REPDIGITS AS SUMS OF TWO FIBONACCI NUMBERS AND TWO LUCAS NUMBERS

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Abstract

In this paper, we completely determine all repdigits in base 10 which can be expressed as sums of two Fibonacci numbers and two Lucas numbers.

1. Introduction

Recall that a positive integer is called a *repdigit* (sequence A010785 in the OEIS [13]), if it has only one distinct digit in its decimal expansion. In particular, a repdigit with base 10 has the form

$$d\left(\frac{10^n-1}{9}\right), \quad \text{for some } n \geq 1 \text{ and } 1 \leq d \leq 9.$$

Questions concerning the Diophantine equations involving repdigits have been studied for a long time, (see [1, 4]). In recent years, there has been quite some interests in computing base 10 repdigits expressible as sums or products of numbers from another sequence. In 2012, D. Marques and A. Togbé determined all the repdigits which are the product of consecutive Fibonacci numbers [9]. In the same year, Luca [6] found all the repdigits as sums of three Fibonacci numbers by following a general method described in [5]. Also, some analogous results were obtained for Lucas numbers and Pell numbers (see [8] and [12]).

Recently, Normenyo, Luca, and Togbé [7, 11] extended these results to repdigits as sums of four numbers in Fibonacci, Lucas or Pell sequences. However, results such as repdigits as sums of numbers from at least two different sequences do not exist. In fact, this came up in a question raised to A. Togbé during his talk at CNTA XV in July 2018 in Quebec City, Canada. The goal of this paper is to provide an answer to that question in the context of Repdigits, Fibonacci numbers and Lucas numbers.

The Fibonacci sequence $(F_n)_n$ and the Lucas sequence $(L_n)_n$ are given, respectively, by

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n, \quad \text{for } \ n \geq 0$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n, \quad \text{for } \ n \geq 0.$$

In this paper, we determine all the base 10 repdigits which can be expressed as the sum of two Fibonacci numbers and two Lucas numbers. In particular, we prove the following theorem.

Theorem 1. *All nonnegative integer solutions* (s_1, s_2, t_1, t_2, n) *of the equation*

$$N = F_{s_1} + F_{s_2} + L_{t_1} + L_{t_2} = d\left(\frac{10^n - 1}{9}\right), \tag{1}$$

with

$$d \in \{1, \ldots, 9\}, \quad n \geq 1, \quad s_1 \leq s_2, \quad \textit{and} \quad t_1 \leq t_2,$$

have

$$\begin{aligned} N \in \{&2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 11,\ 22,\ 33,\ 44,\ 55,\ 66,\ 77,\ 88,\ 99, \\ &111,\ 222,\ 333,\ 444,\ 555,\ 666,\ 777,\ 888,\ 999,\ 1111,\ 2222, \\ &4444,\ 5555,\ 7777,\ 8888,\ 11111,\ 22222,\ 66666,\ 333333\} = R. \end{aligned}$$

Here is the organization of this paper. Our method consists in applying Bugeaud, Mignotte, and Siksek's theory of linear forms in logarithms of algebraic numbers in order to get an absolute bound on the variables. Afterwards, we use reduction procedures to reduce our bounds to some reasonable values. In the next section, we recall some useful results which we need to prove our theorem. Section 3 contains the proof of our main theorem. We divide the proof into several cases depending on the relations among the variables $s_i,\ t_i,\ i = 1, 2$. We explain our work thoroughly for one of the cases. For the remaining cases, we put only the necessary results to avoid any redundancy.

2. Preliminaries

In this section, we recall some results that are useful for the proof of Theorem 1.

Firstly, we discuss a lower bound for linear forms in logarithms due to Bugeaud, Mignotte, and Siksek [2], which is a consequence of the result of Matveev [10].

Let $\mathbb{K}$ be an algebraic number field of degree D over $\mathbb{Q}$, let $\alpha_1, \ldots, \alpha_n \in \mathbb{K} \setminus \{0\}$ and let $b_1, \ldots, b_n \in \mathbb{Z}$. Set

$$B = \max\{|b_1|, \ldots, |b_n|\}$$

and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1.$$

Let $A_1, \ldots, A_n$ be real numbers with

$$\max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\} \leq A_i, \qquad 1 \leq i \leq n,$$

where $h(\eta)$ is the logarithmic height of an algebraic number η which is given by the formula

$$h(\eta) = \frac{1}{d(\eta)} \left(\log |a_0| + \sum_{i=1}^{d(\eta)} \log \left(\max\{|\eta^{(i)}|, 1\} \right) \right),$$

where $d(\eta)$ is the degree of η over $\mathbb{Q}$ and

$$f(X) = a_0 \prod_{i=1}^{d(\eta)} \left(X - \eta^{(i)} \right) \in \mathbb{Z}[X]$$

the minimal polynomial of η of degree $d(\eta)$ over $\mathbb{Z}$.

