A COMBINATORIAL PROBLEM SOLVED BY A META-FIBONACCI RECURRENCE RELATION

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Abstract

We present a natural, combinatorial problem whose solution is given by the meta-Fibonacci recurrence relation $a(n) = \sum_{i=1}^{p} a(n-i+1-a(n-i))$, where p is prime. This combinatorial problem is less general than those given by Jackson and Ruskey, and Ruskey and Deugau, but it has the advantage of having a simpler statement.

1. Introduction

Let M be a matrix with entries in $\mathbb{Z}_2$, such that every column contains at least one 1. We want to pick a subset of the rows such that when they are added together modulo 2, their sum $\vec{s}$ has as many 1's as possible. If M has n columns, what is the largest number of 1's we can guarantee $\vec{s}$ to have? For example, if $n = 5$, we can always find a set of rows whose sum $\vec{s}$ contains at least four 1's. Let $\lambda(n)$ denote the largest number of 1's $\vec{s}$ can be guaranteed to have for any M with n nonzero columns. We will show that $\lambda(n)$ satisfies the recurrence relation

$$\lambda(n) = \lambda(n - \lambda(n-1)) + \lambda(n - 1 - \lambda(n-2)). \tag{1}$$

More generally, for p prime, let $\vec{v} = (v_1, \ldots, v_n)$ satisfy $v_i \in \mathbb{F}_p$ for $1 \leq i \leq n$. Let $\operatorname{supp}(\vec{v}) = \{i \in [n] : v_i \neq 0\}$ and let $\|\vec{v}\| = |\operatorname{supp}(\vec{v})|$, i.e., $\|\vec{v}\|$ is the number of nonzero terms in $\vec{v}$. Let M be an $m \times n$ matrix whose entries are in $\mathbb{F}_p$. Let $\mathbf{row}(M)$ be the rowspace of M, i.e., the set of all linear combinations of the row vectors of M over the field $\mathbb{F}_p$. Let $c(M)$ denote the *capacity* of M, which we define as follows,

$$c(M) = \max_{\vec{v} \in \mathbf{row}(M)} \|\vec{v}\|.$$

For each integer $n \geq 1$, let $\lambda_p(n)$ be the minimum possible capacity of an $\mathbb{F}_p$-matrix consisting of n *nonzero* columns (i.e., no column equals $\vec{0}$). Restated, let

$$\mathcal{M}_n^* = \{M \in \mathbb{F}_p^{m\times n} : 1 \leq m \leq p^n \text{ and no column of } M \text{ equals } \vec{0}\},$$

then

$$\lambda_p(n) = \min_{M\in\mathcal{M}_n^*} c(M).$$

We will see that λ_p satisfies the recurrence relation

$$\lambda_p(n) = \sum_{i=1}^{p} \lambda_p(n-i+1-\lambda_p(n-i)). \tag{2}$$

This type of recurrence relation is called a meta-Fibonacci relation.

Meta-Fibonacci sequences have been studied by various authors, dating at least as far back as 1985, when Hofstadter [4] apparently coined the term "meta-Fibonacci." These are integer sequences defined by "nested, Fibonacci-like" recurrence relations, such as relation (1), which was studied by Conolly [2], and (2). Generalizations of (2) were studied in [1] and [3], and were shown in [5] and [6] to be solutions to certain combinatorial problems involving k-ary infinite trees, and compositions of integers. The "matrix capacity" problem described above is a different combinatorial problem whose solution is also given by relation (2). This combinatorial problem is "natural" in the sense that it arose while the first named author was working on a problem in spatial graph theory. It was only later that we learned (through the OEIS A046699) that it can be characterized as a meta-Fibonacci sequence.

The remainder of the paper is organized as follows. In Section 2, we use Proposition 1 and Claim 1 to obtain a lower bound on $\lambda_p(n)$ for $n \geq 1$. In Lemma 2, we prove a matching upper bound on $\lambda_p(n)$ for the special case when $n = \sum_{j=0}^{k} p^j$ by constructing a matrix with $\sum_{j=0}^{k} p^j$ columns whose capacity matches the lower bound we obtained on $\lambda_p\left(\sum_{j=0}^{k} p^j\right)$. In Proposition 2, we use a generalization of the matrix we use in Lemma 2 to provide the matching upper bound on $\lambda_p(n)$ for all $n \geq 1$. Once we have the exact value of $\lambda_p(n)$ for all $n \geq 1$, we prove that $\lambda_p(n)$ satisfies the meta-Fibonacci recurrence relation in Corollary 3 by using a result from [6].

