ON GENERALIZED DERANGEMENTS AND SOME ORTHOGONAL POLYNOMIALS

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Abstract

We make use of orthogonal polynomials to give new identities for a recently defined class of combinatorial numbers, the r-derangements. In the second part, we carry out the complex analysis of the derangement function. In the third part of the paper we present a combinatorial generalization of the r-derangement numbers.

1. Introduction

The derangement number $D(n)$ denotes the number of fixed point-free permutations (FPF for short) on n letters. In a recent paper [21], the authors defined a new class of derangements: an *r-derangement* is an FPF-permutation such that the first r elements are restricted to be in different cycles. For example,

$$(172)(35)(468)$$

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is not a 2-derangement (1 and 2 share a common cycle), while

$$(197)(2845)(36)$$

is a 2-derangement and even a 3-derangement (but not a 4-derangement). The number of r-derangements on $n+r$ letters is denoted by $D_r(n)$.

First we are going to study the r-derangement numbers $D_r(n)$ by their relation to some orthogonal polynomials. Then we define the complex function $D(z)$, extending $D(n)$ to complex numbers, and study its properties. In the final part of the paper we look at a possible generalization of the r-derangement numbers. This generalization, among other things, leads to the generalization of the recurrence

$$D(n) = nD(n-1) + (-1)^n$$

for the r-derangements (see (17)).

Our motivation comes from the sources [1, 7, 8, 9]. In these papers, M. Ismail et al. applied the theory of orthogonal polynomials to prove identities and develop asymptotic approximations with respect to certain generalized derangement numbers. Those tools can also be applied for the r-derangement numbers, which were defined later.

2. Identities for $D_r(n)$ Related to the Charlier Polynomials

The r-derangements are intimately related to the Charlier polynomials [6], and thus we recall some of the properties of these polynomials.

The Charlier polynomials $C_n^{(a)}(y)$ can be defined by the following equivalent relations:

- (Generating function) $\sum_{n\geq 0} C_n^{(a)}(y)\frac{x^n}{n!} = e^{-ax}(1+x)^y$,
- (Explicit formula) $C_n^{(a)}(y) = \sum_{k=0}^{n}\binom{n}{k}\binom{y}{k}k!(-a)^{n-k}$,
- (Recursion) $C_{n+1}^{(a)}(y) = (y-n-a)C_n^{(a)}(y) - anC_{n-1}^{(a)}(y)$,
- (Hypergeometric representation) $C_n^{(a)}(y) = {}_2F_0\left[\begin{matrix}-n,\ -y\end{matrix};-\frac{1}{a}\right]$,
- (Forward shift) $C_n^{(a)}(y+1) - C_n^{(a)}(y) = nC_{n-1}^{(a)}(y)$ (with appropriate initial conditions).

We apply the Charlier polynomials now to deduce some simple closed form and recursive identities with respect to $D_r(n)$. Instead of giving the detailed symbolic deduction we give only the combinatorial proof wherever we have found one.

It is known [21, Theorem 3] that the exponential generating function of the r-derangement numbers is

$$\sum_{n\geq 0} D_r(n)\frac{x^n}{n!} = \frac{x^r e^{-x}}{(1-x)^{r+1}}. \tag{1}$$

The exponential generating function of the Charlier polynomials immediately gives the relation

$$D_r(n) = (-1)^{n-r}\, n^{\underline{r}}\, C_{n-r}^{(-1)}(-(r+1)). \tag{2}$$

Here $a^{\underline{b}} = a(a-1)\cdots(a-b+1)$ is the falling factorial symbol ($(a)^{\underline{0}} = 1$). We will also use the symbol $a^{\overline{b}} = a(a+1)\cdots(a+b-1)$ for the raising factorial, also known as Pochhammer symbol.

A consequence of this relation and the above explicit formula for the Charlier polynomials results in the next statement that we will prove by combinatorial arguments.

