$	$k=12$	$k=15$
10	0.998905	0.985655	0.929830	0.801477	0.501086
15	0.999741	0.996144	0.977941	0.925497	0.751944
20	0.999910	0.998571	0.991119	0.966957	0.870459
30	0.999981	0.999669	0.997755	0.990752	0.956671
50	0.999997	0.999951	0.999644	0.998405	0.991292
100	0.999999	0.999997	0.999974	0.999676	0.992370
1000	0.999999	0.999999	0.999999	0.999999	0.999999

Table 4: The ratio $\beta(\rho, k)$ for various values of ρ and k

For $|\rho| \gg k$ we may use the first four coefficients derived above as a means of approximating the generalized cosecant numbers. That is, the generalized cosecant numbers can be approximated by

$$c_{\rho,k} \overset{|\rho|\gg k}{\approx} \frac{\rho^k}{(3!)^k k!} + \frac{\rho^{k-1}}{5\cdot(3!)^k(k-2)!} + \frac{(21k+17)\rho^{k-2}}{175\cdot(3!)^{k+1}(k-3)!} + \frac{(k^2+17k/7)\rho^{k-3}}{125\cdot(3!)^{k+1}(k-4)!}. \tag{30}$$

To gain an appreciation of this approximation, let us denote the ratio of (30) to the corresponding values of $c_{\rho,k}$ in Table 2 by $\beta(\rho, k)$. Table 4 presents values of $\beta(\rho, k)$ for integer values of ρ ranging from 10 to 1000. They have been given to six decimal places with no rounding-off. From the table, we see that when ρ is close to k or smaller, which occurs towards the right-hand top corner, $\beta(\rho, k)$ is not close to unity, but for all other values, it is. Therefore, provided ρ is significantly greater than k, (30) represents an accurate approximation for the generalized cosecant numbers.

There is one interesting feature in the table that needs to be mentioned. If one examines the ratios when (1) $k = 10$, $\rho = 20$, (2) $k = 15$ and $\rho = 30$, then it is readily observed that the ratio is more accurate for the first case than in the second case despite the fact they are both good approximations. This means that as k increases, $|\rho/k|$ must also increase in order to obtain the same value of $\beta(\rho, k)$ for the lower values of k. For example, the value of $\beta(20, 10)$ is about 0.991, which in the case of $k = 15$ only is only reached when ρ is about 50. Hence in the $k = 10$ case, it only takes twice the value of k to achieve the same value of $\beta(\rho, k)$ as the $k = 15$ case, which requires at least three times the value of k. This has ramifications when the relationship between the generalized cosecant numbers and the Hurwitz zeta function is discussed next.

4. Connection to Hurwitz Zeta Function

It is found in [14] and [15] that the generalized cosecant numbers can also be expressed as

$$c_{2v,i} = 2^{2i}\frac{\Gamma(2v-2i)}{\Gamma(2v)}s(v,i), \quad i < v, \tag{31}$$

where $s(v,n)$ represents the nth elementary symmetric polynomial obtained by summing over quadratic powers or squared integers, namely, $1^2, 2^2, \ldots, (v-1)^2$. That is,

$$s(v,n) = \sum_{1 \le i_1 < i_2 < \cdots < i_n < v-1} x_{i_1} x_{i_2} \cdots x_{i_n}, \tag{32}$$

where $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ and each x_{i_j} is equal to at least one value in the set $\{1, 2^2, 3^2, \ldots, (v-1)^2\}$. For the three lowest values of n the symmetric polynomials are

$$s(v,0) = 1, \quad s(v,1) = (v-1)v(2v-1)/6,$$

and

$$s(v,2) = \frac{(5v+1)}{4 \cdot 6!}(2v-4)_5, \tag{33}$$

while for the four highest values of n they are given by

$$s(v,v-1) = (v-1)!^2, \quad s(v,v-2) = (v-1)!^2\left(\zeta(2) - \zeta(2,v)\right),$$

$$s(v,v-3) = \frac{(v-1)!^2}{2}\Big((\zeta(2) - \zeta(2,v))^2 + \zeta(4,v) - \zeta(4)\Big),$$

and

$$s(v,v-4) = \frac{(v-1)!^2}{6}\Big((\zeta(2) - \zeta(2,v))^3 - 3(\zeta(4) - \zeta(4,v))(\zeta(2) - \zeta(2,v)) + 2(\zeta(6) - \zeta(6,v))\Big). \tag{34}$$