2. Main Result

We begin with a lemma which allows us to produce a lower bound on $\lambda_p(n)$. For the remainder of this paper, instead of writing λ_p, we will simply write λ. For a matrix M, let $\mathbf{row}^*(M) = \mathbf{row}(M) - \{\vec{0}\}$.

Lemma 1. *Let M be an $\mathbb{F}_p$-matrix with n nonzero columns, i.e., $M \in \mathcal{M}_n^*$. Let $\vec{v} \in \mathbf{row}^*(M)$. If*

$$p\lambda(n - \|\vec{v}\|) > \|\vec{v}\|,$$

then there is a vector $\vec{z} \in \mathbf{row}^*(M)$ such that $\|\vec{z}\| > \|\vec{v}\|$.

Proof. Let M be an $\mathbb{F}_p$-matrix with n nonzero columns. Let $\vec{v} = (v_1, \ldots, v_n) \in \mathbf{row}^*(M)$, and let $k = \|\vec{v}\|$. W.l.o.g., suppose $v_i \neq 0$ for $1 \leq i \leq k$ and $v_i = 0$ for $k+1 \leq i \leq n$. Let $\vec{w} \in \mathbf{row}^*(M)$ be such that $w_i \neq 0$ for at least $\lambda(n-k)$ coordinates i, where $k+1 \leq i \leq n$. In other words, if we let $\vec{w}_L = (w_1, \ldots, w_k)$ and $\vec{w}_R = (w_{k+1}, \ldots, w_n)$, then $\|\vec{w}_R\| \geq \lambda(n-k)$. Since $\|\vec{w}\| = \|\vec{w}_L\| + \|\vec{w}_R\|$, if $\|\vec{w}_L\| \geq (p-1)\lambda(n-k)$, then $\|\vec{w}\| \geq p\lambda(n-k) > \|\vec{v}\|$, and we are done. So we may assume that $\|\vec{w}_L\| < (p-1)\lambda(n-k)$.

Our goal will be to prove that there exists a nonzero constant c such that $\|c\vec{w}_L + \vec{v}_L\| > k - \lambda(n-k)$, where $\vec{v}_L = (v_1, \ldots, v_k)$. Once we establish that such a constant exists, then we will be done, because we will have $\|c\vec{w} + \vec{v}\| = \|c\vec{w}_L + \vec{v}_L\| + \|\vec{w}_R\| > (k - \lambda(n-k)) + \lambda(n-k) = k$.

For $1 \leq a \leq p-1$, let $S_a = \{i \in [k] : aw_i + v_i = 0\}$. Since $v_i \neq 0$ for $1 \leq i \leq k$, then $S_a \subseteq \operatorname{supp}(\vec{w}_L)$. Thus, if $aw_i + v_i = 0 = bw_i + v_i$ with $w_i \neq 0$, then $a = b$. Therefore, if $a \neq b$, then $S_a \cap S_b = \emptyset$. Since $\bigcup_{a=1}^{p-1} S_a \subseteq \operatorname{supp}(\vec{w}_L)$ where the sets S_a are pairwise disjoint, we have

$$\sum_{a=1}^{p-1} |S_a| \leq |\operatorname{supp}(\vec{w}_L)| = \|\vec{w}_L\| < (p-1)\lambda(n-k).$$

Therefore, the average value of $|S_a|$ is strictly less than $\lambda(n-k)$, and if we let $c \in [p-1]$ be such that $|S_c|$ is minimum, then $|S_c| < \lambda(n-k)$. Thus, $\|c\vec{w}_L + \vec{v}_L\| = k - |S_c| > k - \lambda(n-k)$, and as noted above, we are done. Specifically, $\|c\vec{w} + \vec{v}\| > \|\vec{v}\|$. □

Corollary 1. *If $1 \leq n \leq p$, then $\lambda(n) = n$.*

Proof. Suppose $1 \leq n \leq p$, $M \in \mathcal{M}_p^*$ with $c(M) = \lambda(n)$, and $\vec{v} \in \mathbf{row}^*(M)$ with $\|\vec{v}\| = c(M)$. Assume towards a contradiction that $\lambda(n) < n$. Then $\|\vec{v}\| = \lambda(n) < n \leq p$. Since $n - \|\vec{v}\| > 0$, then $\lambda(n - \|\vec{v}\|) \geq 1$ and $\|\vec{v}\| < p\lambda(n - \|\vec{v}\|)$. Thus, Lemma 1 implies there is a vector $\vec{z} \in \mathbf{row}^*(M)$ such that $\|\vec{z}\| > \|\vec{v}\|$, which is a contradiction since $\|\vec{v}\| = c(M)$. Therefore, by contradiction, $\lambda(n) \geq n$. This implies $\lambda(n) = n$, since we naturally have $\lambda(n) \leq n$. □

For an integer $k \geq 0$, let $\sigma_k = \sum_{j=0}^{k} p^j$.

