APPROACHING CUSICK'S CONJECTURE ON THE SUM-OF-DIGITS FUNCTION

Lukas Spiegelhofer[1]
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
lukas.spiegelhofer@tuwien.ac.at

Received: 4/20/19, Accepted: 10/27/19, Published: 11/4/19

Abstract

Cusick's conjecture on the binary sum of digits $s(n)$ of a nonnegative integer n states the following: for all nonnegative integers t we have

$$c_t = \lim_{N\to\infty} \frac{1}{N} |\{n < N : s(n+t) \geq s(n)\}| > 1/2.$$

We prove that for given $\varepsilon > 0$ we have $c_t + c_{t'} > 1 - \varepsilon$ if the binary expansion of t contains enough blocks of consecutive 1s (depending on ε), where $t' = 3 \cdot 2^\lambda - t$ and λ is chosen such that $2^\lambda \leq t < 2^{\lambda+1}$.

1. Introduction

The binary sum-of-digits function s is defined by

$$s\left(\varepsilon_\nu 2^\nu + \cdots + \varepsilon_0 2^0\right) = \varepsilon_\nu + \cdots + \varepsilon_0$$

for all digits $\varepsilon_i \in \{0, 1\}$. It is an elementary yet difficult problem to consider the behavior of s under addition of a constant. T. W. Cusick (private communication, 2011) proposed the following conjecture.

Conjecture 1. For a nonnegative integer t, define

$$c_t = \operatorname{dens}\{n \geq 0 : s(n+t) \geq s(n)\},$$

where $\operatorname{dens} A$ denotes the asymptotic density of a set $A \subseteq \mathbb{N}$. Then $c_t > 1/2$.

[1]The author acknowledges support by the joint project MuDeRa between the Agence Nationale de la Recherche (ANR, France) and the Austrian Science Fund (FWF), project number I-1751-N26; and by Project F5502-N26 (FWF), which is a part of the Special Research Program "Quasi Monte Carlo methods: Theory and Applications".

We note that the set in question is in fact a finite union of arithmetic progressions, so there are no problems of convergence (see Bésineau [1]). This conjecture arose when Cusick was working on a related conjecture due to Tu and Deng [13, 14], which concerns binary addition modulo $2^k - 1$. Tu and Deng's conjecture is of interest since it allows constructing Boolean functions with desirable cryptographic properties. Partial results on the Tu–Deng conjecture have been obtained; see, for example, [2, 4, 9, 11]. Moreover, both Cusick's conjecture and the Tu–Deng conjecture have been proven *asymptotically*, the former by Drmota, Kauers, and the author [5], and the latter by Wallner and the author [12], but the full statements are still open. In particular, we wish to note that no bound of the form $c_t > a$ or $c_t < b$ for some $a > 0$ or $b < 1$, valid for all $t \geq 0$, is known! In this paper we concentrate on Cusick's conjecture. The abovementioned result by Drmota, Kauers and the author [5] is the following: we have $c_t > 1/2$ for almost all t in the sense of asymptotic density, that is,

$$\operatorname{dens}\{t : c_t > 1/2\} = 1.$$

However, this theorem does not tell us anything about the structure of such a set of "good" t; it does not provide a statement allowing to extract many examples of integers t satisfying Cusick's conjecture.

The present paper constitutes a step in this direction. More precisely, since the original conjecture is elusive and too hard, we consider a simplified version. In order to formulate this easier statement and our main theorem, we define $t' = 3 \cdot 2^\lambda - t$, where $2^\lambda \leq t < 2^{\lambda+1}$.

Conjecture 2 (Cusick, simplified). For all $t \geq 0$, we have $c_t + c_{t'} > 1$. In other words, at least one out of t or t' satisfies Cusick's conjecture.

Our main theorem is an approximation to this simplified conjecture.

Theorem 1. *Assume that $\varepsilon > 0$. There exists a constant $C = C(\varepsilon)$ such that*

$$c_t + c_{t'} > 1 - \varepsilon,$$

if the binary expansion of t contains at least C blocks of consecutive `1`*s.*

We note that an admissible value of $C(\varepsilon)$ can be made completely explicit. Note also that this theorem gives a lower bound $c_t > 1/2 - \varepsilon$ for many values of t: in fact, the number of integers $0 \leq t < T$ having less than C blocks of consecutive `1`s in its binary expansion is bounded by T^η for some $\eta < 1$. The important point is the fact that obtain a very efficient method of finding many t such that $c_t > 1/2 - \varepsilon$: we only have to start with an integer having sufficiently many blocks of `1`s and possibly invert the digits between the first and the last `1` (corresponding to $t \mapsto t'$) in order to arrive at such a t.