The $n = v - \ell$ results have been obtained by beginning with the general form given by (32). For $s(v,v-4)$ this becomes

$$s(v,v-4) = \frac{1}{6}\prod_{i=1}^{v-1} i^2 \sum_{\substack{j_1,j_2,j_3=1 \\ j_1 \neq j_2 \neq j_3}}^{v-1} \frac{1}{j_1^2 j_2^2 j_3^2}. \tag{35}$$

To evaluate this result, the constraint that none of the j_i is equal to another must be removed. This is accomplished by dropping it and eliminating all the possibilities where at least one of j_i is equal to another. Consequently, we must consider subtracting all the cases where two of the indices are equal to another and finally when three indices are equal to one another. The product over i yields $\Gamma(v)^2$. Thus, we arrive at

$$s(v, v-4) = \frac{1}{2}\Gamma(v)^2\Big(\sum_{j_1,j_2,j_3=1}^{v-1} \frac{1}{j_1^2 j_2^2 j_3^2} - 3\sum_{j_1,j_2=1}^{v-1} \frac{1}{j_1^4 j_2^2} + 2\sum_{j_1=1}^{v-1} \frac{1}{j_1^6}\Big), \tag{36}$$

which yields the result given by (34). In a similar manner one finds that

$$\begin{aligned} s(v, v-5) = \frac{1}{24}\Gamma(v)^2\Big(&\big(\zeta(2)-\zeta(2,v)\big)^4 - 6\big(\zeta(4)-\zeta(4,v)\big)\big(\zeta(2)-\zeta(2,v)\big)^2 \\ &+ 8\big(\zeta(6)-\zeta(6,v)\big)\big(\zeta(2)-\zeta(2,v)\big) + 3\big(\zeta(4)-\zeta(4,v)\big)^2\Big) \\ &- 6\big(\zeta(8)-\zeta(8,v)\big)\Big). \end{aligned} \tag{37}$$

In general, $s(v, v-\ell)$ is given by

$$s(v, v-\ell) = \sum_{\substack{j_1,\ldots,j_{\ell-1}=1 \\ j_1<j_2,\cdots<j_{\ell-1}}}^{v-1} \prod_{i=1}^{v-1}\prod_{k=1}^{\ell-1} \frac{i^2}{j_k^2} = \frac{\Gamma(v)^2}{(\ell-1)!} \sum_{\substack{j_1,\ldots,j_{\ell-1}=1 \\ j_1\neq j_2\cdots j_{n-2}\neq j_{\ell-1}}}^{v-1} \prod_{k=1}^{\ell-1} \frac{1}{j_k^2}. \tag{38}$$

In order to solve the sum on the right-hand side, one needs to remove the constraint that each of the j_i cannot equal one another. This means that we need to subtract all the possibilities when at least one of the indices is equal to another index from the sum where all the indices are equal each other or $\sum_{j_1,\ldots,j_{\ell-1}=1}^{v-1} \prod_{k=1}^{\ell-1} 1/j_k^2$. The number of sums that appear on the right-hand side becomes $p(\ell-1)$, where $p(k)$ is the partition function or the number of partitions summing to k. For example, in (36), we see that there are three sums for $s(v, v-4)$, since $p(3)$ is equal to three as a result of the partitions, {1,1,1}, {2,1} and {3}. In addition, the powers of the j_i will be twice the value of the parts in the partitions. That is, the sum $\sum_{j_1,j_2=1}^{v-1} 1/(j_1^4 j_2^2)$ corresponds to the partition {2,1}. Moreover, the sign outside each sum is $(-1)^{N_k+1}$, while the factors preceding each sum is the factorial of the length N_k divided by the factorials of the multiplicities of each part and the parts taken to the power of their multiplicities. For example, in the case of the partition {2,1} the sum of the parts is three, while the multiplicities are both equal to 1. Hence the factor outside the sum becomes $3!/(2 \cdot 1! \cdot 1 \cdot 1!) = 3$. See [15] for more details.