Proposition 1. *Suppose*

$$n = \sum_{j=\ell}^{k} b_j \sigma_j,$$

where $b_k \geq 1$, and $0 \leq b_j \leq p-1$ for $j \neq \ell$, and $1 \leq b_\ell \leq p$, and $0 \leq \ell \leq k$. Then

$$\lambda(n) \geq \sum_{j=\ell}^{k} b_j p^j.$$

Proof of Proposition 1. We proceed by induction on k. When $k = 0$, then $n = b_0\sigma_0 = b_0$. Since $1 \leq b_0 \leq p$, then $\lambda(n) = b_0$ by Corollary 1, thus, $\lambda(n) = b_0 p^0$ and the result holds. Now suppose $k \geq 1$. The inductive hypothesis is if

$$n = \sum_{j=\ell}^{m} b_j \sigma_j,$$

where $b_m \geq 1$, and $0 \leq b_j \leq p-1$ for $j \neq \ell$, and $1 \leq b_\ell \leq p$, and $m < k$, then

$$\lambda(n) \geq \sum_{j=\ell}^{m} b_j p^j.$$

Let M be an $\mathbb{F}_p$-matrix with n nonzero columns. Suppose $\vec{v} \in \mathbf{row}^*(M)$ with

$$\|\vec{v}\| < \sum_{j=\ell}^{k} b_j p^j.$$

Then

$$\begin{aligned} n - \|\vec{v}\| > n - \sum_{j=\ell}^{k} b_j p^j &= \sum_{j=\ell}^{k} b_j \sigma_j - \sum_{j=\ell}^{k} b_j p^j \\ &= \sum_{j=\ell}^{k} b_j(\sigma_j - p^j) \\ &= \sum_{j=\ell}^{k} b_j \sigma_{j-1}, \end{aligned}$$

where we define $\sigma_{-1} = 0$ to handle the case $j = 0$, since $\sigma_0 - p^0 = 0$. Thus,

$$n - \|\vec{v}\| \geq \sum_{\ell-1 \leq j \leq k-1} b_{j+1}\sigma_j + 1.$$

We want to determine a lower bound on $p\lambda\left(\sum_{j=\ell-1}^{k-1} b_{j+1}\sigma_j + 1\right)$ that allows us to conclude that $p\lambda(n - \|\vec{v}\|) > \|\vec{v}\|$ so that we may use Lemma 1. We consider the case where $b_\ell = p$ and the case where $1 \leq b_\ell \leq p-1$ separately.

Suppose $b_\ell = p$. Then

$$\begin{aligned}\sum_{\ell-1\le j\le k-1} b_{j+1}\sigma_j + 1 &= \sum_{\ell\le j\le k-1} b_{j+1}\sigma_j + b_\ell\sigma_{\ell-1} + 1\\ &= \sum_{\ell\le j\le k-1} b_{j+1}\sigma_j + (p\sigma_{\ell-1}+1)\\ &= \sum_{\ell\le j\le k-1} b_{j+1}\sigma_j + \sigma_\ell\\ &= \sum_{\ell+1\le j\le k-1} b_{j+1}\sigma_j + b_{\ell+1}\sigma_\ell + \sigma_\ell\\ &= \sum_{\ell+1\le j\le k-1} b_{j+1}\sigma_j + (b_{\ell+1}+1)\sigma_\ell.\end{aligned}$$