The remainder of this paper is dedicated to the proof of Theorem 1. Throughout the proof, we use the common notations $\mathrm{e}(x) = \exp(2\pi i x)$ and $\|x\| = \min_{k\in\mathbb{Z}} |x - k|$.

2. Proof of the Main Theorem

For $t \geq 0$ and $k \in \mathbb{Z}$, we define the densities

$$\delta(k,t) = \operatorname{dens}\{n \in \mathbb{N} : s(n+t) - s(n) = k\}.$$

Again, these densities exist [1]. These values satisfy the recurrence [5]

$$\begin{aligned}
\delta(k,1) &= \begin{cases} 2^{k-2}, & k \leq 1; \\ 0 & \text{otherwise;} \end{cases} \\
\delta(k,2t) &= \delta(k,t); \\
\delta(k,2t+1) &= \frac{1}{2}\delta(k-1,t) + \frac{1}{2}\delta(k+1,t+1).
\end{aligned}$$

Moreover, we define a simplified array φ by modifying the start vector:

$$\begin{aligned}
\varphi(k,1) &= \begin{cases} 1, & k = 0; \\ 0 & \text{otherwise;} \end{cases} \\
\varphi(k,2t) &= \varphi(k,t); \\
\varphi(k,2t+1) &= \frac{1}{2}\varphi(k-1,t) + \frac{1}{2}\varphi(k+1,t+1).
\end{aligned}$$

The reason for the introduction of this array, and in fact also the reason for the definition of t', is the following symmetry property [5]: we have

$$\varphi(k,t) = \varphi(-k,t').$$

Moreover, by linearity we have

$$\delta(k,t) = \sum_{\ell+s=k} \varphi(\ell,t)\delta(s,1) = \sum_{\ell \geq 0} \varphi(k+1-\ell)2^{-\ell-1}.$$

We obtain

$$\begin{aligned}
c_t + c_{t'} &= \sum_{\ell \geq -1} \big(\varphi(\ell,t) + \varphi(-\ell,t)\big)\left(1 - 2^{-\ell-2}\right) \\
&= \frac{3}{2}\varphi(0,t) + \frac{11}{8}\varphi(1,t) + \frac{11}{8}\varphi(-1,t) \\
&\quad + \sum_{\ell \geq 2} \left(1 - 2^{-\ell-2}\right)\big(\varphi(\ell,t) + \varphi(-\ell,t)\big).
\end{aligned} \tag{1}$$

Remark 1. From the above identity we immediately obtain $c_t + c_{t'} \geq 15/16$ by using the identity $\sum_{k\in\mathbb{Z}} \varphi(k,t) = 1$; it is obvious to suspect that the maximum of $\varphi(k,t)$ is attained for $|k| \leq 1$. This would yield $c_t > 1/2$ or $c_{t'} > 1/2$ and thus settle the simplified form of Cusick's conjecture. However, this assumption is wrong: for

$t = 149$ the maximum is attained at the position $k = 2$. Still, there is hope: in fact, it is sufficient to prove that

$$\varphi(-1,t) + \varphi(0,t) + \varphi(1,t) \geq \varphi(k,t) \quad \text{for all } k \text{ such that } |k| \geq 2. \tag{2}$$

This can be seen as follows: under this hypothesis we have

$$\begin{aligned} c_t + c_{t'} &\geq \sum_{|\ell|\geq 2} \varphi(\ell,t) - \sum_{|\ell|\geq 2} 2^{-|\ell|-2} \max_{|\ell|\geq 2} \varphi(\ell,t) \\ &\quad + \varphi(-1,t) + \varphi(0,t) + \varphi(1,t) + \frac{3}{8}\left(\varphi(-1,t) + \varphi(0,t) + \varphi(1,t)\right) \\ &\geq 1 + \max_{|\ell|\geq 2} \varphi(\ell,t) \left(\frac{3}{8} - \sum_{|\ell|\geq 2} 2^{-|\ell|-2} \right) > 1. \end{aligned}$$

It looks like a simple thing to prove (2) by induction on the length of the binary expansion of t, using the recurrence for φ; however, so far we did not succeed.

We are going to work with the following expression, where $\vartheta \in \mathbb{R}$.

$$\omega_t(\vartheta) = \sum_{k\in\mathbb{Z}} \varphi(k,t)\, \mathrm{e}(k\vartheta). \tag{3}$$

Obviously, this sum is absolutely convergent.