Lemma 1. *([2, Theorem 9.4]) Assume that $\Lambda \neq 0$. We then have*

$$\log |\Lambda| > -3 \times 30^{n+4} \times (n+1)^{5.5} D^2 (1 + \log D)(1 + \log nB) A_1 \cdots A_n.$$

Furthermore, if $\mathbb{K}$ is real, we have

$$\log |\Lambda| > -1.4 \times 30^{n+3} \times n^{4.5} D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_n.$$

We also require some properties of the absolute logarithmic height of algebraic numbers. These properties are contained in Lemma 2 below.

Lemma 2. *([15, Property 3.3]) For algebraic numbers α_1 and α_2,*

$$h(\alpha_1 \alpha_2) \leq h(\alpha_1) + h(\alpha_2)$$

and

$$h(\alpha_1 + \alpha_2) \leq \log 2 + h(\alpha_1) + h(\alpha_2).$$

Moreover, for any algebraic number $\alpha \neq 0$ and for any $n \in \mathbb{Z}$,

$$h(\alpha^n) = |n| h(\alpha).$$

We now discuss a computational method for reducing upper bounds for solutions of Diophantine equations.

Let $\vartheta_1, \vartheta_2, \gamma \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \gamma + x_1\vartheta_1 + x_2\vartheta_2. \tag{2}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0 be a (large) positive constant. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{3}$$

$$X \leq X_0. \tag{4}$$

When $\gamma = 0$ in (2), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 3. *([14, Lemma 3.2]) Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1},$$

where k is an integer such that

$$k \leq -1 + \frac{\log\left(1 + X_0\sqrt{5}\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)} := Y_0.$$

If (3) and (4) hold for x_1, x_2 and $\gamma = 0$, then

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\gamma\vartheta_1\vartheta_2 \neq 0$ in (2), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \gamma/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we define $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ to be the distance from x to the nearest integer. We have the following result.

Lemma 4. *([14, Lemma 3.3]) Suppose that*

$$\| q\psi \| > \frac{2X_0}{q}.$$

Then, the solutions of (3) and (4) satisfy

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

3. Proof of Theorem 1

Recall that if s and t are any nonnegative integers, then

$$F_s = \frac{\alpha^s - \beta^s}{\sqrt{5}}, \tag{5}$$

and

$$L_t = \alpha^t + \beta^t, \tag{6}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2},$$

are the solutions of the quadratic equation $x^2 - x - 1 = 0$. Equations (5) and (6) are known as Binet's formula for Fibonacci and Lucas numbers, respectively.

To prove our result, we shall consider six possible cases in the sequel. The first three cases arise when we consider $\max\{s_2, t_2\} = t_2$, and the last three cases arise when $\max\{s_2, t_2\} = s_2$.

`Case 1`: We consider $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2$. Assume that $t_2 \leq 500$. By equation (1), we obtain

$$10^{n-1} \leq d\left(\frac{10^n - 1}{9}\right) = F_{s_1} + F_{s_2} + L_{t_1} + L_{t_2} \leq 4(1 + L_{t_2}) \leq 4(1 + L_{500}),$$

which leads us to the inequality

$$n \leq 1 + \frac{\log(4(1 + L_{500}))}{\log 10},$$

from which it follows that $0 \leq n \leq 106$.

A search in Maple reveals that all the nonnegative integer solutions (s_1, s_2, t_1, t_2, n) of the Diophantine equation

$$N = F_{s_1} + F_{s_2} + L_{t_1} + L_{t_2} = d\left(\frac{10^n - 1}{9}\right),$$

with

$$1 \leq d \leq 9, \quad 0 \leq n \leq 106, \quad \text{and} \quad 0 \leq s_1 \leq s_2 \leq t_1 \leq t_2,$$

have

$$N \in \{2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 222, 333, \\ 555, 666, 777, 888, 2222, 11111, 66666\} = R_1.$$

The set R_1 is a subset of R in Theorem 1.

Next, we assume that $t_2 \geq 501$. If $t_2 \geq 501$, we have

$$L_{501} \leq L_{t_2} \leq F_{s_1} + F_{s_2} + L_{t_1} + L_{t_2} = d\left(\frac{10^n - 1}{9}\right) \leq 10^n - 1,$$

which gives us

$$104 \leq \frac{\log(1 + L_{501})}{\log 10} \leq n.$$

Further, notice that

$$10^{n-1} \leq d\left(\frac{10^n - 1}{9}\right) = F_{s_1} + F_{s_2} + L_{t_1} + L_{t_2} \leq 4(1 + L_{t_2}) < 12\alpha^{t_2} < \alpha^{t_2+5.2}.$$

The last inequality gives us

$$n < 4.78n - 9.98 < t_2,$$

where we used the fact that $n \geq 104$.