If we introduce the results for $s(v, v-n)$ into (31), then for $n = 1$ we obtain

$$\frac{\Gamma(v)}{\Gamma(v+1/2)} = \frac{2}{\sqrt{\pi}} c_{2v,v-1}, \tag{39}$$

which is only valid for $v > 1$. Alternatively, (39) can be expressed as $B(v, 1/2) = 2\, c_{2v,v-1}$, where $B(x, y)$ represents the beta function. Furthermore, according to

No. 2.5.3.1 in [16], we have

$$\int_0^{\pi/2} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2v-1} dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(v)}{\Gamma(v+1/2)} = c_{2v,v-1}, \tag{40}$$

while from Appendix I.1.9 of the same reference, we find that

$$\begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix}^{2v-1} = \frac{1}{2^{2v-2}} \sum_{k=0}^{v-1} (\mp 1)^{v-k-1} \binom{2v-1}{k} \begin{Bmatrix} \sin(2v-2k-1)x \\ \cos(2v-2k-1)x \end{Bmatrix}. \tag{41}$$

A surprising property of the above analysis is that the $c_{2v,k}$ are coefficients in the power series expansion of $(x/\sin x)^{2v}$, but in (40), they are related to positive powers of sine or cosine taken to $2v-1$ when $k = v-1$. Inserting the second result into the first one yields

$$\frac{1}{2^{2v-2}} \sum_{k=0}^{v-1} \frac{(-1)^{v-k-1}}{(2v-2k-1)} \binom{2v-1}{k} = c_{2v,v-1}. \tag{42}$$

This result can be checked by implementing it in Mathematica as follows:

$$\begin{aligned}&\text{C2vvminus1}[v\_] := 2\verb|^|(2-2v)\text{Sum}[(-1)\verb|^|(v-k-1)\text{Binomial}[2v-1,k]/(2v-2k-1),\\ &\qquad \{k,0,v-1\}].\end{aligned} \tag{43}$$

Putting v equal to 5 in this line of code yields 128/315, while putting $\rho = 10$ in the $k = 4$ result of Table 2 gives the same value.

It emerges that the terms at the upper limit of the sum in (42) contribute more to the value of $c_{2v,v-1}$ than those at the lower limit. Therefore, we express (42) as

$$c_{2v,v-1} = 2^{2-2v} \binom{2v-1}{v} \sum_{k=0}^{v-1} \frac{(-1)^k}{(2k+1)} \prod_{j=1}^{k} \Big(\frac{1-j/v}{1+j/v} \Big). \tag{44}$$

Now we expand the terms in the sum or product, thereby obtaining

$$\begin{aligned} c_{2v,v-1} = 2^{2-2v} \binom{2v-1}{v} \Big[&\Big(1 - \frac{2}{v} + \frac{2}{v^2} + O\big(\frac{1}{v^3}\big) \Big) - \frac{1}{3} \Big(1 - \frac{2}{v} + \frac{2}{v^2} + O\big(\frac{1}{v^3}\big) \Big) \Big(1 - \frac{4}{v} \\ &+ \frac{8}{v^2} + O\big(\frac{1}{v^3}\big) \Big) + \frac{1}{5} \Big(1 - \frac{2}{v} + \frac{2}{v^2} + O\big(\frac{1}{v^3}\big) \Big) \Big(1 - \frac{4}{v} + \frac{8}{v^2} + O\big(\frac{1}{v^3}\big) \Big) \\ &\times \Big(1 - \frac{6}{v} + \frac{18}{v^2} + O\big(\frac{1}{v^3}\big) \Big) - \cdots \Big]. \end{aligned} \tag{45}$$

In more compact notation the above result can be written as

$$c_{2v,v-1} = 2^{2-2v} \binom{2v-1}{v} \Big(\sum_{j=0}^{v-1} \frac{(-1)^j}{(2j+1)} - \frac{1}{v} \sum_{j=0}^{v-1} \frac{(-1)^j}{(2j+1)} (j+1)(j+2) + O\big(\frac{1}{v^2}\big) \Big). \tag{46}$$

The first sum is a known result given by No. 4.1.3.4 in [16], while the second sum requires decomposition. Then we arrive at

$$\sum_{j=0}^{v-1} \frac{(-1^j}{2j+1}(j+1)(j+2) = \frac{1}{2}\sum_{j=0}^{v-1}(-1)^j j + \frac{5}{4}\sum_{j=0}^{v-1}(-1)^j + \frac{3}{4}\sum_{j=0}^{v-1}\frac{(-1)^j}{2j+1}. \tag{47}$$

The last sum in (47) is simply another occurrence of the first sum in (46). Consequently, $c_{2v,v-1}$ becomes

$$c_{2v,v-1} = 2^{2-2v}\binom{2v-1}{v}\Big(\frac{\pi}{4} + \frac{(-1)^{v-1}}{2}\,\beta(v+1/2) + \frac{(-1)^{v-1}}{2v}\lfloor v/2\rfloor - \frac{5}{8v}\big(1-(-1)^v\big) + \frac{3}{4v}\frac{(-1)^{v-1}}{2}\,\beta(v+1/2) + O\big(\frac{1}{v^2}\big)\Big), \tag{48}$$

where $\lfloor x\rfloor$ denotes the floor function or the greatest integer less than or equal to x, $\beta(v) = (\psi((v+1)/2) - \psi(v/2))/2$ and $\psi(x)$ is the digamma function. For large values of v, the leading term given by the $\pi/4$ term on the right-hand side is a good asymptotic approximation to $c_{2v,v-1}$.