Theorem 1. *For any $r, n \geq 0$ we have*

$$D_r(n) = n^{\underline{r}} \sum_{k=0}^{n-r} \binom{n-r}{k} (-1)^k (r+1)^{\overline{n-r-k}}. \tag{3}$$

Proof. The number of those permutations in which the first r elements are in disjoint cycles and are not fixed points is, as it is easy to see, equal to

$$n^{\underline{r}}(r+1)^{\overline{n-r}}. \tag{4}$$

Indeed, to guarantee that the elements $1, 2, \ldots, r$ are not fixed points, we choose an element from n to go to the cycle of 1, then we choose another element from $n-1$ to go to the block of 2, and so on. There are $n^{\underline{r}}$ such selections. Denote these r elements by A. The remaining $n-r$ elements go, one by one, to one of the cycles of $1, 2, \ldots, r$ or they form a new permutation σ. In order to assure that σ has no fixed points, we subtract from (4) (the $k = 0$ term in the theorem's statement) those permutations which have one fixed point among the elements $\{r+1, r+2, \ldots, r+n\} \setminus A$. This is the $k = 1$ term. Then we continue by the inclusion-exclusion principle. □

The basic recursion we found for $D_r(n)$ in [21] is

$$D_r(n) = rD_{r-1}(n-1) + (n-1)D_r(n-2) + (n+r-1)D_r(n-1) \quad (n > 2, r > 0). \tag{5}$$

By using the above described recursion for the Charlier polynomials, it is possible to find a recursion for $D_r(n)$ which is simpler than (5). This is the content of the following theorem.

Theorem 2. *We have for $n \geq 2$ and $r \geq 0$*

$$(n-r)D_r(n) = n(n-1)D_r(n-1) + n(n-1)D_r(n-2).$$

Proof. The left-hand side can be interpreted as follows: we choose an r-derangement σ on $n+r$ elements, then we paint (with red, say) one of the n elements which is not the direct successor of the $1, 2, \ldots, r$ elements in the cycle decomposition of σ. For example, in the permutation

$$(159)(264)(387)$$

we can choose $9, 4, 7$ but we are not permitted to choose $5, 6, 8$. There are $n + r - r - r = n - r$ elements to paint, and altogether we have $(n-r)D_r(n)$ choices – this is counted by the left-hand side. We now show that the same objects are counted by the right-hand side.

1) We can paint initially one of the elements, say i in the set $\{r+1, \ldots, n+r\}$ (n options), then we construct an r-derangement on the other $n+r-1$ elements in $D_r(n-1)$ ways. Now we insert i among the $n+r-1$ elements or at the end of the permutation, but i cannot go to the right-hand side of any of the $1, \ldots, r$ elements. We have $n+r-1-r = n-1$ places for i. Thus we have $n(n-1)D_r(n-1)$ options. This is the first term on the right-hand side.

2) It is possible to choose an r-derangement on $n+r-2$ elements in $D_r(n-2)$ ways if i and the other element form a 2-cycle. In order to be sure that i does not go to one of the $1, \ldots, r$ elements, we must choose from the set $\{r+1, \ldots, n+r\} \setminus \{i\}$ ($n-1$ possibilities). As the painted element i can be chosen in n ways, we have $n(n-1)D_r(n-2)$ possibilities, this is what the second term on the right-hand side counts.

It can be seen that we have covered all the possibilities during counting our r-derangements with i painted. □

Note that this theorem, in particular, gives the well known recursion for the classical derangements:

$$D(n) = (n-1)D(n-1) + (n-1)D(n-2).$$

Two results of the Charlier polynomials we have not used yet: the hypergeometric representation and the forward shift. These, together with (2), results in the following two statements.

Theorem 3. *For all $n \geq r \geq 0$ we have the following hypergeometric function representation for the r-derangement numbers:*

$$D_r(n) = (-1)^{n-r} n^{\underline{r}} \cdot {}_2F_0\left[\begin{matrix} -(n-r), \; r+1 \\ - \end{matrix} ; 1\right].$$

The forward difference results in the following recursion, which goes one step back in both n and r.

Theorem 4. *For all $n \geq r \geq 0$*

$$D_r(n) = n(D_r(n-1) + D_{r-1}(n-1)).$$

Proof. We give a combinatorial proof. To do this, we recall the standard form of a permutation: we shift the elements in each of the individual cycles so that the minimal elements (the cycle leaders) are the first, then we order the cycles in decreasing order with respect to the cycle leaders.

Then we have two options for the construction of an r-derangement on $n+r$ letters:

1) the first element 1 is in a cycle of length two. Then we choose an element from n to put it in this cycle by the side of 1, and the rest of the permutation is constructed $D_{r-1}(n-1)$ ways, resulting in the second term of the right-hand side.

