COMPLEX CANTOR SERIES, CANTOR PRODUCTS AND THEIR INDEPENDENCE

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Abstract

Two representations of a complex number via a Cantor series and a Cantor product are introduced. Criteria for independence based on these representations are proved. These criteria are derived from a generalized Liouville's theorem about approximation by numbers from a fixed algebraic number field.

1. Introduction

Independence criteria of real numbers represented by various representations have been of interest for quite some time; see, e.g., [1], [23]. The present work deals with this problem for complex numbers represented via Cantor series and Cantor product expansions. There has been a good deal of work related to representations of real numbers, the irrationality and independence of such representations. Let us briefly review some relevant ones that are accessible to us. Oppenheim in [17], [18] gave several criteria for the irrationality of classes of Cantor series and Cantor products, respectively. In [19] and [20], he developed a number of series representations for real numbers. In [7], [8] and [10], Hančl presented a number of criteria for irrationality of rapidly convergent series representations, while in [9], [11] and [12], he introduced the concept of linearly unrelated sequences and gave various linear independence criteria for series and continued fraction representations of real numbers. In [22], Tijdeman and Yuan established several precise criteria for rationality of Cantor, Ahmed and some other series representations of real numbers and showed that such results are impossible without growth restrictions. For more recent works on transcendence and linear independence of infinite products, see [14] and [13]. The survey work in [16] gives old and new results related to Erdös's results on irrationality.

Using Cantor series expansions for real numbers, we extend them to complex numbers in the usual manner. The representation of real numbers as Cantor products is well-known ([21, Section 33]), while that for complex numbers has only recently been established in [15] and we make use of the results in our work here. Throughout the rest of this section, we recall these preliminary results. Based on the work in [3], in Section 2, we prove some linear independence criteria, introduce the concept of quadratic independence and derive some criteria for it. In the next two sections, we apply the criteria so obtained to complex numbers through their Cantor series and products.

Let us begin with some well-known facts about real Cantor series and products.

Proposition 1. *I)* ([21, Section 33], [4]) *Let* $\mathcal{M} := \{m_k\}_{k\geq 1}$, $m_k \geq 2$, *be a sequence of natural numbers. A real number* $\alpha \in [0,1)$ *is uniquely representable as a (real) Cantor (or* $\mathcal{M}$*-Cantor) series of the form*

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{m_1 m_2 \cdots m_k}, \tag{1}$$

where

$$c_k \in \{0, 1, \ldots, m_k - 1\}, \quad c_k < m_k - 1 \text{ for infinitely many } k. \tag{2}$$

Moreover, assuming that each prime divides infinitely many of the m_k*, then* α *is irrational if and only if both* $c_k > 0$ *and* $c_k < m_k - 1$ *hold infinitely often.*

II) ([21, Section 35]) *A real number* $\alpha > 1$ *is uniquely representable as a (real) Cantor product of the form*

$$\alpha = \prod_{k=1}^{\infty} \left(1 + \frac{1}{a_k}\right),$$

where the integers $a_k \in \mathbb{N}$ *are subject to*

$$a_{k+1} \geq a_k^2, \text{ and there are infinitely many } k \text{ such that } a_k \geq 2. \tag{3}$$

Moreover, α *is rational if and only if* $a_{k+1} = a_k^2$ *for all sufficiently large* k.

III) ([6, Theorem 2.1]) *A necessary and sufficient condition for* α *given by the convergent Cantor series* (1) *to be irrational is that, for every* $B \in \mathbb{N}$*, we can find* *an integer A and a subsequence $k_1, k_2, k_3, \ldots$ such that*

$$\frac{A}{B} < \alpha_{k_j} < \frac{A+1}{B} \qquad (j \in \mathbb{N}),$$

where $\alpha = \alpha_1$ and for $j > 1$,

$$\alpha_j = \frac{c_j}{m_j} + \frac{c_{j+1}}{m_j m_{j+1}} + \frac{c_{j+2}}{m_j m_{j+1} m_{j+2}} + \cdots.$$

IV) ([6, Theorem 2.2]) *Let $1 \le k_1 \le k_2 \le \cdots$ be a sequence of positive integers and let*

$$\frac{C_1}{M_1} = \frac{c_1}{m_1} + \frac{c_2}{m_1 m_2} + \cdots + \frac{c_{k_1}}{m_1 m_2 \cdots m_{k_1}},$$
$$\frac{C_2}{M_2} = \frac{c_{k_1+1}}{m_{k_1+1}} + \frac{c_{k_1+2}}{m_{k_1+1} m_{k_1+2}} + \cdots + \frac{c_{k_2}}{m_{k_1+1} m_{k_1+2} \cdots m_{k_2}}, \ \ldots\ ,$$

where $M_1 = m_1 m_2 \cdots m_{k_1}$, $M_2 = m_{k_1+1} m_{k_1+2} \cdots m_{k_2}$, and so on. Then (1) *becomes*

$$\alpha = \frac{C_1}{M_1} + \frac{C_2}{M_1 M_2} + \cdots + \frac{C_k}{M_1 M_2 \cdots M_k} + \cdots$$

with $M_k \ge 2$ and $0 \le C_k \le M_k - 1$ for each k. That is, from (1) *another Cantor series expansion with respect to the new sequence $M_1, M_2, \ldots$ is introduced; such a procedure is referred to as a condensation. A necessary and sufficient condition for the series* (1)*, under* (2)*, to be rational is that there exist co-prime integers $0 \le a \le b$, a condensation, and an integer N such that $C_k = \frac{a}{b}(M_k - 1)$ for all $k \ge N$.*

1.1. Complex Cantor Series

Each $\beta \in \mathbb{C}$ can be uniquely written as

$$\beta = \alpha_x + i\alpha_y \qquad (i := \sqrt{-1}), \tag{4}$$

where α_x and α_y are real numbers.

Throughout the rest of the paper, we restrict our attention to the case where both α_x and α_y lie in the open interval $(0, 1)$.

From (1), both α_x and α_y can be uniquely represented as

$$\alpha_x = \sum_{k=1}^{\infty} \frac{c_k}{m_1 m_2 \cdots m_k}, \quad \alpha_y = \sum_{k=1}^{\infty} \frac{d_k}{m_1 m_2 \cdots m_k},$$

where

$$c_k \text{ and } d_k \in \{0, \ldots, m_k - 1\}, \text{ and both are less than } m_k - 1 \text{ for infinitely many } k. \tag{5}$$

We define the *complex $\mathcal{M}$-Cantor series expansion* for β in (4) as

$$\beta = \sum_{k=1}^{\infty} \frac{c_k + i\, d_k}{m_1 m_2 \cdots m_k},$$

where c_k, d_k are subject to the conditions in (5).

1.2. Complex Cantor Products

For $D \in \mathbb{N}$, let

$$\theta_D = \begin{cases} \frac{1}{2}\left(1+\sqrt{-D}\right) & \text{if } -D \equiv 1 \mod 4 \\ \sqrt{-D} & \text{if } -D \not\equiv 1 \mod 4 \end{cases},$$

so that the ring of integers of $\mathbb{Q}(\sqrt{-D})$ is $\mathbb{Z}[\theta_D] = \{u + v\theta_D \ ; \ u, v \in \mathbb{Z}\}$. Following [15], each $\beta \in \mathbb{C}$ can be represented as

$$\beta = \prod_{k=1}^{\infty} \left(1 + \frac{1}{a_k}\right),$$

where $a_k \in \mathbb{Z}[\theta_D] \setminus \{0\}$ $(k \geq 1)$, and for k sufficiently large,

$$|a_{k+1}| \geq \begin{cases} \mu^{-1}|a_k|^2 - (\mu^{-1}+1)|a_k| - (1+\mu),\ \mu = \frac{D+1}{4\sqrt{D}} & \text{if } -D \equiv 1 \mod 4 \\ \\ \gamma^{-1}|a_k|^2 - (\gamma^{-1}+1)|a_k| - (1+\gamma),\ \gamma = \frac{\sqrt{D+1}}{2} & \text{if } -D \not\equiv 1 \mod 4. \end{cases}$$

Furthermore, in the case $-D \equiv 1 \mod 4$, for $1 \leq D \leq 13$, the product terminates if and only if $\beta \in \mathbb{Q}(\sqrt{-D})$, while in the case $-D \not\equiv 1 \mod 4$, for $D = 1$ or 2, the product terminates if and only if $\beta \in \mathbb{Q}(\sqrt{-D})$.