Now, we examine equation (1) in four possible ways, as captured in the following four steps.

`Step 1`: We express (1) in the form

$$\alpha^{t_2}\left(\frac{\alpha^{s_1-t_2}}{\sqrt{5}} + \frac{\alpha^{s_2-t_2}}{\sqrt{5}} + \alpha^{t_1-t_2} + 1\right) - \frac{d \times 10^n}{9} = -\frac{d}{9} + \frac{\beta^{s_1}}{\sqrt{5}} + \frac{\beta^{s_2}}{\sqrt{5}} - \beta^{t_1} - \beta^{t_2}, \quad (7)$$

which gives us

$$\left|\alpha^{t_2}\left(\frac{\alpha^{s_1-t_2}}{\sqrt{5}} + \frac{\alpha^{s_2-t_2}}{\sqrt{5}} + \alpha^{t_1-t_2} + 1\right) - \frac{d \times 10^n}{9}\right| < \alpha^{2.83}. \quad (8)$$

Thus, we arrive at

$$\left|1 - \alpha^{-s_1}10^n\left(\frac{d\sqrt{5}}{9\left(1 + \alpha^{s_2-s_1} + \sqrt{5}\left(\alpha^{t_1-s_1} + \alpha^{t_2-s_1}\right)\right)}\right)\right| < \alpha^{2.83-t_2}. \quad (9)$$

Put

$$\Gamma_1 := 1 - \alpha^{-s_1}10^n\left(\frac{d\sqrt{5}}{9\left(1 + \alpha^{s_2-s_1} + \sqrt{5}\left(\alpha^{t_1-s_1} + \alpha^{t_2-s_1}\right)\right)}\right).$$

We wish to apply Lemma 1 on Γ_1. First, we need to prove that $\Gamma \neq 0$. If indeed it were zero, then

$$\alpha^{s_1} + \alpha^{s_2} + \sqrt{5}\left(\alpha^{t_1} + \alpha^{t_2}\right) = \frac{10^n \times d\sqrt{5}}{9},$$

which implies

$$\beta^{s_1} + \beta^{s_2} - \sqrt{5}\left(\beta^{t_1} + \beta^{t_2}\right) = -\frac{10^n \times d\sqrt{5}}{9},$$

by conjugating in $\mathbb{Q}\left(\sqrt{5}\right)$. As a result, we obtain

$$\begin{aligned}\alpha^{501} < \alpha^{t_2} &< \alpha^{s_1} + \alpha^{s_2} + \sqrt{5}\left(\alpha^{t_1} + \alpha^{t_2}\right)\\ &= \left|\beta^{s_1} + \beta^{s_2} - \sqrt{5}\left(\beta^{t_1} + \beta^{t_2}\right)\right| < 2\left(1+\sqrt{5}\right),\end{aligned}$$

which is not possible as $\alpha^{501} > 2(1+\sqrt{5})$. Therefore, we find that $\Gamma_1 \neq 0$.

In the notation of Lemma 1, we set

$$\mathbb{K} = \mathbb{Q}\left(\sqrt{5}\right), \quad \alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d\sqrt{5}}{9\left(1 + \alpha^{s_2-s_1} + \sqrt{5}\left(\alpha^{t_1-s_1} + \alpha^{t_2-s_1}\right)\right)},$$

$$D = 2, \quad b_1 = -s_1, \quad b_2 = n, \quad b_3 = 1, \quad B = \max\{s_1, n, 1\} \leq t_2.$$

We find that

$$\max\{2h(\alpha_1), |\log\alpha_1|, 0.16\} = \log\alpha < 0.49 =: A_1,$$

$$\max\{2h(\alpha_2), |\log\alpha_2|, 0.16\} = 2\log 10 < 4.61 =: A_2.$$

Let us set

$$C_1 = 2.3 \times 10^{12} > 1.4 \times 30^6 \times 3^{4.5} \times D^2 \times (1+\log D) \times A_1 \times A_2.$$

We observe that,

$$\alpha_3 = \frac{d\sqrt{5}}{9\left(1 + \alpha^{s_2-s_1} + \sqrt{5}\left(\alpha^{t_1-s_1} + \alpha^{t_2-s_1}\right)\right)} < \sqrt{5},$$

and

$$\alpha_3^{-1} = \frac{9\left(1 + \alpha^{s_2-s_1} + \sqrt{5}\left(\alpha^{t_1-s_1} + \alpha^{t_2-s_1}\right)\right)}{d\sqrt{5}} \leq \frac{18\left(5+\sqrt{5}\right)}{5}\alpha^{t_2-s_1}.$$