2) the first element 1 is in a cycle of length greater than two. Then we construct an arbitrary permutation on $n-1+r$ elements. In any of these 1 will have at least one companion. We insert the remaining element (which can be one of the n) so 1 will have at least two companions in its cycle. This gives the first term on the right-hand side. □

To close this section we give another closed form representation for the r-derangement sequence. The Laguerre polynomials can be defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right).$$

Putting $x = 1-t$ and multiplying by t^r, we can transform this generating function into the exponential generating function (1) of the r-derangement numbers. By comparison of the coefficients, and recalling that

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!},$$

we arrive at the following statement.

Theorem 5. *For any $n, r \geq 0$*

$$D_r(n) = n! \sum_{k=0}^{n} \frac{(r+1)_{k-r}}{(k-r)!} \sum_{\ell=n-k}^{k-r} \binom{\ell}{n-k} \frac{(r-k)^{\overline{\ell}}}{\ell!(r+1)^{\overline{\ell}}} (-1)^{n-k}.$$

3. An Integral Formula and Some of Its Corollaries

We now give an integral representation formula for $D_r(n+r)$ which is a corollary of (3).

Corollary 1. *For any $n, r \geq 0$ we have*

$$D_r(n+r) = (-1)^n (n+r)^{\underline{r}} \int_0^\infty e^{-t} {}_2F_1\left[\begin{matrix} -n,\ r+1 \\ 1 \end{matrix}; t\right] \mathrm{d}t.$$

If $0 \leq n \leq r$, we additionally have

$$D_r(n+r) = (-1)^n (n+r)^{\underline{r}} \frac{(-r)^{\overline{n}}}{n!} \int_0^\infty e^{-t} {}_2F_1\left[\begin{matrix} -n,\ r+1 \\ -n+r+1 \end{matrix}; 1-t\right] \mathrm{d}t. \tag{6}$$

Here ${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}; t\right]$ is the Gaussian hypergeometric function.

Proof. It is not hard to see that (3) is equivalent to

$$D_r(n+r) = (-1)^n (n+r)^{\underline{r}} \sum_{k=0}^{n} \binom{n}{k} \binom{-(r+1)}{k} k!.$$

Substituting the standard integral $k! = \int_0^\infty e^{-t} t^k \mathrm{d}t$, we have

$$D_r(n+r) = (-1)^n (n+r)^{\underline{r}} \int_0^\infty e^{-t} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{-(r+1)}{k} t^k \right) \mathrm{d}t.$$

The inner sum can be shown to be equal to

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{-(r+1)}{k} t^k = {}_2F_1\left[\begin{matrix} -n,\ r+1 \\ 1 \end{matrix}; t\right],$$

and the result comes.

To see the validity if (6), we recall the known transformation formula

$${}_2F_1\left[\begin{matrix} -n,\ b \\ c \end{matrix}; t\right] = \frac{(c-b)^{\overline{n}}}{c^{\overline{n}}} {}_2F_1\left[\begin{matrix} -n,\ b \\ -n+b+c-1 \end{matrix}; 1-t\right].$$

This was proven combinatorially by Labelle and Yeh [15]. Setting $b = r+1$ and $c = 1$ we get (6) when $n \leq r$ (when $n > r$ the rising factorial $(-r)^{\overline{n}}$ is obviously zero, while the left-hand side is not). □

Note that when $r = 0$, the hypergeometric function encodes the binomial expansion:

$${}_2F_1\left[\begin{matrix} -n,\ 1 \\ 1 \end{matrix}; t\right] = {}_1F_0\left[\begin{matrix} -n \\ \ \end{matrix}; t\right] = (1-t)^n,$$

and we get a well known fact [1, p. 855]:

$$D_n = \int_0^\infty e^{-t}(t-1)^n \mathrm{d}t. \tag{7}$$

We can deduce a summation formula from (7):

$$\sum_{n=0}^{\infty} D_n a_n = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} a_n (t-1)^n \mathrm{d}t.$$

To make this summation work, a_n must be appropriate in the sense that the sum on the left-hand side converges, as well as the integral on the right-hand side so that the integral and summation can be interchanged.