For $n = 2$, (31) yields

$$\sum_{k=1}^{v-1}\frac{1}{k^2} = \frac{\pi^2}{6} - \zeta(2,v) = \frac{2}{3}\frac{c_{2v,v-2}}{c_{2v,v-1}}, \tag{49}$$

where $\zeta(x,y)$ represents the Hurwitz zeta function and $v > 2$. By adopting the same approach for $n = 3$ to $n = 6$, we arrive at

$$\sum_{k=1}^{v-1}\frac{1}{k^4} = \frac{\pi^4}{90} - \zeta(4,v) = \frac{4}{9}\Big(\frac{c_{2v,v-2}}{c_{2v,v-1}}\Big)^2 - \frac{4}{15}\frac{c_{2v,v-3}}{c_{2v,v-1}}, \tag{50}$$

$$\sum_{k=1}^{v-1}\frac{1}{k^6} = \frac{\pi^6}{945} - \zeta(6,v) = \frac{4}{105}\frac{c_{2v,v-4}}{c_{2v,v-1}} - \frac{4}{15}\frac{c_{2v,v-3}}{c_{2v,v-1}}\frac{c_{2v,v-2}}{c_{2v,v-1}} + \frac{8}{27}\Big(\frac{c_{2v,v-2}}{c_{2v,v-1}}\Big)^3, \tag{51}$$

$$\sum_{k=1}^{v-1}\frac{1}{k^8} = \frac{\pi^8}{9450} - \zeta(8,v) = \frac{8}{14175}\left(350\left(\frac{c_{2v,v-2}}{c_{2v,v-1}}\right)^4 - 420\,\frac{c_{2v,v-3}}{c_{2v,v-1}}\left(\frac{c_{2v,v-2}}{c_{2v,v-1}}\right)^2 + 63\Big(\frac{c_{2v,v-3}}{c_{2v,v-1}}\Big)^2 + 60\,\frac{c_{2v,v-4}}{c_{2v,v-1}}\frac{c_{2v,v-2}}{c_{2v,v-1}} - 5\frac{c_{2v,v-5}}{c_{2v,v-1}}\right), \tag{52}$$

and

$$\sum_{k=1}^{v-1}\frac{1}{k^{10}} = \frac{\pi^{10}}{93555} - \zeta(10,v) = \frac{4}{93555}\left(3080\Big(\frac{c_{2v,v-2}}{c_{2v,v-1}}\Big)^5 - 4620\,\frac{c_{2v,v-3}}{c_{2v,v-1}}\Big(\frac{c_{2v,v-2}}{c_{2v,v-1}}\Big)^3 + 1386\Big(\frac{c_{2v,v-3}}{c_{2v,v-1}}\Big)^2\frac{c_{2v,v-2}}{c_{2v,v-1}} + 660\frac{c_{2v,v-4}}{c_{2v,v-1}}\left(\frac{c_{2v,v-2}}{c_{2v,v-1}}\right)^2 - 198\frac{c_{2v,v-4}}{c_{2v,v-1}}\frac{c_{2v,v-3}}{c_{2v,v-1}} - 55\frac{c_{2v,v-5}}{c_{2v,v-1}}\frac{c_{2v,v-2}}{c_{2v,v-1}} + 3\,\frac{c_{2v,v-6}}{c_{2v,v-1}}\right). \tag{53}$$

where $v > 2$ for the first result, $v > 3$ in the second, etc. In principle, this process can be continued for higher powers of the sum on the left-hand side by determining larger values of ℓ in the symmetric polynomials, $s(v, v-\ell)$. Consequently, we see that integer values of the Hurwitz zeta function for even powers can now be expressed in terms of ratios of the generalized cosecant numbers, which is indeed fascinating in view of the intractability of this famous function. Moreover, in the limit as $v \to \infty$, we obtain new results for the Riemann zeta function such as

$$\zeta(4) = \frac{\pi^4}{90} = \lim_{v\to\infty} \Big\{ \frac{4}{9} \Big(\frac{c_{2v,v-2}}{c_{2v,v-1}} \Big)^2 - \frac{4}{15} \frac{c_{2v,v-3}}{c_{2v,v-1}} \Big\}. \tag{54}$$