Notice that the sum satisfies all of the criteria for the inductive hypothesis. Specifically, the coefficient of its lowest sigma-term σ_ℓ is $b_{\ell+1}+1$, which satisfies $1 \le b_{\ell+1}+1 \le p$; the coefficient of σ_j is b_{j+1} and $0 \le b_{j+1} \le p-1$ for $j \ne \ell$; the coefficient of the largest sigma-term σ_{k-1} is b_k, which satisfies $b_k \ge 1$; and finally, the index of its largest sigma term is $k-1$ which is strictly less than k. Therefore, by the inductive hypothesis,

$$\begin{aligned}p\lambda\left(\sum_{\ell+1\le j\le k-1} b_{j+1}\sigma_j + (b_{\ell+1}+1)\sigma_\ell\right) &\ge p\left(\sum_{\ell+1\le j\le k-1} b_{j+1}p^j + (b_{\ell+1}+1)p^\ell\right)\\ &= \sum_{\ell+1\le j\le k-1} b_{j+1}p^{j+1} + (b_{\ell+1}+1)p^{\ell+1}\\ &= \sum_{\ell\le j\le k-1} b_{j+1}p^{j+1} + p\cdot p^\ell\\ &= \sum_{\ell+1\le j\le k} b_j p^j + p\cdot p^\ell\\ &= \sum_{\ell\le j\le k} b_j p^j,\end{aligned}$$

where the last equality holds because $b_\ell = p$. Since λ is a nondecreasing function, our previous work implies

$$\begin{aligned}p\lambda(n-\|\vec{v}\|) &\ge p\lambda\left(\sum_{\ell+1\le j\le k-1} b_{j+1}\sigma_j + (b_{\ell+1}+1)\sigma_\ell\right)\\ &\ge \sum_{\ell\le j\le k} b_j p^j > \|\vec{v}\|.\end{aligned}$$

Thus, by Lemma 1, there is a vector $\vec{z} \in \mathbf{row}^*(M)$ such that $\|\vec{z}\| > \|\vec{v}\|$.

Now suppose $1 \le b_\ell \le p-1$. Recall that the sum is

$$\sum_{\ell-1\le j\le k-1} b_{j+1}\sigma_j + 1 \;=\; \sum_{\ell-1\le j\le k-1} b_{j+1}\sigma_j + 1\cdot\sigma_0.$$

In this case, the smallest sigma-term is σ_0, and its coefficient is b_1+1, where $b_1 = 0$ if $\ell \ge 2$. We note that the sum satisfies all of the criteria for the inductive hypothesis. Since each b_j satisfies $0 \le b_j \le p-1$, then $1 \le b_1+1 \le p$; when $j \ge 1$, the coefficient of each σ_j is b_{j+1} and $0 \le b_{j+1} \le p-1$; the coefficient of the largest sigma-term σ_{k-1} is b_k, which satisfies $b_k \ge 1$; and finally, the index of its largest sigma term is $k-1$ which is strictly less than k.

When $\ell \ge 2$, the coefficient of σ_0 is 1, and we apply the inductive hypothesis to obtain

$$\begin{aligned} p\lambda\left(\sum_{\ell-1\le j\le k-1} b_{j+1}\sigma_j + 1\right) &\ge p\left(\sum_{\ell-1\le j\le k-1} b_{j+1}p^j + 1\right)\\ &= \sum_{\ell-1\le j\le k-1} b_{j+1}p^{j+1} + p\\ &= \sum_{\ell\le j\le k} b_j p^j + p. \end{aligned}$$

Thus,

$$\begin{aligned} p\lambda(n - \|\vec{v}\|) &\ge p\lambda\left(\sum_{\ell-1\le j\le k-1} b_{j+1}\sigma_j + 1\right)\\ &\ge \sum_{\ell\le j\le k} b_j p^j + p > \|\vec{v}\|. \end{aligned}$$

When $\ell \in \{0,1\}$, the sum is $\sum_{j=0}^{k-1} b_{j+1}\sigma_j + 1$, and we apply the inductive hypothesis to obtain

$$\begin{aligned} p\lambda\left(\sum_{0\le j\le k-1} b_{j+1}\sigma_j + 1\right) &= p\lambda\left(\sum_{1\le j\le k-1} b_{j+1}\sigma_j + (b_1+1)\right)\\ &\ge p\left(\sum_{1\le j\le k-1} b_{j+1}p^j + b_1 + 1\right)\\ &= \sum_{1\le j\le k-1} b_{j+1}p^{j+1} + b_1 p + p\\ &= \sum_{1\le j\le k} b_j p^j + p. \end{aligned}$$

Thus,

$$\begin{aligned} p\lambda(n - \|\vec{v}\|) &\geq p\lambda\left(\sum_{0\leq j\leq k-1} b_{j+1}\sigma_j + 1\right) \\ &\geq \sum_{1\leq j\leq k} b_j p^j + p \\ &> \sum_{\ell\leq j\leq k} b_j p^j > \|\vec{v}\|. \end{aligned}$$