For example, taking $a_n = 1/(n!)^2$, we get

$$\sum_{n=0}^{\infty} \frac{D_n}{n!^2} = \int_0^\infty e^{-t} I_0(2\sqrt{t-1})\mathrm{d}t,$$

where I_0 is the modified Bessel function of the first kind [3, 22]. More generally, for $m \geq 0$

$$\sum_{n=0}^{\infty} \frac{D_n}{n!(n+m)!} = \int_0^\infty \frac{e^{-t}}{(x-1)^{m/2}} I_m(2\sqrt{t-1})\mathrm{d}t.$$

4. The Derangement Function $D(z)$

A quick look at (7) reveals how the sequence $D(n)$ can be extended to complex arguments. Let us define the function

$$D(z) = \int_0^\infty e^{-t}(t-1)^z \mathrm{d}t \quad (\Re(z) > -1),$$

and outside of the region of validity of the integral by analytic extension via the recursion

$$D(z+1) = zD(z) + zD(z-1). \tag{8}$$

We must be careful with this definition, because at the negative integers $D(z)$ is not defined (see the below Theorem for the full statement).

Note that the $D(z)$ function is not completely new, by the integral transformation one can see that

$$D(z) = \int_0^\infty e^{-t}(t-1)^z \mathrm{d}t = \frac{1}{e}\int_{-1}^\infty e^{-t} t^z \mathrm{d}t = \frac{1}{e}\Gamma(z+1,-1),$$

where the last function is the complementary incomplete Gamma function

$$\Gamma(a,z) = \int_z^\infty e^{-t} t^{a-1} \mathrm{d}t. \tag{9}$$

The idea of studying this particular incomplete Gamma (or derangement) function came from the study of the related left factorial function

$$K(n)=\sum_{i=0}^{n-1} i! \quad (n>0),$$

done by Kurepa [13, 14]. Kurepa extended $K(n)$ to the complex plane, and studied the analyticity of the $K(z)$ function. In the more recent paper [10], the main properties of this function are listed. Moreover, in [17] the alternating sum

$$A(n)=\sum_{i=1}^{n}(-1)^{n-i} i!,$$

as well as the function

$$A_1(z)=\sum_{n=0}^{\infty}(-1)^n \Gamma(z+1-n)$$

are studied.

Our main observations with respect to the function $D(z)$ are contained in the following two theorems.

Theorem 6. *The function $D(z)$ is an analytic function on the domain $z \in \mathbb{C} \setminus \{-1,-2,\dots\}$, and can be holomorphically extended to the whole complex plane $\mathbb{C}$ to a function $\overline{D}$ such that*

$$\overline{D}(z)=\begin{cases} D(z), & \text{if } z \neq -1,-2,\dots; \\ \overline{D}(k), & \text{if } z=k,\ k=-1,-2,\dots, \end{cases}$$

where

$$\overline{D}(-1)=-\frac{\mathrm{Ei}(1)}{e}-\frac{\pi}{e} i, \quad \overline{D}(-2)=-1+\frac{\mathrm{Ei}(1)}{e}+\frac{\pi}{e} i,$$

and

$$\overline{D}(k)=\frac{1}{k+1} \overline{D}(k+2)-\overline{D}(k+1) \quad (k=-3,-4,\dots).$$

Moreover, $z=1$ is a zero of $D(z)$ of multiplicity one. More concretely,

$$\lim_{z \to 1} \frac{D(z)}{z-1}=1-\frac{\mathrm{Ei}(1)}{e}-\frac{\pi}{e} i.$$

Here and above

$$\mathrm{Ei}(1)=-\int_{-1}^{\infty} \frac{e^{-t}}{t} \mathrm{d}t \approx 1.8951178$$

is the exponential integral function at $z=1$.

Proof. That the integral is convergent for $\Re(z) > -1$ is easy to see. Now let $z = -1 + is$ with some $s \in \mathbb{R}$. Then, by (8),

$$D(-1+is) = \frac{1}{is}D(is+1) - D(is).$$

The right-hand side is regular and defined by the integral representation, except the only point when $s = 0$. At this point the function $D(z)$ has a removable singularity, and the limit $\lim_{z\to-1+} D(z)$ will be denoted by $\overline{D}(-1)$. To prove this statement we calculate the following limit:

$$\begin{aligned}\overline{D}(-1) &:= \lim_{z\to-1+} D(z) = \lim_{z\to-1+}\left(\frac{D(z+2)}{z+1} - D(z+1)\right)\\ &= \lim_{z\to0+}\left(\frac{D(z+1)}{z} - D(z)\right) = \lim_{z\to0+}\frac{D(z+1)}{z} - 1. \qquad (10)\end{aligned}$$