Our strategy is to give an upper bound for $|\omega_t(\vartheta)|$, where $\vartheta = j/m$. This will give us some information on the behavior of $k \mapsto \varphi(k,t)$ on residue classes $b + m\mathbb{Z}$. For each b, we take the least weight appearing in (1) for $\ell \in b + m\mathbb{Z}$ and multiply it with the sum $\sum_{\ell\in b+m\mathbb{Z}} \varphi(\ell,t)$; afterwards, we sum the contributions of the different residue classes, using the argument on the smallness of $\omega_t(j/m)$.

Lemma 1. *We have the following recurrence for the values $\omega_t(\vartheta)$:*

$$\begin{aligned} \omega_1(\vartheta) &= 1, \\ \omega_{2t}(\vartheta) &= \omega_t(\vartheta), \\ \omega_{2t+1} &= \frac{\mathrm{e}(\vartheta)}{2}\omega_t(\vartheta) + \frac{\mathrm{e}(-\vartheta)}{2}\omega_{t+1}(\vartheta). \end{aligned}$$

The proof is straightforward, using the recurrence for φ. This new recurrence can be written using 2×2-matrices [10]: define

$$A_0 = \begin{pmatrix} 1 & 0 \\ \mathrm{e}(\vartheta)/2 & \mathrm{e}(-\vartheta)/2 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} \mathrm{e}(\vartheta)/2 & \mathrm{e}(-\vartheta)/2 \\ 0 & 1 \end{pmatrix}.$$

If $t = (\varepsilon_\nu \cdots \varepsilon_0)_2$ is the binary representation of $t \geq 1$, we have

$$\omega_t(\vartheta) = \begin{pmatrix} 1 & 0 \end{pmatrix} A_{\varepsilon_0} \cdots A_{\varepsilon_{\nu-1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Lemma 2. *Assume that the binary expansion of $t \geq 1$ contains at least $2M+1$ blocks of consecutive* `1`*s. Then*

$$|\omega_t(\vartheta)| \leq \left(1 - \frac{1}{2}\|\vartheta\|^2\right)^M.$$

Proof. There are at least M positions $j \in \{0, \dots, \nu-3\}$, having distance ≥ 3 from each other, such that $(\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}) = (\mathtt{100})$ or $(\varepsilon_j, \varepsilon_{j+1}, \varepsilon_{j+2}) = (\mathtt{101})$. We show that for such a block we gain a factor of $1 - \|\vartheta\|^2/2$. Using the row-sum norm $\|\cdot\|_\infty$ for matrices, which is derived from the maximum norm for vectors and which is sub-multiplicative, it is sufficient to prove that

$$\|A_1A_0A_0\|_\infty \leq 1 - \frac{1}{2}\|\vartheta\|^2 \quad \text{and} \quad \|A_1A_0A_1\|_\infty \leq 1 - \frac{1}{2}\|\vartheta\|^2.$$

After a short calculation we obtain

$$A_1A_0A_0 = \begin{pmatrix} \frac{\mathrm{e}(\vartheta)}{2} + \frac{1}{4} + \frac{\mathrm{e}(-\vartheta)}{8} & \frac{\mathrm{e}(-3\vartheta)}{8} \\ \frac{\mathrm{e}(\vartheta)}{2} + \frac{1}{4} & \frac{\mathrm{e}(-2\vartheta)}{4} \end{pmatrix} \quad \text{and}$$

$$A_1A_0A_1 = \begin{pmatrix} \frac{\mathrm{e}(2\vartheta)}{4} + \frac{\mathrm{e}(\vartheta)}{8} & \frac{1}{4} + \frac{\mathrm{e}(-2\vartheta)}{4} + \frac{\mathrm{e}(-\vartheta)}{8} \\ \frac{\mathrm{e}(2\vartheta)}{4} & \frac{\mathrm{e}(-\vartheta)}{2} + \frac{1}{4} \end{pmatrix},$$

and we see that the row-sum norm is strictly below 1 as soon as $\vartheta \notin \mathbb{Z}$. More precisely, we use [3, Lemme 3], stating that

$$\left|\frac{1}{q}(1 + z_1 + \cdots + z_{q-1})\right| \leq 1 - \frac{1}{2q} \max_{1 \leq j < q} (1 - \Re z_j)$$

for $|z_1|, \dots, |z_{q-1}| \leq 1$. For example, the left upper entry of $A_1A_0A_0$ can be bounded as follows:

$$\left|\frac{\mathrm{e}(\vartheta)}{2} + \frac{1}{4} + \frac{\mathrm{e}(-\vartheta)}{8}\right| = \frac{7}{8}\left|\frac{1}{7}\left(1 + 1 + \mathrm{e}(-\vartheta) + 4 \cdot \mathrm{e}(\vartheta)\right)\right| \leq \frac{7}{8}\left(1 - \frac{1}{14}\left(1 - \Re\,\mathrm{e}(\vartheta)\right)\right).$$