2. Independence Criteria

The classical Liouville's theorem [2, Theorem 1.2] gives a sufficient condition for a real number to be transcendental. Liouville's theorem has been extended in various directions; the one due to Fel'dman [5] is:

Lemma 1 (Liouville-type estimate). *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be algebraic numbers with $D := [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]$. If $f \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ satisfies $f(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq 0$, then*

$$|f(\alpha_1, \alpha_2, \ldots, \alpha_n)| \geq L(f)^{1-D} \prod_{\sigma=1}^{n} \left(H(\alpha_\sigma)(1 + \partial(\alpha_\sigma))\right)^{-\partial_\sigma(f) D/\partial(\alpha_\sigma)},$$

where $L(f)$, the length of f, denotes the sum of the absolute values of the coefficients of f, $H(\alpha)$, the height of α, denotes the maximum of the absolute values of the coefficients in the minimal polynomial of α over $\mathbb{Z}$, $\partial_\sigma(f)$ denotes the degree of f with respect to x_σ, and $\partial(\alpha)$ denotes the degree of an algebraic number α.

In 1988, Bundschuh [3], using Lemma 1, proved the following criterion for algebraic independence.

Theorem 1 (Criterion for algebraic independence). *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and let $g : \mathbb{N} \to \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ satisfy $g(t) \to \infty$ as $t \to \infty$. Suppose that for each $\tau \in \{1, 2, \ldots, n\}$, there exist an infinite set $N_\tau \subseteq \mathbb{N}$ and τ sequences $(\alpha_{1,t})_{t \in N_\tau}, \ldots, (\alpha_{\tau,t})_{t \in N_\tau}$ of algebraic numbers such that for each $t \in N_\tau$, the inequalities*

$$g(t) \sum_{\sigma=1}^{\tau-1} |\alpha_\sigma - \alpha_{\sigma,t}| < |\alpha_\tau - \alpha_{\tau,t}| \leq \exp\left(-g(n)\left[\mathbb{Q}(\alpha_{1,t}, \ldots, \alpha_{\tau,t}) : \mathbb{Q}\right] \sum_{\sigma=1}^{\tau} \frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right)$$

hold, where

$$s(\beta) := \partial(\beta) + \log(H(\beta))$$

is the so-called size of an algebraic number β. Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraically independent over $\mathbb{Q}$.

2.1. Linear Independence

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be complex numbers. We say that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are *linearly dependent* over $\mathbb{Q}$ if there exist $A_1, A_2, \ldots, A_{n+1} \in \mathbb{Z}$ not all zero such that

$$A_{n+1} + A_n\alpha_n + \cdots + A_2\alpha_2 + A_1\alpha_1 = 0.$$

We say that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are *linearly independent* over $\mathbb{Q}$ if they are not linearly dependent over $\mathbb{Q}$.

We now establish a criterion for linear independence along the same line as that in Theorem 1.

Theorem 2 (Criterion for linear independence). *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and let $g_i : \mathbb{N} \to \mathbb{R}^+ \quad (i = 1, 2)$ satisfy $g_i(t) \to \infty$ as $t \to \infty$. Suppose that for each $\tau \in \{1, 2, \ldots, n\}$, there exist an infinite set $N_\tau \subseteq \mathbb{N}$ and τ sequences $(\alpha_{1,t})_{t \in N_\tau}, \ldots, (\alpha_{\tau,t})_{t \in N_\tau}$ of algebraic numbers such that for each $t \in N_\tau$, the inequalities*

$$g_1(t) \sum_{\sigma=1}^{\tau-1} |\alpha_\sigma - \alpha_{\sigma,t}| < |\alpha_\tau - \alpha_{\tau,t}| \leq \frac{1}{g_2(t)} \exp\left(-\left[\mathbb{Q}(\alpha_{1,t}, \ldots, \alpha_{\tau,t}) : \mathbb{Q}\right] \sum_{\sigma=1}^{\tau} \frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right) \tag{6}$$

hold. Then $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$.

Proof. If $1, \alpha_1, \ldots, \alpha_n$ are linearly dependent over $\mathbb{Q}$, then there are $A_1, \ldots, A_{n+1} \in \mathbb{Z}$ not all zero such that

$$A_{n+1} + A_n\alpha_n + \cdots + A_1\alpha_1 = 0. \tag{7}$$

Not all of $A_1, \ldots, A_n$ are 0, for otherwise, $A_{n+1} = 0$ which is not possible. Without loss of generality assume $A_n \neq 0$. Let

$$f(x_1, \ldots, x_n) := A_{n+1} + A_n x_n + \cdots + A_1 x_1 \in \mathbb{Z}[x_1, \ldots, x_n] \setminus \{0\}.$$

Using (7), we rewrite this polynomial as

$$f(x_1, x_2, \ldots, x_n) = A_n(x_n - \alpha_n) + \cdots + A_2(x_2 - \alpha_2) + A_1(x_1 - \alpha_1).$$

Invoking upon (6), for $t \to \infty,\ t \in N_\tau$, we have

$$\begin{aligned} f(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t}) &= A_n(\alpha_{n,t} - \alpha_n) + \cdots + A_2(\alpha_{2,t} - \alpha_2) + A_1(\alpha_{1,t} - \alpha_1) \\ &= A_n(\alpha_{n,t} - \alpha_n)\left(1 + \frac{A_{n-1}(\alpha_{n-1,t} - \alpha_{n-1})}{A_n(\alpha_{n,t} - \alpha_n)} + \cdots + \frac{A_1(\alpha_{1,t} - \alpha_1)}{A_n(\alpha_{n,t} - \alpha_n)}\right) \\ &= A_n(\alpha_{n,t} - \alpha_n)(1 + o(1)) \neq 0 \end{aligned} \tag{8}$$

for all large $t \in N_\tau$. From Lemma 1, we get

$$|f(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t})| \geq L(f)^{1-D_t} \prod_{\sigma=1}^{n} \left(H(\alpha_{\sigma,t})(1 + \partial(\alpha_{\sigma,t}))\right)^{-\partial_\sigma(f) D_t/\partial(\alpha_{\sigma,t})} =: e^B,$$

where $D_t := [\mathbb{Q}(\alpha_{1,t}, \ldots, \alpha_{\tau,t}) : \mathbb{Q}]$. Since

$$\begin{aligned} B &= (1 - D_t)\log(L(f)) + \log\left(\prod_{\sigma=1}^{n}\left(e^{s(\alpha_{\sigma,t}) - \partial(\alpha_{\sigma,t})}(1 + \partial(\alpha_{\sigma,t}))\right)^{-D_t/\partial(\alpha_{\sigma,t})}\right) \\ &= (1 - D_t)\log(L(f)) - D_t\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})} + D_t\sum_{\sigma=1}^{n}\left(1 - \frac{\log(1 + \partial(\alpha_{\sigma,t}))}{\partial(\alpha_{\sigma,t})}\right) \\ &> -D_t\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})} - (D_t - 1)\log(L(f)), \end{aligned}$$

we get

$$|f(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t})| > \exp\left(-D_t\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})} - C(f)\right), \tag{9}$$

where $C(f) := (D_t - 1)\log(L(f)) > 0$ is a constant depending only on f. On the other hand, by (8) and (6), we get

$$|f(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t})| \leq 2|A_n||\alpha_{n,t} - \alpha_n| \leq \frac{2|A_n|}{g_2(t)}\exp\left(-D_t\sum_{\sigma=1}^{\tau}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right) \tag{10}$$

for all large $t \in N_\tau$. The bounds in (9) and (10) are contradictory for large $t \in N_\tau$, and the proof is complete. □

2.2. Quadratic Independence

The complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are said to be *quadratically dependent* over $\mathbb{Q}$ if there exists $f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n] \setminus \{0\}$ of exact degree 2 such that

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0,$$

and we say that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are quadratically independent over $\mathbb{Q}$ otherwise.