This means that $|\log\alpha_3| < 4 + (t_2 - s_1)\log\alpha$. Furthermore, we have

$$\begin{aligned}h\left(\alpha_3\right) &\leq h\left(d\sqrt{5}\right) + h(9) + h\left(1 + \alpha^{s_2-s_1} + \sqrt{5}\left(\alpha^{t_1-s_1} + \alpha^{t_2-s_1}\right)\right)\\ &\leq h\left(9\sqrt{5}\right) + h(9) + \log 2 + h\left(\alpha^{s_2-s_1}\left(1 + \sqrt{5}\left(\alpha^{t_1-s_2} + \alpha^{t_2-s_2}\right)\right)\right)\\ &\leq h(9) + 2h(\sqrt{5}) + h(9) + 2\log 2 + h(\alpha^{s_2-s_1}) + h(\alpha^{t_1-s_2}(1+\alpha^{t_2-t_1}))\\ &\leq 2h(\sqrt{5}) + 2h(9) + 3\log 2 + h(\alpha^{s_2-s_1}) + h(\alpha^{t_1-s_2}) + h(\alpha^{t_2-t_1})\\ &\leq 2h(\sqrt{5}) + 2h(9) + 3\log 2 + (s_2-s_1)h(\alpha) + (t_1-s_2)h(\alpha) + (t_2-t_1)h(\alpha)\\ &= \log 5 + 2\log 9 + 3\log 2 + \frac{1}{2}(t_2-s_1)\log\alpha.\end{aligned}$$

Therefore, $2h(\alpha_3) \leq 17 + (t_2 - s_1)\log\alpha$. As a consequence, we obtain

$$\max\{2h(\alpha_3), |\log\alpha_3|, 0.16\} \leq 17 + (t_2 - s_1)\log\alpha =: A_3.$$

The application of Lemma 1 to Γ_1 and the use of (9) yield

$$t_2 \log\alpha < 2.83\log\alpha + (17 + (t_2 - s_1)\log\alpha)C_1(1 + \log t_2)). \tag{10}$$

`Step 2`: Here, begin with the equation

$$\alpha^{t_2}\left(\frac{\alpha^{s_2-t_2}}{\sqrt{5}} + \alpha^{t_1-t_2} + 1\right) - \frac{d \times 10^n}{9} = -\frac{d}{9} + \frac{\beta^{s_1}}{\sqrt{5}} + \frac{\beta^{s_2}}{\sqrt{5}} - \beta^{t_1} - \beta^{t_2} - \frac{\alpha^{s_1}}{\sqrt{5}}, \tag{11}$$

from which we deduce that

$$\left|\alpha^{t_2}\left(\frac{\alpha^{s_2-t_2}}{\sqrt{5}} + \alpha^{t_1-t_2} + 1\right) - \frac{d \times 10^n}{9}\right| < \alpha^{s_1+3.06}. \tag{12}$$

This means that

$$\left|1 - \alpha^{-s_2}10^n\left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1-s_2} + \alpha^{t_2-s_2}\right)\right)}\right)\right| < \alpha^{s_1-t_2+3.06}. \tag{13}$$

Put

$$\Gamma_2 := 1 - \alpha^{-s_2}10^n\left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1-s_2} + \alpha^{t_2-s_2}\right)\right)}\right).$$

Let us assume, if possible, that $\Gamma_2 = 0$. Then, we observe that

$$\alpha^{501} < \alpha^{t_2} < \alpha^{s_2} + \sqrt{5}\left(\alpha^{t_1} + \alpha^{t_2}\right) < 1 + 2\sqrt{5},$$

which is a contradiction as $\alpha^{501} > 1 + 2\sqrt{5}$. This shows that $\Gamma_2 \neq 0$.

To apply Lemma 1 to Γ_2, we set

$$\mathbb{K} = \mathbb{Q}\left(\sqrt{5}\right), \quad \alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d\sqrt{5}}{9(1 + \sqrt{5}\left(\alpha^{t_1-s_2} + \alpha^{t_2-s_2}\right))},$$

$$b_1 = -s_2, \quad b_2 = n, \quad b_3 = 1, \quad B = \max\{s_2, n, 1\} \leq t_2.$$

Next, we find A_3. Notice that

$$\alpha_3 = \frac{d\sqrt{5}}{9(1 + \sqrt{5}\left(\alpha^{t_1-s_2} + \alpha^{t_2-s_2}\right))} < \sqrt{5}$$

and

$$\alpha_3^{-1} = \frac{9(1 + \sqrt{5}\left(\alpha^{t_1-s_2} + \alpha^{t_2-s_2}\right))}{d\sqrt{5}} \leq \frac{9\left(10 + \sqrt{5}\right)}{5}\alpha^{t_2-s_2},$$

from which we see that $|\log \alpha_3| < 4 + (t_2 - s_2)\log \alpha$. We also have that

$$\begin{aligned} h(\alpha_3) &\le h\left(d\sqrt{5}\right) + h(9) + \log 2 + h\left(\sqrt{5}\right) + h\left(\alpha^{t_1-s_2}\left(1+\alpha^{t_2-t_1}\right)\right) \\ &\le \log 5 + 2h(9) + 2\log 2 + h\left(\alpha^{t_1-s_2}\right) + h\left(\alpha^{t_2-t_1}\right) \\ &\le \log 5 + 2h(9) + 2\log 2 + (t_1 - s_2)h(\alpha) + (t_2 - t_1)h(\alpha) \\ &= \log 5 + 2\log 9 + 2\log 2 + \frac{1}{2}(t_2 - s_2)\log \alpha. \end{aligned}$$