Thus, by Lemma 1, there is a vector $\vec{z} \in \mathbf{row}^*(M)$ such that $\|\vec{z}\| > \|\vec{v}\|$. Therefore $\lambda(n) \geq \sum_{j=\ell}^{k} b_j p^j$. □

Now we show that every $n \geq 1$ can be written in the form described in Proposition 1. This is analogous to determining a base q representation of a number, i.e., $n = \sum_{j=0}^{k} c_j q^j$, where $q > 1$, and $0 \leq c_j \leq q-1$. A simple algorithm for finding the base q representation of n is to determine an integer k such that $q^k \leq n < q^{k+1}$. Then we determine the largest coefficient c_k such that $c_k q^k \leq n$. We then repeat this process on the remainder $(n-c_k q^k)$. Below, in the proof of Claim 1, we describe the analogous technique for writing $n = \sum_{j=0}^{k} b_j\sigma_j$, where $0 \leq b_j \leq p$.

Claim 1. *Let $n \in \mathbb{Z}^+$. Suppose $n < \sigma_{k+1}$. Then there exist coefficients b_j such that*

$$n = \sum_{j=0}^{k} b_j\sigma_j,$$

where $0 \leq b_j \leq p$ for $0 \leq j \leq k$, and if $b_j = p$ for some j, then $b_i = 0$ for all $i < j$.

Proof of Claim 1. Suppose $n \in \mathbb{Z}^+$ and $n < \sigma_{k+1}$. Then $n \leq \sigma_{k+1} - 1 = p\sigma_k$. We describe how to inductively define the b_j terms, starting with $j = k$ and decreasing to $j = 0$. To help us do this, we introduce remainder terms n_j. We start with $n_{k+1} = n$. Then, for $0 \leq j \leq k$, let b_j be the largest integer such that $b_j\sigma_j \leq n_{j+1}$, and let $n_j = n_{j+1} - b_j\sigma_j$. For $0 \leq j \leq k+1$, we will show that $0 \leq n_j \leq p\sigma_{j-1}$ (where we define $\sigma_{-1} = 0$), and for $0 \leq j \leq k$, we will show that $0 \leq b_j \leq p$. When $j = k+1$, we have $n_{k+1} = n$, and we have the desired bound $0 \leq n_{k+1} \leq p\sigma_k$. We use the inductive step below to show that $0 \leq b_j \leq p$ and $0 \leq n_j \leq p\sigma_{j-1}$ for $0 \leq j \leq k$.

We now prove the inductive step. Let $0 \leq j \leq k$. Assume $0 \leq n_{j+1} \leq p\sigma_j$. Since b_j is the largest integer such that $b_j\sigma_j \leq n_{j+1}$ and $0 \leq n_{j+1}$, then $b_j \geq 0$. Since $b_j\sigma_j \leq n_{j+1} \leq p\sigma_j$ and $\sigma_j \geq 1$, then $b_j \leq p$. Now we let $n_j = n_{j+1} - b_j\sigma_j$, and we show that $0 \leq n_j \leq p\sigma_{j-1}$. Since $b_j\sigma_j \leq n_{j+1}$, then $n_j \geq 0$. Since $n_{j+1} < (b_j+1)\sigma_j$, then $n_{j+1} - b_j\sigma_j < \sigma_j$, i.e., $n_j \leq \sigma_j - 1 = p\sigma_{j-1}$. (This works even in the special case $j = 0$, where $p\sigma_{-1} = 0$). Therefore, by induction, $0 \leq n_j \leq p\sigma_{j-1}$, for $0 \leq j \leq k+1$, and $0 \leq b_j \leq p$ for $0 \leq j \leq k$.

Now suppose $b_j = p$. Since $b_j\sigma_j \le n_{j+1} \le p\sigma_j$, then $n_{j+1} = p\sigma_j$ and $n_j = n_{j+1} - b_j\sigma_j = 0$. Moreover, $b_i = 0$ and $n_i = 0$ for all $i < j$.

To see that $n = \sum_{j=0}^{k} b_j\sigma_j$, observe that $b_j\sigma_j = n_{j+1} - n_j$ for $0 \le j \le k$, because of the definition of n_j. Thus,

$$\sum_{j=0}^{k} b_j\sigma_j = \sum_{j=0}^{k}(n_{j+1} - n_j) = n_{k+1} - n_0 = n - n_0.$$

Since $0 \le n_0 \le p\sigma_{-1}$ and $\sigma_{-1} = 0$, then $n_0 = 0$. Thus, $\sum_{j=0}^{k} b_j\sigma_j = n$.