Theorem 3 (Criterion for quadratic independence). *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$. Assume that*

- *the numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are quadratically independent over $\mathbb{Q}$ and*
- *the numbers $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$.*

Let $g_i : \mathbb{N} \to \mathbb{R}^+$ $(i = 1, 2)$ satisfy $g_i(t) \to \infty$ as $t \to \infty$. Suppose that there exist an infinite set $N \subseteq \mathbb{N}$, and n sequences $(\alpha_{1,t})_{t\in N}, (\alpha_{2,t})_{t\in N}, \ldots, (\alpha_{n,t})_{t\in N}$ of algebraic numbers such that for each $t \in N$ the inequalities

$$g_1(t)\left(\sum_{\nu_1+\cdots+\nu_n=2} |\alpha_1 - \alpha_{1,t}|^{\nu_1} \cdots |\alpha_n - \alpha_{n,t}|^{\nu_n} + \sum_{\sigma=1}^{n-1} |\alpha_\sigma - \alpha_{\sigma,t}|\right) < |\alpha_n - \alpha_{n,t}| \tag{11}$$

$$\leq \frac{1}{g_2(t)} \exp\left(-2\left[\mathbb{Q}(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t}) : \mathbb{Q}\right] \sum_{\sigma=1}^{n} s(\alpha_{\sigma,t}) / \partial(\alpha_{\sigma,t})\right) \tag{12}$$

hold. Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are quadratically independent over $\mathbb{Q}$.

Proof. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are quadratically dependent over $\mathbb{Q}$, then there exists $f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n] \setminus \{0\}$ of exact degree 2 such that $f(\alpha_1, \ldots, \alpha_n) = 0$. Expanding the polynomial f about the point $(\alpha_1, \ldots, \alpha_n)$, we get

$$f(x_1, x_2, \ldots, x_n) = \sum c_{\nu_1,\ldots,\nu_n} (x_1 - \alpha_1)^{\nu_1} \cdots (x_n - \alpha_n)^{\nu_n},$$

where all the exponents satisfy $0 \leq \nu_1 + \cdots + \nu_n \leq 2$, and one particular set of exponents $\nu_1^*, \ldots, \nu_n^*$ with $\nu_1^* + \cdots + \nu_n^* = 2$ has a non-zero coefficient. Clearly, $c_{0,\ldots,0,0} = f(\alpha_1, \ldots, \alpha_n) = 0$. From

$$\frac{\partial f}{\partial x_n} = \sum \nu_n c_{\nu_1,\ldots,\nu_n} (x_1 - \alpha_1)^{\nu_1} \cdots (x_{n-1} - \alpha_{n-1})^{\nu_{n-1}} (x_n - \alpha_n)^{\nu_n - 1},$$

we see that $\partial f / \partial x_n (\alpha_1, \alpha_2, \ldots, \alpha_n) = c_{0,\ldots,0,1}$. If $\partial f / \partial x_n \equiv 0$, then f is independent of x_n, contradicting the assumption that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are quadratically independent, showing that f truly contains x_n. If $c_{0,\ldots,0,1} = 0$, then $\partial f / \partial x_n$ is a polynomial of degree less than 2 with $\partial f / \partial x_n (\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$, contradicting the assumption that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent; henceforth, we must have $c_{0,\ldots,0,1} \neq 0$.

Using (11) and (12), for $t \in N$, we have

$$f\left(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t}\right) = c_{0,\ldots,0,1}\left(\alpha_{n,t} - \alpha_n\right)\left(1 + o\left(1\right)\right) \neq 0 \quad (t \to \infty). \tag{13}$$

From Lemma 1, we get

$$\left|f\left(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t}\right)\right| \geq L\left(f\right)^{1-D_t} \prod_{\sigma=1}^{n} \left(H\left(\alpha_{\sigma,t}\right)\left(1 + \partial\left(\alpha_{\sigma,t}\right)\right)\right)^{-\partial_\sigma(f)D_t/\partial\left(\alpha_{\sigma,t}\right)} =: e^{B_2},$$

where

$$\begin{aligned}
B_2 &= \left(1 - D_t\right)\log\left(L\left(f\right)\right) + \log\left(\prod_{\sigma=1}^{n}\left(e^{s(\alpha_{\sigma,t})-\partial(\alpha_{\sigma,t})}\left(1 + \partial\left(\alpha_{\sigma,t}\right)\right)\right)^{-\partial_\sigma(f)D_t/\partial(\alpha_{\sigma,t})}\right) \\
&= \left(1 - D_t\right)\log\left(L\left(f\right)\right) - D_t\sum_{\sigma=1}^{n}\frac{\partial_\sigma\left(f\right)s\left(\alpha_{\sigma,t}\right)}{\partial\left(\alpha_{\sigma,t}\right)} \\
&\quad + D_t\sum_{\sigma=1}^{n}\partial_\sigma\left(f\right)\left(1 - \frac{\log\left(1 + \partial\left(\alpha_{\sigma,t}\right)\right)}{\partial\left(\alpha_{\sigma,t}\right)}\right) \\
&> -\left(D_t - 1\right)\log\left(L\left(f\right)\right) - 2D_t\sum_{\sigma=1}^{n}\frac{s\left(\alpha_{\sigma,t}\right)}{\partial\left(\alpha_{\sigma,t}\right)}.
\end{aligned}$$

Thus,

$$\left|f\left(\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{n,t}\right)\right| > \exp\left(-2D_t\sum_{\sigma=1}^{n}\frac{s\left(\alpha_{\sigma,t}\right)}{\partial\left(\alpha_{\sigma,t}\right)} - C\left(f\right)\right), \tag{14}$$

where $C\left(f\right) := \left(D_t - 1\right)\log\left(L\left(f\right)\right) > 0$ is a constant depending only on f. On the other hand, by (13), (11) and (12), we get

$$\left|f\left(\alpha_{1,t}, \alpha_{2,t} \ldots, \alpha_{n,t}\right)\right| \leq 2\left|c_{0,\ldots,0,1}\right|\left|\alpha_{n,t} - \alpha_n\right| \leq \frac{2\left|c_{0,\ldots,0,1}\right|}{g_2\left(t\right)}\exp\left(-2D_t\sum_{\sigma=1}^{n}\frac{s\left(\alpha_{\sigma,t}\right)}{\partial\left(\alpha_{\sigma,t}\right)}\right) \tag{15}$$

for all large $t \in N$. The two bounds in (14) and (15) are untenable for sufficiently large $t \in N$. □

3. Independence of Cantor Series

We start with the case of real numbers.

3.1. Real Cantor Series

The following notation and conditions will be kept fixed throughout this subsection. Let $\alpha_1, \ldots, \alpha_n$ be nonzero real numbers in the interval $(0,1)$ whose infinite $\mathcal{M}$-Cantor series are

$$\alpha_\tau := \sum_{k=1}^{\infty} \frac{c_{\tau,k}}{m_1 m_2 \cdots m_k} \qquad (1 \le \tau \le n)$$

subject to the digit conditions (2). For $t \in \mathbb{N}$, let

$$\alpha_{\tau,t} = \sum_{k=1}^{t} \frac{c_{\tau,k}}{m_1 m_2 \cdots m_k} \quad (1 \le \tau \le n).$$

Theorem 4. *If there exist functions $g_1, g_2 : \mathbb{N} \to \mathbb{R}^+$ such that*

i) the function g_1 is non-decreasing,

ii) both functions $g_1(k), g_2(k) \to \infty \qquad (k \to \infty)$,

iii) $g_1(k) \sum_{\sigma=1}^{\tau-1} c_{\sigma,k} \le c_{\tau,k} \qquad (2 \le \tau \le n,\ k \in \mathbb{N})$, and

iv) $\frac{m_1 m_2 \cdots m_{k+1}}{1+c_{\tau,k+1}} \ge g_2(k) \exp\left([\mathbb{Q}(\alpha_{1,k}, \ldots, \alpha_{\tau,k}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\alpha_{\sigma,k})}{\partial(\alpha_{\sigma,k})} \right) \qquad (1 \le \tau \le n,\ k \in \mathbb{N})$,

then $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$.