Hence, $2h(\alpha_3) \le 15 + (t_2 - s_2)\log \alpha$. This leads us to conclude that

$$\max\{2h(\alpha_3), |\log \alpha_3|, 0.16\} \le 15 + (t_2 - s_2)\log \alpha =: A_3.$$

By applying Lemma 1 to Γ_2, we obtain

$$(t_2 - s_1)\log \alpha < 3.06 \log \alpha + (15 + (t_2 - s_2)\log \alpha)C_1(1 + \log t_2), \tag{14}$$

where we used the inquality (13).

`Step 3`: In this case, we write equation (1) in the form

$$\alpha^{t_2}\left(\alpha^{t_1-t_2} + 1\right) - \frac{d \times 10^n}{9} = -\frac{d}{9} + \frac{\beta^{s_1}}{\sqrt{5}} + \frac{\beta^{s_2}}{\sqrt{5}} - \beta^{t_1} - \beta^{t_2} - \frac{\alpha^{s_1}}{\sqrt{5}} - \frac{\alpha^{s_2}}{\sqrt{5}}, \tag{15}$$

from which we obtain

$$\left|\alpha^{t_2}\left(\alpha^{t_1-t_2} + 1\right) - \frac{d \times 10^n}{9}\right| < \alpha^{s_2+3.26}, \tag{16}$$

which means that

$$\left|1 - \alpha^{-t_1}10^n\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right)\right| < \alpha^{s_2-t_2+3.26}. \tag{17}$$

Put

$$\Gamma_3 := 1 - \alpha^{-t_1}10^n\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right).$$

Suppose that $\Gamma_3 = 0$. Then we obtain

$$\alpha^{501} \le \alpha^{t_2} < \alpha^{t_1} + \alpha^{t_2} < |\beta|^{t_1} + |\beta|^{t_2} < 2,$$

which implies that $\alpha^{501} < 2$, an impossibility. Hence, $\Gamma_3 \ne 0$. In order to use Lemma 1, we put

$$\alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d}{9(1+\alpha^{t_2-t_1})}, \quad b_1 = -t_1, \quad b_2 = n, \quad b_3 = 1,$$

$$B = \max\{t_1, n, 1\} \le t_2.$$

We find that

$$\alpha_3 = \frac{d}{9(1+\alpha^{t_2-t_1})} \leq 1 \ \text{ and } \ \alpha_3^{-1} = \frac{9(1+\alpha^{t_2-t_1})}{d} \leq 18\alpha^{t_2-t_1}.$$

Hence, we get $|\log \alpha_3| < 3 + (t_2 - t_1)\log \alpha$. Next, we obtain

$$\begin{aligned} h(\alpha_3) &\leq h(d) + h(9) + \log 2 + h(\alpha^{t_2-t_1}) \\ &\leq 2h(9) + \log 2 + (t_2 - t_1)h(\alpha) \\ &= 2h(9) + \log 2 + \frac{1}{2}(t_2 - t_1)\log \alpha. \end{aligned}$$

Hence, $2h(\alpha_3) \leq 11 + (t_2 - t_1)\log \alpha$. This implies that

$$\max\{2h(\alpha_3), |\log \alpha_3|, 0.16\} < 11 + (t_2 - t_1)\log \alpha =: A_3.$$

Applying Lemma 1 to Γ_3 yields

$$(t_2 - s_2)\log \alpha < 3.26 \log \alpha + (11 + (t_2 - t_1)\log \alpha)C_1(1 + \log t_2). \tag{18}$$

`Step 4`: In the final step, we have

$$\alpha^{t_2} - \frac{d \times 10^n}{9} = -\frac{d}{9} + \frac{\beta^{s_1}}{\sqrt{5}} + \frac{\beta^{s_2}}{\sqrt{5}} - \beta^{t_1} - \beta^{t_2} - \frac{\alpha^{s_1}}{\sqrt{5}} - \frac{\alpha^{s_2}}{\sqrt{5}} - \alpha^{t_1}, \tag{19}$$

which leads to

$$\left|1 - \alpha^{-t_2}10^n \left(\frac{d}{9}\right)\right| < \alpha^{t_1-t_2+3.65}. \tag{20}$$