Analogously, the entry below, and also the right lower entry of $A_1A_0A_1$, may be bounded by

$$\frac{3}{4}\left|\frac{1}{3}\left(1 + 2 \cdot \mathrm{e}(\vartheta)\right)\right| \leq \frac{3}{4}\left(1 - \frac{1}{6}\left(1 - \Re\,\mathrm{e}(\vartheta)\right)\right),$$

while the right upper entry of this second matrix can be bounded by

$$\frac{5}{8}\left(1 - \frac{1}{10}\left(1 - \Re\,\mathrm{e}(\vartheta)\right)\right).$$

It follows that $\|B\|_\infty \leq 1 - \frac{1}{16}\left(1 - \Re\,\mathrm{e}(\vartheta)\right)$ for $B \in \{A_1A_0A_0, A_1A_0A_1\}$. Moreover, we have the elementary inequality $\Re\,\mathrm{e}(\vartheta) = \cos(2\pi\vartheta) \leq 1 - 8\|\vartheta\|^2$, so that $\|B\| \leq 1 - \frac{1}{2}\|\vartheta\|^2$. This proves the lemma. □

We are interested in the quantity

$$\psi(b,m,t)=\sum_{\ell\in b+m\mathbb{Z}}\varphi(\ell,t).$$

Moreover, we define

$$\tilde{a}_\ell=\begin{cases}3/2, & \text{if } \ell=0;\\ 11/8, & \text{if } |\ell|=1;\\ 1-2^{-|\ell|-2}, & \text{if } |\ell|\geq 2.\end{cases}$$

Clearly $\tilde{a}_\ell\geq a_\ell\coloneqq 1-2^{-|\ell|-2}$ for all $\ell\in\mathbb{Z}$. By (1), we obtain

$$c_t+c_{t'}\geq\sum_{0\leq b<m}\psi(b,m,t)\min_{\ell\in b+m\mathbb{Z}}a_\ell. \tag{4}$$

By monotonicity and symmetry of a_ℓ, we obtain

$$\min_{\ell\in b+m\mathbb{Z}}a_\ell=\begin{cases}1-2^{-b-2} & \text{if } 0\leq b<m/2\\ 1-2^{-(m-b)-2} & \text{if } m/2\leq b<m.\end{cases} \tag{5}$$

Moreover,

$$\begin{aligned}\psi(b,m,t)=\sum_{\ell\in b+m\mathbb{Z}}\varphi(\ell,t)&=\sum_{k\in\mathbb{Z}}\varphi(k,t)\frac{1}{m}\sum_{0\leq j<m}\mathrm{e}\left(j\frac{k-b}{m}\right)\\&=\frac{1}{m}\sum_{0\leq j<m}\mathrm{e}\left(-\frac{jb}{m}\right)\omega_t(j/m).\end{aligned}$$

By Lemma 2 it follows (using the abbreviation $x\pm y$ to stand for $x+O(y)$ with an implied constant 1) that

$$\begin{aligned}\psi(b,m,t)&=\frac{1}{m}\pm\max_{1\leq j<m}|\omega_t(j/m)|\\&=\frac{1}{m}\pm\left(1-1/(2m^2)\right)^M=\frac{1}{m}\pm e^{-M/(2m^2)},\end{aligned} \tag{6}$$

if t has at least $2M+1$ blocks of consecutive 1s in its binary expansion.