We note that b_k could equal 0, but if $\sigma_k \le n$, then $b_k > 0$. □

With Proposition 1 and Claim 1, we have established a lower bound on $\lambda(n)$ for all $n \ge 1$. We need to prove the corresponding upper bound. We will do so by constructing a matrix with n columns whose capacity equals the lower bound given in Proposition 1. We begin by constructing such a matrix for certain values of n, namely, when $n = \sigma_k$ for some $k \ge 0$.

For each integer $k \ge 0$, we define a $(k+1)\times\sigma_k$ matrix B_k, recursively, as follows. The matrix B_0 is the 1×1 matrix whose sole entry is 1. For $k \ge 1$, B_k can be defined as a block matrix with a "row" consisting of p copies of B_{k-1} followed by a $k \times 1$ column of 0's, then one more row of dimensions $1 \times \sigma_k$ with its first σ_{k-1} entries equal to 0 (below the first B_{k-1}), then σ_{k-1} entries equal to 1 (below the next B_{k-1}), ..., then σ_{k-1} entries equal to $p-1$ (below the last B_{k-1}), and one last entry equal to 1, i.e.,

$$B_k = \left[\begin{array}{c|c|c|c|c} B_{k-1} & B_{k-1} & \cdots & B_{k-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0\ldots0 & 1\ldots1 & \cdots & (p-1)\ldots(p-1) & 1 \end{array}\right].$$

For $k \ge 1$, let B'_k be the $k \times \sigma_k$ matrix obtained from B_k by removing its last row, i.e.,

$$B'_k = \left[\begin{array}{c|c|c|c|c} B_{k-1} & B_{k-1} & \cdots & B_{k-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \end{array}\right].$$

Lemma 2. *For each $\vec{v} \in \mathbf{row}^*(B_k)$, $\|\vec{v}\| = p^k$.*

Proof. We proceed by induction on k. When $k = 0$, the result is trivial. Let $k \ge 1$. Assume the result for $j < k$. Let $\vec{v} \in \mathbf{row}^*(B_k)$. We first consider the case where $\vec{v} \in \mathbf{row}^*(B'_k)$. Then we can write

$$\vec{v} = (v_1^{(0)}, \ldots, v_{\sigma_{k-1}}^{(0)}, v_1^{(1)}, \ldots, v_{\sigma_{k-1}}^{(1)}, \ldots, v_1^{(p-1)}, \ldots, v_{\sigma_{k-1}}^{(p-1)}, 0).$$

To shorten notation, we will write

$$\vec{v} = (\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{p-1}, 0), \tag{3}$$

where $\vec{v}_i = (v_1^{(i)}, \ldots, v_{\sigma_{k-1}}^{(i)})$ for $0 \leq i \leq p-1$. Technically, in equation (3), $\vec{v}_i$ simply represents the coordinates $v_1^{(i)}, \ldots, v_{\sigma_{k-1}}^{(i)}$. We observe that $\vec{v}_0 = \vec{v}_1 = \cdots = \vec{v}_{p-1}$ based on how B'_k and $\vec{v}$ are defined. We also observe that $\vec{v}_i \in \mathbf{row}^*(B_{k-1})$. By the inductive hypothesis, $\|\vec{v}_i\| = p^{k-1}$, therefore, $\|\vec{v}\| = p^k$.

We now show the result holds for $\vec{w} \in \mathbf{row}^*(B_k) - \mathbf{row}^*(B'_k)$. Let $\vec{u}$ be the last row in B_k, i.e., $\vec{u} = (0, \ldots, 0, 1, \ldots, 1, \ldots, p-1, \ldots, p-1, 1)$. We observe that $\|\vec{u}\| = \sigma_k - \sigma_{k-1} = p^k$, thus, the result holds when $\vec{w} = \vec{u}$. To illustrate our argument, we next consider the special case where $\vec{w} = \vec{v} + \vec{u}$ for some $\vec{v} \in \mathbf{row}^*(B'_k)$. Again, we slightly abuse notation and write $\vec{u} = (\vec{0}, \vec{1}, \ldots, (p-1)\vec{1}, 1)$, where $\vec{0} = (0, \ldots, 0)$ and $\vec{1} = (1, \ldots, 1)$ are σ_{k-1}-dimensional vectors, and for a scalar c, we have $c\vec{1} = c(1, \ldots, 1) = (c, \ldots, c)$. Then we can write $\vec{v} + \vec{u} = (\vec{v}_0 + \vec{0}, \vec{v}_1 + \vec{1}, \ldots, \vec{v}_{p-1} + (p-1)\vec{1}, 1)$. Since we are working modulo p, a coordinate of $\vec{v}_j + j\vec{1}$ is congruent to 0 if and only if the corresponding coordinate of $\vec{v}_j$ is congruent to $p - j$. Thus, we can count the total number of coordinates that are congruent to 0 in $\vec{v} + \vec{u}$ as follows