Put

$$\Gamma_4 := 1 - \alpha^{-t_2}10^n \left(\frac{d}{9}\right).$$

Suppose that $\Gamma_4 = 0$. Then

$$\alpha^{501} < \alpha^{t_2} = \frac{d \times 10^n}{9} = |\beta^{t_2}| < 1,$$

which is impossible as $\alpha^{501} > 1$. Hence, $\Gamma_4 \neq 0$. We apply Lemma 1 to Γ_4 by setting

$$\alpha_1 = \alpha, \ \ \alpha_2 = 10, \ \ \alpha_3 = \frac{d}{9}, \ \ b_1 = -t_2, \ \ b_2 = n, \ \ b_3 = 1, \ \ A_3 = 2.2$$

Using Lemma 1 we find that

$$(t_2 - t_1)\log \alpha < 3.65 \log \alpha + 2.2C_1(1 + \log t_2) < 2.21C_1(1 + \log t_2). \tag{21}$$

Putting together (21) and (18) yields

$$\begin{aligned} (t_2 - s_2)\log \alpha &< 3.26 \log \alpha + (11 + 2.21C_1(1 + \log t_2))C_1(1 + \log t_2) \\ &< 2.22C_1^2(1 + \log t_2)^2. \end{aligned}$$

This inequality, together with (14), yield

$$(t_2 - s_1)\log\alpha < 2.23C_1^3(1+\log t_2)^3,$$

which combines with (10) to give us

$$t_2 < 1.31\cdot 10^{50}(1+\log t_2)^4. \tag{22}$$

Therefore, we obtain $t_2 < 4.49\times 10^{58}$. We now employ the reduction method in three steps as follows.

Let

$$\Lambda_1 = -t_2\log\alpha + n\log 10 + \log\left(\frac{d}{9}\right).$$

We see from (19) that

$$\alpha^{t_2}\left(1-e^{\Lambda_1}\right) = -\frac{d}{9} - F_{s_1} - F_{s_2} - L_{t_1} - \beta^{t_2} \leq -\frac{1}{9} + |\beta|^{501} < 0,$$

since $t_2 \geq 501$. This implies that $\Lambda_1 > 0$. It follows that

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = \left|1 - \alpha^{-t_2}10^n\left(\frac{d}{9}\right)\right| < \alpha^{t_1-t_2+3.65},$$

which leads to

$$\log\left(\frac{d}{9}\right) - t_2\log\alpha + n\log 10 < \alpha^{3.66}\exp(-0.48(t_2-t_1)),$$

with $X = \max\{t_2, n\} = t_2 \leq 4.49\times 10^{58}$. It can also be seen that

$$\frac{\Lambda_1}{\log 10} = \frac{\log(d/9)}{\log 10} - t_2\frac{\log\alpha}{\log 10} + n.$$

In order to apply Lemma 4, we set

$$c = \alpha^{3.66},\ \ \delta = 0.48,\ \ X_0 = 4.49\times 10^{58},\ \ \psi = \frac{\log(d/9)}{\log 10},\quad Y = t_2 - t_1,$$

$$\vartheta = \frac{\log\alpha}{\log 10},\ \ \vartheta_1 = -\log\alpha,\ \ \vartheta_2 = \log 10,\ \ \gamma = \log(d/9).$$

For $\gamma \neq 0$, which occurs when $d \neq 9$, computations reveal that the smallest value of q such that $q > X_0$ is $q = q_{124}$, and that $q = q_{125}$ satisfies the hypothesis of Lemma 4 for $d = 1,\ldots,8$. Application of Lemma 4 leads us to $t_2 - t_1 \leq 299$ and $t_1 \geq 202$.

For $\gamma = 0$, which occurs when $d = 9$, we deduce that $0 \leq k \leq 281$ and $A = a_{138} = 770$ using the notation of Lemma 3. Lemma 3 then gives us $t_2 - t_1 \leq 297$.

Next, we consider

$$\Lambda_2 = -t_1\log\alpha + n\log 10 + \log\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right),$$

where $1 \le d \le 9$ and $0 \le t_2 - t_1 \le 299$. We see from equation (15) that

$$\begin{aligned}\alpha^{t_2}(\alpha^{t_1-t_2}+1)\left(1-e^{\Lambda_2}\right) &= -\frac{d}{9} - F_{s_1} - F_{s_2} - \beta^{t_1} - \beta^{t_2} \\ &\le -\frac{1}{9} + |\beta|^{202} + |\beta|^{501} \\ &< 0,\end{aligned}$$

and so $\Lambda_2 > 0$. Thus, we obtain

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = \left|1 - \alpha^{-t_1} 10^n \left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right)\right| < \alpha^{s_2-t_2+3.26}.$$

from which comes

$$\log\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right) - t_1 \log\alpha + n\log 10 < \alpha^{3.27}\exp(-0.48(t_2 - s_2)),$$

where $X = \max\{t_1, n\} \le t_2 \le 4.49 \times 10^{58}$. We also have that

$$\frac{\Lambda_2}{\log 10} = \frac{1}{\log 10}\log\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right) - t_1\frac{\log\alpha}{\log 10} + n.$$