Proof. For $1 \le \tau \le n$, using the digit conditions (2), we have

$$|\alpha_\tau - \alpha_{\tau,t}| = \frac{1}{m_1 m_2 \cdots m_{t+1}} \left\{ c_{\tau,t+1} + \frac{c_{\tau,t+2}}{m_{t+2}} + \frac{c_{\tau,t+3}}{m_{t+2} m_{t+3}} + \cdots \right\} \tag{16}$$

$$\le \frac{1}{m_1 m_2 \cdots m_{t+1}} \left\{ c_{\tau,t+1} + \left(1 - \frac{1}{m_{t+2}}\right) + \left(\frac{1}{m_{t+2}} - \frac{1}{m_{t+2} m_{t+3}}\right) + \cdots \right\}$$

$$= \frac{c_{\tau,t+1} + 1}{m_1 m_2 \cdots m_{t+1}} \le \frac{1}{g_2(t)} \exp\left(-[\mathbb{Q}(\alpha_{1,t}, \ldots, \alpha_{\tau,t}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})} \right). \tag{17}$$

From (16) and the hypotheses, we deduce that

$$\frac{\sum_{\sigma=1}^{\tau-1} |\alpha_\sigma - \alpha_{\sigma,t}|}{|\alpha_\tau - \alpha_{\tau,t}|}$$

$$= \frac{\left(c_{1,t+1} + \frac{c_{1,t+2}}{m_{t+2}} + \frac{c_{1,t+3}}{m_{t+2} m_{t+3}} + \cdots \right) + \cdots + \left(c_{\tau-1,t+1} + \frac{c_{\tau-1,t+2}}{m_{t+2}} + \frac{c_{\tau-1,t+3}}{m_{t+2} m_{t+3}} + \cdots \right)}{c_{\tau,t+1} + \frac{c_{\tau,t+2}}{m_{t+2}} + \frac{c_{\tau,t+3}}{m_{t+2} m_{t+3}} + \cdots}$$

$$\le \frac{\frac{1}{g_1(t+1)} c_{\tau,t+1} + \frac{1}{g_1(t+2)} \frac{c_{\tau,t+2}}{m_{t+2}} + \frac{1}{g_1(t+3)} \frac{c_{\tau,t+3}}{m_{t+2} m_{t+3}} + \cdots}{c_{\tau,t+1} + \frac{c_{\tau,t+2}}{m_{t+2}} + \frac{c_{\tau,t+3}}{m_{t+2} m_{t+3}} + \cdots}$$

$$\leq \frac{\frac{1}{g_1(t+1)}\left(c_{\tau,t+1}+\frac{c_{\tau,t+2}}{m_{t+2}}+\frac{c_{\tau,t+3}}{m_{t+2}m_{t+3}}+\cdots\right)}{c_{\tau,t+1}+\frac{c_{\tau,t+2}}{m_{t+2}}+\frac{c_{\tau,t+3}}{m_{t+2}m_{t+3}}+\cdots}=\frac{1}{g_1(t+1)}. \tag{18}$$

Noting (18), Theorem 2 then implies that $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$. □

Theorem 5. *Assume that $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are quadratically independent over $\mathbb{Q}$, and that $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$. If there are functions $h_1, h_2 : \mathbb{N} \to \mathbb{R}^+$ such that*

i) the function h_1 is increasing,

ii) both the functions $h_1(k), h_2(k) \to \infty \qquad (k \to \infty)$,

iii) $h_1(k)\sum_{\sigma=1}^{n-1} c_{\sigma,k} \leq c_{n,k} \qquad (k \in \mathbb{N})$, and

iv) $\frac{m_1 m_2 \cdots m_{k+1}}{1+c_{n,k+1}} \geq h_2(k)\exp\left(2[\mathbb{Q}(\alpha_{1,k},\dots,\alpha_{n,k}):\mathbb{Q}]\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,k})}{\partial(\alpha_{\sigma,k})}\right) \quad (k \in \mathbb{N})$,

then $\alpha_1, \alpha_2, \dots, \alpha_n$ are quadratically independent over $\mathbb{Q}$.

Proof. We make use of Theorem 3. Proceed as in the proof of Theorem 4. Choosing $\tau = n$ and using hypothesis (iv), we get, similar to (17) and (18),

$$|\alpha_n - \alpha_{n,t}| \leq \frac{c_{n,t+1}+1}{m_1 m_2 \cdots m_{t+1}} \leq \frac{1}{h_2(t)}\exp\left(-2[\mathbb{Q}(\alpha_{1,t},\dots,\alpha_{n,t}):\mathbb{Q}]\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right), \tag{19}$$

and

$$\frac{1}{|\alpha_n-\alpha_{n,t}|}\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}| \leq \frac{1}{h_1(t+1)}. \tag{20}$$

Next, since $0 < |\alpha_\tau - \alpha_{\tau,t}| < 1$ $(1 \leq \tau \leq n)$, we have

$$\begin{aligned}
&\sum_{\nu_1+\cdots+\nu_n=2}|\alpha_1-\alpha_{1,t}|^{\nu_1}\cdots|\alpha_n-\alpha_{n,t}|^{\nu_n}\\
&=\sum_{\sigma=1}^{n}|\alpha_\sigma-\alpha_{\sigma,t}|^2+\sum_{\substack{\nu_1+\cdots+\nu_n=2\\ \nu_i\neq 2}}|\alpha_1-\alpha_{1,t}|^{\nu_1}\cdots|\alpha_n-\alpha_{n,t}|^{\nu_n}\\
&\leq\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}|+|\alpha_n-\alpha_{n,t}|^2+(n-1)\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}|\\
&=n\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}|+|\alpha_n-\alpha_{n,t}|^2.
\end{aligned} \tag{21}$$

Using (19), (20) and (21), we get

$$\frac{1}{|\alpha_n - \alpha_{n,t}|}\left(\sum_{\nu_1+\cdots+\nu_n=2} |\alpha_1-\alpha_{1,t}|^{\nu_1}\cdots|\alpha_n-\alpha_{n,t}|^{\nu_n} + \sum_{\sigma=1}^{n-1}|\alpha_\sigma - \alpha_{\sigma,t}|\right)$$
$$\leq \frac{n+1}{h_1(t+1)} + \frac{1}{h_2(t)} =: \frac{1}{g_1(t)}. \tag{22}$$

Invoking upon (19) and (22), the desired result now follows from Theorem 3. □

As for algebraic independence, we have the following result.

Theorem 6. *If there exist functions $g_1, g_2 : \mathbb{N} \to \mathbb{R}^+$ such that*

- *g_1 is increasing,*
- *both $g_1(k), g_2(k) \to \infty \qquad (k \to \infty)$,*
- *$g_1(k)\sum_{\sigma=1}^{\tau-1} c_{\sigma,k} \leq c_{\tau,k} \qquad (2 \leq \tau \leq n,\ k \in \mathbb{N})$, and*
- *$\frac{m_1 m_2\cdots m_{k+1}}{1+c_{\tau,k+1}} \geq \exp\left(g_2(k)[\mathbb{Q}(\alpha_{1,k},\ldots,\alpha_{\tau,k}):\mathbb{Q}]\sum_{\sigma=1}^{\tau}\frac{s(\alpha_{\sigma,k})}{\partial(\alpha_{\sigma,k})}\right) \qquad (1 \leq \tau \leq n,\ k \in \mathbb{N})$,*

then $\alpha_1, \ldots, \alpha_n$ are algebraically independent over $\mathbb{Q}$.