$$\begin{pmatrix} \text{Total \# of 0-coordinates} \\ \text{in } \vec{v} + \vec{u} \end{pmatrix} = \sum_{j=0}^{p-1} (\text{\# of } (p-j)\text{-coordinates in } \vec{v}_j). \tag{4}$$

Since $\vec{v}_0 = \vec{v}_1 = \cdots = \vec{v}_{p-1}$, equation (4) reduces to

$$\begin{pmatrix} \text{Total \# of 0-coordinates} \\ \text{in } \vec{v} + \vec{u} \end{pmatrix} = \begin{pmatrix} \text{Total \# of coordinates} \\ \text{in } \vec{v}_0 \end{pmatrix} = \sigma_{k-1}.$$

Thus, $\|\vec{v} + \vec{u}\| = \sigma_k - \sigma_{k-1} = p^k$. In general, $\vec{w} \in \mathbf{row}^*(B_k) - \mathbf{row}^*(B'_k)$ satisfies $\vec{w} = \vec{v} + c\vec{u}$ for some $\vec{v} \in \mathbf{row}^*(B'_k)$ and $c \not\equiv 0 \pmod{p}$. In this case, $\vec{w} = (\vec{v}_0 + c\vec{0}, \vec{v}_1 + c\vec{1}, \ldots, \vec{v}_{p-1} + c(p-1)\vec{1}, 1)$, and equation (4) becomes

$$\begin{pmatrix} \text{Total \# of 0-coordinates} \\ \text{in } \vec{w} \end{pmatrix} = \sum_{j=0}^{p-1} (\text{\# of } (p-cj)\text{-coordinates in } \vec{v}_j), \tag{5}$$

where arithmetic is modulo p. Since $\vec{v}_0 = \vec{v}_1 = \cdots = \vec{v}_{p-1}$, we obtain

$$\begin{pmatrix} \text{Total \# of 0-coordinates} \\ \text{in } \vec{w} \end{pmatrix} = \sum_{j=0}^{p-1} (\text{\# of } (p-cj)\text{-coordinates in } \vec{v}_0).$$

Since p is prime and $c \not\equiv 0 \pmod{p}$, then $\{p, p-c, p-2c, \ldots, p-(p-1)c\}$ is equivalent to $\{0, 1, \ldots, p-1\}$ modulo p, thus,

$$\begin{pmatrix} \text{Total \# of 0-coordinates} \\ \text{in } \vec{w} \end{pmatrix} = \begin{pmatrix} \text{Total \# of coordinates} \\ \text{in } \vec{v}_0 \end{pmatrix} = \sigma_{k-1}.$$

Therefore, $\|\vec{w}\| = \sigma_k - \sigma_{k-1} = p^k$, and we can conclude that for each $\vec{v} \in \mathbf{row}^*(B_k)$, $\|\vec{v}\| = p^k$. □

Since B_k has σ_k columns, Lemma 2 implies that $\lambda(n) \leq p^k$ when $n = \sigma_k$ for some nonnegative integer k. We would like a similar upper bound on $\lambda(n)$ for all positive integers n. Thus, we provide the following proposition.

Proposition 2. *If* $n = \sum_{j=0}^{k} b_j \sigma_j$, *then* $\lambda(n) \leq \sum_{j=0}^{k} b_j p^j$.