We take

$$c = \alpha^{3.27}, \quad \delta = 0.48, \quad X_0 = 4.49 \times 10^{58}, \quad \psi = \frac{1}{\log 10}\log\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right),$$

$$Y = t_2 - s_2, \quad \vartheta = \frac{\log\alpha}{\log 10}, \quad \vartheta_1 = -\log\alpha, \quad \vartheta_2 = \log 10, \quad \gamma = \log\left(\frac{d}{9(1+\alpha^{t_2-t_1})}\right).$$

We find that $q = q_{131}$ satisfies the hypothesis of Lemma 4 for $d = 1, \ldots, 9$ and $0 \le t_2 - t_1 \le 299$. Applying Lemma 4, we get $t_2 - s_2 \le 321$. Hence, $s_2 \ge 180$.

For $1 \le d \le 9$, $0 \le t_1 - s_2 \le t_2 - s_2 \le 321$, we let

$$\Lambda_3 = -s_2\log\alpha + n\log 10 + \log\left(\frac{d\sqrt{5}}{9\left(1+\sqrt{5}\left(\alpha^{t_1-s_2}+\alpha^{t_2-s_2}\right)\right)}\right).$$

Using (11) we arrive at

$$\begin{aligned}\alpha^{t_2}\left(\frac{\alpha^{s_2-t_2}}{\sqrt{5}} + \alpha^{t_1-t_2} + 1\right)\left(1-e^{\Lambda_3}\right) &= -\frac{d}{9} - F_{s_1} + \frac{\beta^{s_2}}{\sqrt{5}} - \beta^{t_1} - \beta^{t_2} \\ &\le -\frac{1}{9} + \frac{|\beta|^{180}}{\sqrt{5}} + |\beta|^{202} + |\beta|^{501} \\ &< 0.\end{aligned}$$

Hence, $\Lambda_3 > 0$. Thus, we have

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = \left|1 - \alpha^{-s_2} 10^n \left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1 - s_2} + \alpha^{t_2 - s_2}\right)\right)}\right)\right| < \alpha^{s_1 - t_2 + 3.06}.$$

This means that

$$\log\left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1 - s_2} + \alpha^{t_2 - s_2}\right)\right)}\right) - s_2 \log\alpha + n \log 10 < \alpha^{3.07} \exp(-0.48(t_2 - s_1)),$$

where $X = \max\{s_2, n\} \leq t_2 \leq 4.49 \times 10^{58}$. We note also that

$$\frac{\Lambda_3}{\log 10} = \frac{1}{\log 10} \log\left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1 - s_2} + \alpha^{t_2 - s_2}\right)\right)}\right) - s_2 \frac{\log\alpha}{\log 10} + n.$$

Hence, we put

$$c = \alpha^{3.07}, \quad \delta = 0.48, \quad X_0 = 4.49 \times 10^{58}, \quad Y = t_2 - s_1,$$

$$\psi = \frac{1}{\log 10} \log\left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1 - s_2} + \alpha^{t_2 - s_2}\right)\right)}\right), \quad \vartheta = \frac{\log\alpha}{\log 10}, \quad \vartheta_1 = -\log\alpha,$$

$$\vartheta_2 = \log 10, \quad \gamma = \log\left(\frac{d\sqrt{5}}{9\left(1 + \sqrt{5}\left(\alpha^{t_1 - s_2} + \alpha^{t_2 - s_2}\right)\right)}\right).$$

Computations with Maple indicate that $q = q_{134}$ satisfies the hypothesis of Lemma 4 for $1 \leq d \leq 9$, $0 \leq t_1 - s_2 \leq t_2 - s_2 \leq 321$. We further deduce that $t_2 - s_1 \leq 335$ and hence $s_1 \geq 166$ upon application of Lemma 4.