Sketch of proof. Proceeding as in the proof of Theorem 4, we get, similar to (17) and (18),

$$|\alpha_\tau - \alpha_{\tau,t}| \leq \frac{c_{\tau,t+1}+1}{m_1 m_2\cdots m_{t+1}} \leq \exp\left(-g_2(t)[\mathbb{Q}(\alpha_{1,t},\ldots,\alpha_{\tau,t}):\mathbb{Q}]\sum_{\sigma=1}^{\tau}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right),$$

and

$$\frac{\sum_{\sigma=1}^{\tau-1}|\alpha_\sigma - \alpha_{\sigma,t}|}{|\alpha_\tau - \alpha_{\tau,t}|} \leq \frac{1}{g_1(t+1)}.$$

The desired result follows by invoking upon Theorem 1. □

3.2. Complex Cantor Series

Throughout this subsection, the following notation and restrictions will be kept standard. Let $\beta_\tau := x^{(\tau)} + i\ y^{(\tau)} \quad (1 \leq \tau \leq n)$ be n nonzero complex numbers whose Cantor series representations, as defined in Subsection 1.1, are

$$\beta_\tau = x^{(\tau)} + i\ y^{(\tau)} := \sum_{k=1}^{\infty}\frac{c_{\tau,k}}{m_1 m_2\cdots m_k} + i\sum_{k=1}^{\infty}\frac{d_{\tau,k}}{m_1 m_2\cdots m_k} \qquad (1 \leq \tau \leq n)$$

where $c_{\tau,k}, d_{\tau,k}$ are subject to the digit conditions (5). For $t \in \mathbb{N}$, let

$$x_{\tau,t} = \sum_{k=1}^{t}\frac{c_{\tau,k}}{m_1 m_2\cdots m_k}, \quad y_{\tau,t} = \sum_{k=1}^{t}\frac{d_{\tau,k}}{m_1 m_2\cdots m_k} \quad (1 \leq \tau \leq n).$$

Theorem 7. *If there exist functions $g_1, g_2 : \mathbb{N} \to \mathbb{R}^+$ such that*

i) the function g_1 is non-decreasing,

ii) both functions $g_1(k), g_2(k) \to \infty \quad (k \to \infty)$, and

iii) for $k \in \mathbb{N}$, either

$$g_1(k) \sum_{\sigma=1}^{\tau-1} c_{\sigma,k} \leq c_{\tau,k} \quad (2 \leq \tau \leq n)$$

$$\frac{m_1 m_2 \cdots m_{k+1}}{1 + c_{\tau,k+1}} \geq g_2(k) \exp\left([\mathbb{Q}(x_{1,k}, \ldots, x_{\tau,k}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(x_{\sigma,k})}{\partial(x_{\sigma,k})} \right) \quad (1 \leq \tau \leq n)$$

or

$$g_1(k) \sum_{\sigma=1}^{\tau-1} d_{\sigma,k} \leq d_{\tau,k} \quad (2 \leq \tau \leq n)$$

$$\frac{m_1 m_2 \cdots m_{k+1}}{1 + d_{\tau,k+1}} \geq g_2(k) \exp\left([\mathbb{Q}(y_{1,k}, \ldots, y_{\tau,k}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(y_{\sigma,k})}{\partial(y_{\sigma,k})} \right) \quad (1 \leq \tau \leq n),$$

then $1, \beta_1, \beta_2, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$.

Proof. Appealing to Theorem 4, the hypotheses ensure that either $1, x^{(1)}, \ldots, x^{(n)}$ or $1, y^{(1)}, \ldots, y^{(n)}$ are linearly independent over $\mathbb{Q}$. Should $1, \beta_1, \beta_2, \ldots, \beta_n$ be linearly dependent over $\mathbb{Q}$, then there would exist $A_1, A_2, \ldots, A_n, A_{n+1} \in \mathbb{Z}$ not all zero such that

$$\begin{aligned} 0 &= A_{n+1} + A_n \beta_n + \cdots + A_2 \beta_2 + A_1 \beta_1 \\ &= A_{n+1} + A_n x^{(n)} + \cdots + A_2 x^{(2)} + A_1 x^{(1)} + i \left(A_n y^{(n)} + \cdots + A_2 y^{(2)} + A_1 y^{(1)} \right), \end{aligned}$$

showing that both the real and imaginary parts are linearly dependent over $\mathbb{Q}$, which is a contradiction. □

For convenience, we relabel the digits (in any fixed order) as

$$\{c_{1,k}, c_{2,k}, \ldots, c_{n,k}, d_{1,k}, d_{2,k}, \ldots, d_{n,k}\} =: \{e_{1,k}, e_{2,k}, \ldots, e_{2n,k}\} \qquad (k \in \mathbb{N}).$$

For $t \in \mathbb{N}$, let

$$\beta_{\tau,t} = \sum_{k=1}^{t} \frac{e_{\tau,k}}{m_1 m_2 \cdots m_k} \quad (1 \leq \tau \leq 2n).$$

Theorem 8. *If there exist functions $g_1, g_2 : \mathbb{N} \to \mathbb{R}^+$ such that*

i) the function g_1 is increasing,

ii) both functions $g_1(k), g_2(k) \to \infty \qquad (k \to \infty)$, and *iii) for $k \in \mathbb{N}$, the following two inequalities hold*

$$g_1(k) \sum_{\sigma=1}^{\tau-1} e_{\sigma,k} \le e_{\tau,k} \quad (2 \le \tau \le 2n)$$

$$\frac{m_1 m_2 \cdots m_{k+1}}{1 + e_{\tau,k+1}} \ge \exp\left(g_2(k) [\mathbb{Q}(\beta_{1,k}, \ldots, \beta_{\tau,k}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\beta_{\sigma,k})}{\partial(\beta_{\sigma,k})} \right) \quad (1 \le \tau \le 2n),$$

then $\beta_1, \beta_2, \ldots, \beta_n$ are algebraically independent over $\mathbb{Q}$.

Proof. By Theorem 6, we see that $x^{(1)}, \ldots, x^{(n)}, y^{(1)}, \ldots, y^{(n)}$ are algebraically independent over $\mathbb{Q}$. If $\beta_1, \beta_2, \ldots, \beta_n$ are algebraically dependent over $\mathbb{Q}$, then there would exist

$$f(x_1, x_2, \ldots, x_n) := \sum A_{(v_1, v_2, \ldots, v_n)} x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n} \in \mathbb{Z}[x_1, x_2, \ldots, x_n] \setminus \{0\}$$

such that $0 = f(\beta_1, \beta_2, \ldots, \beta_n) = \sum A_{(v_1, v_2, \ldots, v_n)} \beta_1^{v_1} \beta_2^{v_2} \cdots \beta_n^{v_n}$. Simplifying, we get an equation of the form

$$\begin{aligned} 0 = &\sum B_{(r_1, \ldots, r_n, r_{n+1}, \ldots, r_{2n})} (x^{(1)})^{r_1} \cdots (x^{(n)})^{r_n} (y^{(1)})^{r_{n+1}} \cdots (y^{(n)})^{r_{2n}} \\ &+ i \sum C_{(s_1, \ldots, s_n, s_{n+1}, \ldots, s_{2n})} (x^{(1)})^{s_1} \cdots (x^{(n)})^{s_n} (y^{(1)})^{s_{n+1}} \cdots (y^{(n)})^{s_{2n}} \end{aligned}$$

whose coefficients $B_{(r_1, \ldots, r_n, r_{n+1}, \ldots, r_{2n})}, C_{(s_1, \ldots, s_n, s_{n+1}, \ldots, s_{2n})}$ are all in $\mathbb{Z}$ and not all simultaneously 0. Equating the real and imaginary parts, we have

$$\sum B_{(r_1, \ldots, r_n, r_{n+1}, \ldots, r_{2n})} (x^{(1)})^{r_1} \cdots (x^{(n)})^{r_n} (y^{(1)})^{r_{n+1}} \cdots (y^{(n)})^{r_{2n}} = 0$$

and

$$\sum C_{(s_1, \ldots, s_n, s_{n+1}, \ldots, s_{2n})} (x^{(1)})^{s_1} \cdots (x^{(n)})^{s_n} (y^{(1)})^{s_{n+1}} \cdots (y^{(n)})^{s_{2n}} = 0,$$

contradicting the algebraic independence of $x^{(1)}, \ldots, x^{(n)}, y^{(1)}, \ldots, y^{(n)}$. □

Regarding the quadratic independence, we have not yet obtained suitable criteria for both complex Cantor series and complex Cantor products. We plan to take up this question later.

4. Independence of Cantor Products

As in the case of series representations, we start with real numbers.