Proof of Proposition 2. We will construct a matrix M with n columns such that $c(M) = \sum_{j=0}^{k} b_j p^j$. The matrix M will essentially be a block matrix with b_j copies of B_j for $0 \leq j \leq k$. However, the number of rows of B_j does not equal the number of rows of B_ℓ when $j \neq \ell$. Thus, for $0 \leq j \leq k$, we define the $(k+1) \times \sigma_j$ matrix $B_j^{(k)}$ where the first j rows of $B_j^{(k)}$ match the first j rows of B_j and the last $k+1-j$ rows of $B_j^{(k)}$ all equal the last row of B_j. Thus, $B_0^{(k)}$ is a $(k+1) \times 1$ column of 1's, and for $1 \leq j \leq k$,

$$B_j^{(k)} = \left(B_j^{(k)}\right)_{(k+1)\times\sigma_j} = \left[\begin{array}{c|c|c|c|c} B_{j-1} & B_{j-1} & \cdots & B_{j-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0\ldots0 & 1\ldots1 & \cdots & (p-1)\ldots(p-1) & 1 \\ \hline 0\ldots0 & 1\ldots1 & \cdots & (p-1)\ldots(p-1) & 1 \\ \hline \vdots & \vdots & \ldots & \vdots & \vdots \\ \hline 0\ldots0 & 1\ldots1 & \cdots & (p-1)\ldots(p-1) & 1 \end{array}\right]$$

where the last row is repeated $(k+1) - j$ times. After comparing $B_j^{(k)}$ with B_j, it is easy to see that $\mathbf{row}^*(B_j^{(k)}) = \mathbf{row}^*(B_j)$.

Let n be a positive integer such that $n = \sum_{j=0}^{k} b_j \sigma_j$. Let M be the $(k+1) \times n$ matrix defined as a block matrix with b_j copies of $B_j^{(k)}$ for $0 \leq j \leq k$, where the blocks appear in a single row in nondecreasing order according to their lower index, i.e.,

$$M = \Big[\ \underbrace{B_0^{(k)} \ \cdots \ B_0^{(k)}}_{b_0} \ \Big|\ \underbrace{B_1^{(k)} \ \cdots \ B_1^{(k)}}_{b_1} \ \Big|\ \cdots \ \Big|\ \underbrace{B_k^{(k)} \ \cdots \ B_k^{(k)}}_{b_k}\ \Big].$$

Let $\vec{v} \in \mathbf{row}^*(M)$. Then we can (essentially) write

$$\vec{v} = (\vec{v}_1^{(0)}, \ldots, \vec{v}_{b_0}^{(0)}, \vec{v}_1^{(1)}, \ldots, \vec{v}_{b_1}^{(1)}, \ldots, \vec{v}_1^{(k)}, \ldots, \vec{v}_{b_k}^{(k)})$$

where $\vec{v}_i^{(j)} \in \mathbf{row}^*(B_j)$ for $0 \leq j \leq k$ and $1 \leq i \leq b_j$. Moreover, for $1 \leq i \leq b_j$, we have $\vec{v}_i^{(j)} = \vec{v}_{b_j}^{(j)}$. Thus,

$$\|\vec{v}\| = \sum_{j=0}^{k} b_j \|\vec{v}_{b_j}^{(j)}\|.$$

Because $\vec{v}_{b_j}^{(j)} \in \mathbf{row}^*(B_j)$, Lemma 2 implies $\|\vec{v}_{b_j}^{(j)}\| = p^j$, therefore, $\|\vec{v}\| = \sum_{j=0}^{k} b_j p^j$. Thus, $c(M) = \sum_{j=0}^{k} b_j p^j$, and $\lambda(n) \leq \sum_{j=0}^{k} b_j p^j$. □

Thus, we can combine Propositions 1 and 2 with Claim 1 to obtain the following corollary.

Corollary 2. *Let $n \in \mathbb{Z}^+$. Suppose $n < \sigma_{k+1}$. Then $n = \sum_{j=0}^{k} b_j \sigma_j$, where $0 \leq b_j \leq p$ for $0 \leq j \leq k$, and if $b_j = p$ for some j, then $b_i = 0$ for all $i < j$. Moreover,*

$$\lambda(n) = \sum_{j=0}^{k} b_j p^j.$$

Corollary 3. *The sequence $\lambda(n)$ satisfies the meta-Fibonacci recurrence relation*

$$\lambda(n) = \sum_{i=1}^{p} \lambda(n - i + 1 - \lambda(n - i)).$$

Proof of Corollary 3. We refer to Corollary 32 in [6], which implies that a sequence which is defined by the meta-Fibonacci recurrence relation (2) is also defined by the recurrence relation

$$\lambda(n) = p^k + \lambda(n - \sigma_k), \tag{6}$$

for $\sigma_k \leq n < \sigma_{k+1}$. Based on Corollary 2, it is clear that $\lambda(n)$ satisfies recurrence (6). Therefore, $\lambda(n)$ satisfies the meta-Fibonacci recurrence (2). □

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