Finally, we consider

$$\Lambda_4 = -s_1 \log\alpha + n \log 10 + \log\left(\frac{d\sqrt{5}}{9\left(1 + \alpha^{s_2 - s_1} + \sqrt{5}\left(\alpha^{t_1 - s_1} + \alpha^{t_2 - s_1}\right)\right)}\right),$$

with $1 \leq d \leq 9$, $0 \leq s_2 - s_1 \leq t_1 - s_1 \leq t_2 - s_1 \leq 335$. Using equation (7), we obtain

$$\begin{aligned}\alpha^{t_2}\left(\frac{\alpha^{s_1 - t_2}}{\sqrt{5}} + \frac{\alpha^{s_2 - t_2}}{\sqrt{5}} + \alpha^{t_1 - t_2} + 1\right)\left(1 - e^{\Lambda_4}\right)& \\ \leq -\frac{1}{9} + \frac{1}{\sqrt{5}}\left(|\beta|^{166} + |\beta|^{180}\right) + |\beta|^{202} + |\beta|^{501} < 0.&\end{aligned}$$

Hence, $\Lambda_4 > 0$. We have that

$$\begin{aligned}0 < \Lambda_4 < e^{\Lambda_4} - 1 &= \left|1 - \alpha^{-s_1} 10^n \left(\frac{d\sqrt{5}}{9\left(1 + \alpha^{s_2 - s_1} + \sqrt{5}\left(\alpha^{t_1 - s_1} + \alpha^{t_2 - s_1}\right)\right)}\right)\right| \\ &< \alpha^{2.83 - t_2},\end{aligned}$$

from which it follows that

$$\log\left(\frac{d\sqrt{5}}{9\left(1+\alpha^{s_2-s_1}+\sqrt{5}\left(\alpha^{t_1-s_1}+\alpha^{t_2-s_1}\right)\right)}\right)-s_1\log\alpha+n\log 10<\alpha^{2.84}\exp(-0.48t_2),$$

where $X=\max\{s_1,n\}\leq t_2<4.49\times 10^{58}$. In addition,

$$\frac{\Lambda_4}{\log 10}=\frac{1}{\log 10}\log\left(\frac{d\sqrt{5}}{9\left(1+\alpha^{s_2-s_1}+\sqrt{5}\left(\alpha^{t_1-s_1}+\alpha^{t_2-s_1}\right)\right)}\right)-s_1\frac{\log\alpha}{\log 10}+n.$$

Thus,

$$c=\alpha^{2.84},\quad \delta=0.48,\quad X_0=4.49\times 10^{58},\quad Y=t_2$$

$$\psi=\frac{1}{\log 10}\log\left(\frac{d\sqrt{5}}{9\left(1+\alpha^{s_2-s_1}+\sqrt{5}\left(\alpha^{t_1-s_1}+\alpha^{t_2-s_1}\right)\right)}\right),\quad \vartheta=\frac{\log\alpha}{\log 10},$$

$$\vartheta_1=-\log\alpha,\ \vartheta_2=\log 10,\ \gamma=\log\left(\frac{d\sqrt{5}}{9\left(1+\alpha^{s_2-s_1}+\sqrt{5}\left(\alpha^{t_1-s_1}+\alpha^{t_2-s_1}\right)\right)}\right).$$

We find that $q=q_{138}$ satisfies the hypothesis of Lemma 4 for $1\leq d\leq 9$, $0\leq s_2-s_1\leq t_1-s_1\leq t_2-s_1\leq 335$. Applying Lemma 4, we get $t_2\leq 374$, which contradicts the assumption that $t_2\geq 501$. And the result follows.

In the remaining five cases, we proceed as in the first case. The following are the results.

`Case 2`: $0\leq s_1\leq t_1\leq s_2\leq t_2$. We obtain the set R_2 given by

$$N\in\{3,4,5,6,7,8,9,11,22,33,44,55,66,77,88,99,111,222,333,\\ 555,666,777,888,999,1111,2222,8888,22222,66666\}=R_2.$$

`Case 3`: $0\leq t_1\leq s_1\leq s_2\leq t_2$. We obtain the set R_3 given by

$$N\in\{3,4,5,6,7,8,9,11,22,33,44,55,66,77,88,99,111,222,333,\\ 555,666,777,888,999,1111,2222,11111,66666\}=R_3.$$

`Case 4`: $0\leq t_1\leq t_2\leq s_1\leq s_2$. Here, we get the set R_4 given by

$$N\in\{4,5,6,7,8,9,11,22,33,44,55,66,77,88,99,111,222,\\ 333,444,555,666,777,888,999,2222,4444,7777,11111,66666\}=R_4.$$

`Case 5`: $0\leq t_1\leq s_1\leq t_2\leq s_2$. Next, we get the set R_5 given by

$$N\in\{4,5,6,7,8,9,11,22,33,44,55,66,77,88,99,111,222,333,\\ 444,555,666,777,999,5555,7777,11111,333333\}=R_5.$$

Case 6: $0 \leq s_1 \leq t_1 \leq t_2 \leq s_2$. Here, we have the set R_6 given by

$$N \in \{4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 222, 333, 444, \\ 555, 666, 777, 999, 5555, 7777, 11111, 333333\} = R_6.$$

Finally, we observe that the union of the sets $R_i,\ i = 1, \ldots, 6$, is the set R as in Theorem 1.

This completes the proof of Theorem 1.

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