4.1. Real Cantor Products

The following notation and restrictions will be kept standard throughout this section. Let $\alpha_\tau > 1 \ \ (\tau = 1, \ldots, n(\geq 2))$ be real numbers whose infinite Cantor product representations, as described in Subsection 1.2, are

$$\alpha_\tau := \prod_{k=1}^{\infty} \left(1 + \frac{1}{a_{\tau,k}}\right) \quad (1 \leq \tau \leq n),$$

where for each τ, the positive integers $a_{\tau,k}$ are subject to the restrictions (3). For $t \in \mathbb{N}, \ 1 \leq \tau \leq n$, let

$$\alpha_{\tau,t} = \prod_{k=1}^{t} \left(1 + \frac{1}{a_{\tau,k}}\right).$$

Theorem 9. *Assume that*

$$a_{\tau,k} \sum_{\sigma=1}^{\tau-1} \left(\frac{1}{a_{\sigma,k} - 1}\right) \to 0 \quad (2 \leq \tau \leq n, \ k \to \infty). \tag{23}$$

If there exists $g : \mathbb{N} \to \mathbb{R}^+$ *with* $g(k) \to \infty \ \ (k \to \infty)$ *such that*

$$\frac{a_{\tau,k+1} - 1}{\alpha_{\tau,k}} \geq g(k) \exp\left([\mathbb{Q}(\alpha_{1,k}, \ldots, \alpha_{\tau,k}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\alpha_{\sigma,k})}{\partial(\alpha_{\sigma,k})}\right) \quad (1 \leq \tau \leq n), \tag{24}$$

then $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ *are linearly independent over* $\mathbb{Q}$.

Proof. For $1 \leq \tau \leq n$, using (3), we have

$$\begin{aligned} \left|\frac{1}{\alpha_{\tau,t}}\right| |\alpha_\tau - \alpha_{\tau,t}| &= \prod_{k \geq t+1} \left(1 + \frac{1}{a_{\tau,k}}\right) - 1 \\ &\leq \left(1 + \frac{1}{a_{\tau,t+1}}\right)\left(1 + \frac{1}{a_{\tau,t+1}^2}\right)\left(1 + \frac{1}{a_{\tau,t+1}^{2^2}}\right) \cdots - 1 \\ &= \frac{1}{1 - 1/a_{\tau,t+1}} - 1 = \frac{1}{a_{\tau,t+1} - 1}. \end{aligned} \tag{25}$$

Making use of (24), we get

$$|\alpha_\tau - \alpha_{\tau,t}| \leq \left|\frac{\alpha_{\tau,t}}{a_{\tau,t+1} - 1}\right| \leq \frac{1}{g(t)} \exp\left(-[\mathbb{Q}(\alpha_{1,t}, \ldots, \alpha_{\tau,t}) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right). \tag{26}$$

In another direction, for $1 \leq \tau \leq n, \ t \in \mathbb{N}$, since $a_{\tau,k} \in \mathbb{N}$, we have

$$\left|\prod_{k \geq t+1} \left(1 + \frac{1}{a_{\tau,k}}\right)\right| > \left|1 + \frac{1}{a_{\tau,t+1}}\right| = \frac{a_{\tau,t+1} + 1}{a_{\tau,t+1}}, \tag{27}$$

so that

$$|\alpha_{\tau,t}| = \frac{|\alpha_\tau|}{\left|\prod_{k\geq t+1}(1+1/a_{\tau,k})\right|} < \frac{|\alpha_\tau|\ a_{\tau,t+1}}{a_{\tau,t+1}+1},$$

which yields for sufficiently large t,

$$|\alpha_\tau - \alpha_{\tau,t}| \leq \left|\frac{\alpha_{\tau,t}}{a_{\tau,t+1}-1}\right| < \frac{|\alpha_\tau|\ a_{\tau,t+1}}{(a_{\tau,t+1}+1)(a_{\tau,t+1}-1)} < \frac{|\alpha_\tau|}{2},$$

and so

$$\frac{|\alpha_\tau|}{2} < |\alpha_{\tau,t}| < \frac{3|\alpha_\tau|}{2}. \tag{28}$$

From (25), (28) and (23), we get

$$\begin{aligned}
\frac{1}{|\alpha_\tau - \alpha_{\tau,t}|}\sum_{\sigma=1}^{\tau-1}|\alpha_\sigma - \alpha_{\sigma,t}| &\leq \frac{|\alpha_{1,t}|\ \frac{1}{a_{1,t+1}-1} + \cdots + |\alpha_{\tau-1,t}|\ \frac{1}{a_{\tau-1,t+1}-1}}{|\alpha_{\tau,t}|\ \frac{1}{a_{\tau,t+1}}} \\
&< \frac{3\,|\alpha_1|\ \frac{1}{a_{1,t+1}-1} + \cdots + 3\,|\alpha_{\tau-1}|\ \frac{1}{a_{\tau-1,t+1}-1}}{|\alpha_\tau|\ \frac{1}{a_{\tau,t+1}}} \\
&\leq \frac{3M}{|\alpha_\tau|}\cdot a_{\tau,t+1}\sum_{\sigma=1}^{\tau-1}\frac{1}{a_{\sigma,t+1}-1} \to 0 \quad (t\to\infty),
\end{aligned} \tag{29}$$

where $M := \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{\tau-1}|\}$. Noting (26) and (29), Theorem 2 then implies that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. □

Theorem 10. *Assume that*

- *the numbers* $\alpha_1, \ldots, \alpha_{n-1}$ *are quadratically independent over* $\mathbb{Q}$*,*
- *the numbers* $1, \alpha_1, \ldots, \alpha_n$ *are linearly independent over* $\mathbb{Q}$*, and*
- *the values* $a_{n,k}\sum_{\sigma=1}^{n-1}\left(\frac{1}{a_{\sigma,k}-1}\right) \to 0 \qquad (k\to\infty)$*.*

If there exists $g : \mathbb{N} \to \mathbb{R}^+$ *with* $g(k) \to \infty \quad (k\to\infty)$ *such that*

$$\frac{a_{n,k+1}-1}{\alpha_{n,k}} \geq g(k)\exp\left(2[\mathbb{Q}(\alpha_{1,k},\ldots,\alpha_{n,k}):\mathbb{Q}]\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,k})}{\partial(\alpha_{\sigma,k})}\right), \tag{30}$$

then $\alpha_1, \ldots, \alpha_n$ *are quadratically independent over* $\mathbb{Q}$*.*

Proof. To apply Theorem 3, we only need to check the conditions (11) and (12). Choosing $\tau = n$, proceeding as in the proof of Theorem 9 up to (26), and using (30) instead of (24), we get

$$|\alpha_n - \alpha_{n,t}| \leq \frac{\alpha_{n,t}}{a_{n,t+1}-1} \leq \frac{1}{g(t)}\exp\left(-2[\mathbb{Q}(\alpha_{1,t},\ldots,\alpha_{n,t}):\mathbb{Q}]\sum_{\sigma=1}^{n}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right) \tag{31}$$

which shows that the condition (12) is fulfilled. Similarly, proceeding as in the proof of Theorem 9 up to (29), we get

$$\frac{\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}|}{|\alpha_n-\alpha_{n,t}|}<\frac{3M}{|\alpha_n|}\cdot a_{n,t+1}\sum_{\sigma=1}^{n-1}\frac{1}{a_{\sigma,t+1}-1}=:\frac{1}{H(t)}\to 0 \quad (t\to\infty). \tag{32}$$

Using the inequality (21) in the proof of Theorem 5, we get

$$\begin{aligned}&\frac{1}{|\alpha_n-\alpha_{n,t}|}\sum_{\nu_1+\cdots+\nu_n=2}|\alpha_1-\alpha_{1,t}|^{\nu_1}\cdots|\alpha_n-\alpha_{n,t}|^{\nu_n}\\ &\le\frac{n}{|\alpha_n-\alpha_{n,t}|}\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}|+|\alpha_n-\alpha_{n,t}|.\end{aligned} \tag{33}$$