From (4) and (6) it follows that

$$c_t+c_{t'}\geq\frac{1}{m}\sum_{0\leq b<m}\min_{\ell\in b+m\mathbb{Z}}a_\ell\pm me^{-M/(2m^2)}.$$

It remains to consider mean values of the quantity in (5). It is obvious that this mean value converges to 1 for $m\to\infty$; quantitatively, we get for all $N\leq m$

$$\sum_{0\leq b<m}\min_{\ell\in b+m\mathbb{Z}}a_\ell\geq\sum_{N\leq b<m-N}\min_{\ell\in b+m\mathbb{Z}}a_\ell\geq(m-2N)\left(1-2^{-N-2}\right)$$

$$\geq m\left(1-2^{-N-2}\right)-2N.$$

We obtain

$$c_t + c_{t'} \geq 1 - 2^{-N-2} - \frac{2N}{m} - me^{-M/(2m^2)}.$$

Let $\varepsilon \in (0,1)$ be given. We aim to define a bound C as in the statement of the theorem. Let $N = \lfloor -\log_2 \varepsilon \rfloor + 1$. Then clearly $2^{-N-2} < \varepsilon/3$. Moreover, choose $m = \lfloor 6N/\varepsilon \rfloor + 1$, then $2N/m < \varepsilon/3$. Finally, let $M = \lfloor -2m^2 \log(\varepsilon/(3m)) \rfloor + 1$, so that $me^{-M/(2m^2)} < \varepsilon/3$. The choice $C = 2M + 1$ satisfies the claim of the theorem. Asymptotically, an admissible choice for C is given by $\alpha(\log \varepsilon)^3/\varepsilon^2$ for some constant $\alpha < 0$ that can be given explicitly.

Remark 2. It would be desirable to improve our theorem in one or more of the following three aspects:

1. Prove statements on individual c_t instead of the combined quantity $c_t + c_{t'}$.
2. Eliminate the quantity ε appearing in our lower bound.
3. Prove statements for *all* $t \geq 0$ instead of demanding the existence of many blocks of 1s in the binary expansion of t.

Of course, Cusick's original conjecture corresponds to (1)$\wedge$(2)$\wedge$(3), while the simplified form given above corresponds to (2)$\wedge$(3).

We expect that progress on (1) can be made by appealing to the study of moments of the probability distribution defined by $k \mapsto \delta(k,t)$ initiated by Emme and Prikhod'ko [8] and pursued by Emme and Hubert [7, 6]. This will be the subject of a future research paper.

References

[1] J. Bésineau, Indépendance statistique d'ensembles liés à la fonction "somme des chiffres", *Acta Arith.* **20** (1972), 401–416.

[2] T. W. Cusick, Y. Li, and P. Stănică, On a combinatorial conjecture, *Integers* **11** (2011), #A17.

[3] H. Delange, Sur les fonctions q-additives ou q -multiplicatives, *Acta Arith.* **21** (1972), 285–298 (errata insert).

[4] G. Deng and P. Yuan, On a combinatorial conjecture of Tu and Deng, *Integers* **12** (2012), #A48.

[5] M. Drmota, M. Kauers, and L. Spiegelhofer, On a Conjecture of Cusick Concerning the Sum of Digits of n and $n+t$, *SIAM J. Discrete Math.* **30(2)** (2016), 621–649.

[6] J. Emme and P. Hubert, Normal distribution of correlation measures of binary sum-of-digits functions, preprint, `http://arxiv.org/abs/1810.11234`.

[7] J. Emme and P. Hubert, Central limit theorem for probability measures defined by sum-of-digits function in base 2, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **19(2)** (2019), 757–780.

[8] J. Emme and A. Prikhod'ko, On the Asymptotic Behavior of Density of Sets Defined by Sum-of-digits Function in Base 2, *Integers* **17** (2017), #A58.

[9] J.-P. Flori, H. Randriambololona, G. Cohen, and S. Mesnager, On a Conjecture about Binary Strings Distribution, *Sequences and Their Applications - SETA 2010 Springer Berlin/Heidelberg (Ed.)* (2010), 346–358.

[10] J. F. Morgenbesser and L. Spiegelhofer, A reverse order property of correlation measures of the sum-of-digits function, *Integers* **12** (2012), #A47.

[11] S. Qarboua, J. Schrek, and C. Fontaine, New results about Tu-Deng's conjecture, *2016 IEEE International Symposium on Information Theory (ISIT)* (2016), 485–489.

[12] L. Spiegelhofer and M. Wallner, The Tu–Deng conjecture holds almost surely, *Electron. J. Combin.* **26(1)**, P1.28.

[13] Z. Tu and Y. Deng, A conjecture about binary strings and its applications on constructing Boolean functions with optimal algebraic immunity, *Des. Codes Cryptogr.* **60(1)** (2011), 1–14.

[14] Z. Tu and Y. Deng, Boolean functions optimizing most of the cryptographic criteria, *Discrete Appl. Math.* **160(4-5)** (2012), 427–435.