Unfortunately, we cannot introduce (30) into the above result because $|\rho/k|$ is approximately equal to 2 when k is equal to either $v-1$ and $v-2$, whereas we have observed in Table 4 that as ρ or $2v$ increases, the ratio $2v/k$ needs to increase dramatically in order to ensure that $\beta(\rho, k)$ remains close to unity. Otherwise, (30) does not represent a good approximation for the generalized cosecant numbers. Therefore, we require asymptotic forms as in (48) for the generalized cosecant numbers of the form, $c_{2v,v-\ell}$, where $\ell = 2, 3, \ldots$. These have yet to be developed.

To conclude, it should be mentioned that the series on the left-hand sides of (49) to (53) also represent specific values of the generalized harmonic numbers, which are defined as $H_{n,r} = \sum_{k=1}^{n} 1/k^r$ [17]. In particular, for the case of $r = 2$ given by (49) the numbers are known as Wolstenholme numbers, which appear as Sequences A007406, A007408, A11354 and A123751 in the Online Encyclopedia of Integer Sequences [18]. Although this paper has only been able to deal with the highest order coefficients in the generalized cosecant numbers, in a future paper, the partition method for a power series expansion will be applied to an alternative formulation of $z^\rho/\sin^\rho z$, thereby enabling general formulas for the lowest order coefficients to be derived. There it will be observed that the coefficients of the generalized cosecant numbers are sums of products involving the even integer values of the Riemann zeta function.

Acknowledgement. The author is grateful to Professor Carlos M. da Fonseca, Kuwait College of Science and Technology, Kuwait, without whose support and encouragement this paper would not exist.

References

[1] V. Kowalenko, Applications of the cosecant and related numbers, *Acta Appl. Math.* **114** (2011), No. 1-2, 15-134.

[2] V. Kowalenko, *The Partition Method for a Power Series Expansion: Theory and Applications*, Academic Press/Elsevier, Oxford, 2017.

[3] I.S. Gradshteyn, I.M. Ryzhik, A. Jeffrey (ed.), *Table of Integrals, Series, and Products*, Fifth Ed., Academic Press, Boston, 1994.

[4] V. Kowalenko, Towards a theory of divergent series and its importance to asymptotics, in *Recent Research Developments in Physics*, Vol. 2, Transworld Research Network, Trivandrum, India, 2001, 17-68.

[5] V. Kowalenko, Exactification of the asymptotics for Bessel and Hankel functions, *Appl. Math. and Comp.* **133** (2002), 487-518.

[6] V. Kowalenko, *The Stokes Phenomenon, Borel Summation and Mellin-Barnes Regularisation*, Bentham ebooks, http://www.bentham.org, 2009.

[7] V. Kowalenko, Properties and applications of the reciprocal logarithm numbers, *Acta Appl. Math.* **109** (2010), 413-437.

[8] V. Kowalenko, Generalizing the reciprocal logarithm numbers by adapting the partition method for a power series expansion, *Acta Appl. Math.* **106** (2009), No.3, 369-420.

[9] V. Kowalenko, Euler and divergent series, *Eur. J. Pure Appl. Math.* **4** (2011), 370-423.

[10] S. Wolfram, *Mathematica-A System for Doing Mathematics by Computer*, Addison-Wesley, Reading, 1992.

[11] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1964.

[12] J. Spanier and K.B. Oldham, *An Atlas of Functions*, Hemisphere Publishing, New York, 1987.

[13] V. Kowalenko, Developments from programming the partition method for a power series expansion, arXiv:1203.4967v1, (2012).

[14] C.M. da Fonseca, M.L. Glasser and V. Kowalenko, Generalized cosecant numbers and trigonometric inverse power sums, *Appl. Anal. Discrete Math.* (2018) **12**, 70-109.

[15] C.M. da Fonseca, M.L. Glasser and V. Kowalenko, An integral approach to the Gardner-Fisher and untwisted Dowker sums, arXiv:1603.03700, (2016).

[16] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series, Vol. 1: Elementary Functions*, Gordon & Breach, New York, 1986.

[17] J. Sondow and E.W. Weisstein, Harmonic number, From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/HarmonicNumber.html, Oct. 30, (2017).

[18] N.J.A. Sloane, Sequences A007406/M4004, A111354, and A123751 in *The On-Line Encyclopedia of Integer Sequences*, Oct. 31, (2017).