Combining (32), (33) and using (31), we arrive at

$$\begin{aligned}&\frac{1}{|\alpha_n-\alpha_{n,t}|}\left(\sum_{\nu_1+\cdots+\nu_n=2}|\alpha_1-\alpha_{1,t}|^{\nu_1}\cdots|\alpha_n-\alpha_{n,t}|^{\nu_n}+\sum_{\sigma=1}^{n-1}|\alpha_\sigma-\alpha_{\sigma,t}|\right)\\ &\le\frac{n+1}{H(t)}+\frac{1}{g(t)},\end{aligned}$$

which shows that (11) is also fulfilled. □

As for algebraic independence, we prove:

Theorem 11. *Assume that*

$$a_{\tau,k}\sum_{\sigma=1}^{\tau-1}\left(\frac{1}{a_{\sigma,k}-1}\right)\to 0 \qquad (2\le\tau\le n,\ k\to\infty).$$

If there exists $g:\mathbb{N}\to\mathbb{R}^+$ *with* $g(k)\to\infty \ \ (k\to\infty)$ *such that*

$$\frac{a_{\tau,k+1}-1}{\alpha_{\tau,k}}\ge\exp\left(g(k)[\mathbb{Q}(\alpha_{1,k},\ldots,\alpha_{\tau,k}):\mathbb{Q}]\sum_{\sigma=1}^{\tau}\frac{s(\alpha_{\sigma,k})}{\partial(\alpha_{\sigma,k})}\right) \qquad (1\le\tau\le n),$$

then $\alpha_1,\alpha_2,\ldots,\alpha_n$ *are algebraically independent over* $\mathbb{Q}$.

Sketch of proof. Proceeding as in the proof of Theorem 9, we get, similar to (26) and (29),

$$|\alpha_\tau-\alpha_{\tau,t}|\le\left|\frac{\alpha_{\tau,t}}{a_{\tau,t+1}-1}\right|\le\exp\left(-g(t)[\mathbb{Q}(\alpha_{1,t},\ldots,\alpha_{\tau,t}):\mathbb{Q}]\sum_{\sigma=1}^{\tau}\frac{s(\alpha_{\sigma,t})}{\partial(\alpha_{\sigma,t})}\right),$$

and

$$\frac{1}{|\alpha_\tau-\alpha_{\tau,t}|}\sum_{\sigma=1}^{\tau-1}|\alpha_\sigma-\alpha_{\sigma,t}|<\frac{3M}{|\alpha_\tau|}\cdot a_{\tau,t+1}\sum_{\sigma=1}^{\tau-1}\frac{1}{a_{\sigma,t+1}-1}\to 0 \qquad (t\to\infty),$$

where $M:=\max\{|\alpha_1|,|\alpha_2|,\ldots,|\alpha_{\tau-1}|\}$. The desired result now follows by invoking upon Theorem 1. □

4.2. Complex Cantor Products

We begin by stating two useful identities whose straightforward proofs are omitted. These identities will be used to obtain upper and lower bounds for a complex infinite product.

Lemma 2. *For $z \in \mathbb{C}$, if $|z| < 1$, then*

(a) $\prod_{j=0}^{\infty}\left(1+z^{2^j}\right) = (1+z)\left(1+z^2\right)\left(1+z^{2^2}\right)\left(1+z^{2^3}\right)\cdots = \frac{1}{1-z}$

(b) $\prod_{j=0}^{\infty}\left(1+z^{2^j}\right) - 1 = \left(z+z^2+z^{2^2}+\cdots\right)+\left(z\cdot z^2+z\cdot z^{2^2}+\cdots\right)+\cdots = \frac{z}{1-z}.$

Consider the case $D = 1$ of Knopfmacher's result mentioned in Subsection 1.2. The following notation and restrictions will be kept fixed throughout this subsection. Let $\beta_1, \ldots, \beta_n$ be n nonzero complex numbers having infinite Cantor product representations of the form

$$\beta_\tau = \prod_{k=1}^{\infty}\left(1+\frac{1}{a_{\tau,k}}\right) \qquad (1 \le \tau \le n),$$

where $a_{\tau,k}$ are nonzero Gaussian integers subject to the condition that

$$|a_{\tau,k+1}| \ge \sqrt{2}\,|a_{\tau,k}|^2 - \left(\sqrt{2}+1\right)|a_{\tau,k}| - \left(1+1/\sqrt{2}\right) \tag{34}$$

for k sufficiently large. For $t \in \mathbb{N},\ 1 \le \tau \le n$, let

$$\alpha_{\tau,t} = \prod_{k=1}^{t}\left(1+\frac{1}{a_{\tau,k}}\right).$$

Theorem 12. *Assume that there exists a large $K_0 \in \mathbb{N}$ such that*

$$\begin{aligned} &|a_{\tau,k}| \ge 3 \qquad (k \ge K_0, 1 \le \tau \le n), \quad \textit{and} \\ &\frac{|a_{\tau,k}|\left(|a_{\tau,k}|-1\right)}{|a_{\tau,k}|-2}\sum_{\sigma=1}^{\tau-1}\frac{1}{|a_{\sigma,k}|-1} \to 0 \qquad (2 \le \tau \le n,\ k \to \infty). \end{aligned} \tag{35}$$

If there exists $g : \mathbb{N} \to \mathbb{R}^+$ with $g(k) \to \infty \ \ (k \to \infty)$ such that

$$\frac{|a_{\tau,k+1}|-1}{|\alpha_{\tau,k}|} \ge g(k)\exp\left([\mathbb{Q}\left(\alpha_{1,k},\ldots,\alpha_{\tau,k}\right):\mathbb{Q}]\sum_{\sigma=1}^{\tau}\frac{s\left(\alpha_{\sigma,k}\right)}{\partial\left(\alpha_{\sigma,k}\right)}\right) \qquad (1 \le \tau \le n), \tag{36}$$

then $1, \beta_1, \beta_2, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$.

Proof. Using (34) and (35), we get

$$|a_{\tau,t+1}| \geq 4, \quad |a_{\tau,t+2}| \geq 12 \qquad (t \geq K_0,\ 1 \leq \tau \leq n),$$

and so

$$|a_{\tau,t+1}| \geq |a_{\tau,t}|^2 \qquad (t \geq K_0 + 2). \tag{37}$$

For $1 \leq \tau \leq n$ and $t \geq K_0 + 2$, using Lemma 2(b), (37) and putting $\delta := |1/a_{\tau,t+1}|$, we have

$$\left|\frac{1}{\alpha_{\tau,t}}\right| |\beta_\tau - \alpha_{\tau,t}| = \left|\prod_{k \geq t+1}\left(1 + \frac{1}{a_{\tau,k}}\right) - 1\right| \tag{38}$$

$$= \left|\left(\frac{1}{a_{\tau,t+1}} + \frac{1}{a_{\tau,t+2}} + \frac{1}{a_{\tau,t+3}} + \cdots\right) + \left(\frac{1}{a_{\tau,t+1}a_{\tau,t+2}} + \frac{1}{a_{\tau,t+1}a_{\tau,t+3}} + \cdots\right) + \cdots\right|$$

$$\leq \left(\delta + \delta^2 + \delta^{2^2} + \cdots\right) + \left(\delta \cdot \delta^2 + \delta \cdot \delta^{2^2} + \cdots\right) + \cdots$$

$$= \frac{\delta}{1 - \delta} = \frac{1}{|a_{\tau,t+1}| - 1}. \tag{39}$$

Applying (36), we get

$$|\beta_\tau - \alpha_{\tau,t}| \leq \frac{|\alpha_{\tau,t}|}{|a_{\tau,t+1}| - 1} \leq \frac{\exp\left(-[\mathbb{Q}\left(\alpha_{1,t}, \ldots, \alpha_{\tau,t}\right) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} s\left(\alpha_{\sigma,t}\right)/\partial\left(\alpha_{\sigma,t}\right)\right)}{g\left(t\right)}. \tag{40}$$

In another direction, for $1 \leq \tau \leq n,\ t \geq K_0 + 2$, using (39), we get

$$\left|\prod_{k \geq t+1}\left(1 + \frac{1}{a_{\tau,k}}\right)\right| \geq 1 - \left|\prod_{k \geq t+1}\left(1 + \frac{1}{a_{\tau,k}}\right) - 1\right| \geq 1 - \frac{1}{|a_{\tau,t+1}| - 1} = \frac{|a_{\tau,t+1}| - 2}{|a_{\tau,t+1}| - 1},$$

and so

$$|\alpha_{\tau,t}| = \frac{|\beta_\tau|}{\left|\prod_{k \geq t+1}\left(1 + 1/a_{\tau,k}\right)\right|} \leq \frac{|\beta_\tau|(|a_{\tau,t+1}| - 1)}{|a_{\tau,t+1}| - 2}.$$

Combining this last inequality with (38), for sufficiently large t, we have

$$|\beta_\tau - \alpha_{\tau,t}| \leq \frac{|\alpha_{\tau,t}|}{|a_{\tau,t+1}| - 1} \leq \frac{|\beta_\tau|}{|a_{\tau,t+1}| - 2} < \frac{|\beta_\tau|}{2}, \tag{41}$$

and so

$$\frac{|\beta_\tau|}{2} < |\alpha_{\tau,t}| < \frac{3|\beta_\tau|}{2} \quad (1 \leq \tau \leq n). \tag{42}$$

For $t \geq K_0 + 2$, first as in (39), using (37) with $\delta = |1/a_{\tau,t+1}|$, we get

$$\begin{aligned}
&\left| \prod_{k \geq t+1} \left(1 + \frac{1}{a_{\tau,k}}\right) - 1 \right| \\
&= \left| \left(\frac{1}{a_{\tau,t+1}} + \frac{1}{a_{\tau,t+2}} + \frac{1}{a_{\tau,t+3}} + \cdots \right) + \left(\frac{1}{a_{\tau,t+1}a_{\tau,t+2}} + \frac{1}{a_{\tau,t+1}a_{\tau,t+3}} + \cdots \right) + \cdots \right| \\
&\geq \left| \frac{1}{a_{\tau,t+1}} \right| \\
&\quad - \left\{ \left(\left| \frac{1}{a_{\tau,t+2}} \right| + \left| \frac{1}{a_{\tau,t+3}} \right| + \cdots \right) + \left(\left| \frac{1}{a_{\tau,t+1}a_{\tau,t+2}} \right| + \left| \frac{1}{a_{\tau,t+1}a_{\tau,t+3}} \right| + \cdots \right) + \cdots \right\} \\
&\geq \delta - \left\{ \left(\delta^2 + \delta^{2^2} + \cdots \right) + \left(\delta \cdot \delta^2 + \delta \cdot \delta^{2^2} + \cdots \right) \right\} = \delta - \left\{ \frac{\delta}{1-\delta} - \delta \right\} \\
&= \frac{|a_{\tau,t+1}| - 2}{|a_{\tau,t+1}|(|a_{\tau,t+1}| - 1)} \qquad (1 \leq \tau \leq n). \qquad (43)
\end{aligned}$$

From (39), (42), (43) and (41), we arrive at

$$\begin{aligned}
&\frac{\sum_{\sigma=1}^{\tau-1} |\beta_\sigma - \alpha_{\sigma,t}|}{|\beta_\tau - \alpha_{\tau,t}|} = \frac{|\alpha_{1,t}| \left| \frac{\beta_1}{\alpha_{1,t}} - 1 \right| + \cdots + |\alpha_{\tau-1,t}| \left| \frac{\beta_{\tau-1}}{\alpha_{\tau-1,t}} - 1 \right|}{|\alpha_{\tau,t}| \left| \frac{\beta_\tau}{\alpha_{\tau,t}} - 1 \right|} \\
&= \frac{|\alpha_{1,t}| \left| \prod_{k \geq t+1} \left(1 + \frac{1}{a_{1,k}}\right) - 1 \right| + \cdots + |\alpha_{\tau-1,t}| \left| \prod_{k \geq t+1} \left(1 + \frac{1}{a_{\tau-1,k}}\right) - 1 \right|}{|\alpha_{\tau,t}| \left| \prod_{k \geq t+1} \left(1 + \frac{1}{a_{\tau,k}}\right) - 1 \right|} \\
&< \frac{|\alpha_{1,t}| \frac{1}{|a_{1,t+1}|-1} + \cdots + |\alpha_{\tau-1,t}| \frac{1}{|a_{\tau-1,t+1}|-1}}{|\alpha_{\tau,t}| \frac{|a_{\tau,t+1}|-2}{|a_{\tau,t+1}|(|a_{\tau,t+1}|-1)}} \\
&< \frac{3|\beta_1| \frac{1}{|a_{1,t+1}|-1} + \cdots + 3|\beta_{\tau-1}| \frac{1}{|a_{\tau-1,t+1}|-1}}{|\beta_\tau| \frac{|a_{\tau,t+1}|-2}{|a_{\tau,t+1}|(|a_{\tau,t+1}|-1)}} \\
&< \frac{3M \sum_{\sigma=1}^{\tau-1} \frac{1}{|a_{\sigma,t+1}|-1}}{|\beta_\tau| \frac{|a_{\tau,t+1}|-2}{|a_{\tau,t+1}|(|a_{\tau,t+1}|-1)}} \qquad (M := \max\{|\beta_1|, |\beta_2|, \ldots, |\beta_{\tau-1}|\}) \\
&= \frac{3M}{|\beta_\tau|} \cdot \frac{|a_{\tau,t+1}|(|a_{\tau,t+1}|-1)}{|a_{\tau,t+1}|-2} \sum_{\sigma=1}^{\tau-1} \frac{1}{|a_{\sigma,t+1}|-1} \to 0 \qquad (t \to \infty). \qquad (44)
\end{aligned}$$

Noting (40) and (44), Theorem 2 then implies that $1, \beta_1, \beta_2, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$. □

As for algebraic independence, we have the following result.

Theorem 13. *Assume that there exists a large* $K_0 \in \mathbb{N}$ *such that*

$$|a_{\tau,k}| \geq 3 \qquad (k \geq K_0, 1 \leq \tau \leq n),$$
$$\frac{|a_{\tau,k}|\left(|a_{\tau,k}|-1\right)}{|a_{\tau,k}|-2} \sum_{\sigma=1}^{\tau-1} \frac{1}{|a_{\sigma,k}|-1} \to 0 \qquad (2 \leq \tau \leq n,\ k \to \infty).$$

If there exists $g : \mathbb{N} \to \mathbb{R}^+$ *with* $g(k) \to \infty \ \ (k \to \infty)$ *such that*

$$\frac{|a_{\tau,k+1}|-1}{|\alpha_{\tau,k}|} \geq \exp\left(g(k) [\mathbb{Q}\left(\alpha_{1,k}, \ldots, \alpha_{\tau,k}\right) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s\left(\alpha_{\sigma,k}\right)}{\partial\left(\alpha_{\sigma,k}\right)} \right) \qquad (1 \leq \tau \leq n),$$

then $\beta_1, \beta_2, \ldots, \beta_n$ *are algebraically independent over* $\mathbb{Q}$.

Sketch of proof. Proceeding as in the proof of Theorem 12, we get, similar to (40) and (44),

$$|\beta_\tau - \alpha_{\tau,t}| \leq \frac{|\alpha_{\tau,t}|}{|a_{\tau,t+1}|-1} \leq \exp\left(-g(t) [\mathbb{Q}\left(\alpha_{1,t}, \ldots, \alpha_{\tau,t}\right) : \mathbb{Q}] \sum_{\sigma=1}^{\tau} \frac{s\left(\alpha_{\sigma,t}\right)}{\partial\left(\alpha_{\sigma,t}\right)} \right),$$

and

$$\frac{\sum_{\sigma=1}^{\tau-1} |\beta_\sigma - \alpha_{\sigma,t}|}{|\beta_\tau - \alpha_{\tau,t}|} < \frac{3M}{|\beta_\tau|} \cdot \frac{|a_{\tau,t+1}|\left(|a_{\tau,t+1}|-1\right)}{|a_{\tau,t+1}|-2} \sum_{\sigma=1}^{\tau-1} \frac{1}{|a_{\sigma,t+1}|-1} \to 0 \qquad (t \to \infty),$$

where $M := \max\left\{|\beta_1|, |\beta_2|, \ldots, |\beta_{\tau-1}|\right\}$). The desired result follows by invoking Theorem 1. □

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