The limit can be determined by making the expansion of $(t-1)^{z+1}$ around the point $z = 0$ behind the integral defining $D(z+1)$:

$$\begin{aligned}D(z+1) &= \int_0^\infty e^{-t}((t-1) + (t-1)\log(t-1)z + O(z^2))\mathrm{d}t\\ &= \int_0^\infty e^{-t}(t-1)\mathrm{d}t + z\int_0^\infty e^{-t}(t-1)\log(t-1)\mathrm{d}t + O(z^2).\end{aligned}$$

The first integral is zero, thus

$$\lim_{z\to0+}\frac{D(z+1)}{z} = \int_0^\infty e^{-t}(t-1)\log(t-1)\mathrm{d}t + O(z) = 1 - \frac{\mathrm{Ei}(1)}{e} - \frac{\pi}{e}i.$$

Substituting this into (10) we see that

$$\overline{D}(-1) = 1 - \frac{\mathrm{Ei}(1)}{e} - \frac{\pi}{e}i - 1 = -\frac{\mathrm{Ei}(1)}{e} - \frac{\pi}{e}i.$$

We define the holomorphic extension of $D(z)$ at this point by the value $\overline{D}(-1)$. In any vertical line the function is determined by the function values taken on the line shifted to the right by one and two. Thus $D(z)$ is well defined on all the complex plane except the points $z = -1, -2, \dots$. Here it has removable singularities, where the limits

$$\overline{D}(-k) := \lim_{z=-k} D(z)$$

exist. We have just seen that $\overline{D}(-1) = -\frac{\mathrm{Ei}(1)}{e} - \frac{\pi}{e}i$, and we can also see that

$$\overline{D}(-2) = -D(0) - \overline{D}(-1) = -1 + \frac{\mathrm{Ei}(1)}{e} + \frac{\pi}{e}i.$$

In general, by the recursion (8),

$$\overline{D}(k) = \frac{1}{k+1}\overline{D}(k+2) - \overline{D}(k+1) \quad (k = -3, -4, \dots).$$

Now we turn to the only zero we know of the $D(z)$ function, $D(1) = 0$, as there are no fixed point free partitions on one element. Now we prove that $z = 1$ is a zero of the complex $D(z)$ function of multiplicity one by showing that

$$\lim_{z\to 1} \frac{D(z)}{z-1} \neq 0.$$

Indeed,

$$\lim_{z\to 1} \frac{D(z)}{z-1} = \lim_{z\to 1} \frac{1}{z-1} \int_0^\infty e^{-t}(t-1)^z \mathrm{d}t = \lim_{z\to 1} \frac{1}{z-1} \int_0^\infty e^{-t}(t-1)\mathrm{d}t$$
$$+ \lim_{z\to 1} \frac{1}{z-1} \int_0^\infty e^{-t}(t-1)\log(t-1)(z-1)\mathrm{d}t + \lim_{z\to 1} \frac{1}{z-1} O(z-1)^2.$$

The integral $\int_0^\infty e^{-t}(t-1)\mathrm{d}t$ vanishes as well as the $\lim_{z\to 1} \frac{1}{z-1} O(z-1)^2$ term. Thus

$$\lim_{z\to 1} \frac{D(z)}{z-1} = \lim_{z\to 1} \frac{1}{z-1} \int_0^\infty e^{-t}(t-1)\log(t-1)(z-1)\mathrm{d}t$$
$$= \int_0^\infty e^{-t}(t-1)\log(t-1)\mathrm{d}t = 1 - \frac{\mathrm{Ei}(1)}{e} - \frac{\pi}{e}i \neq 0,$$

as we stated. □

The Taylor coefficients of $D(z)$ around the origin can be expressed by the derivatives of the incomplete Gamma function. The exact statement is in the following theorem.

Theorem 7. *Around the origin we have*

$$D(z) = \sum_{n=0}^\infty D^{(n)}(0)\frac{z^n}{n!},$$

where $D^{(0)}(0) = 1$*, and*

$$D^{(n)}(0) = (i\pi)^n + \frac{n}{e}\Gamma^{(n-1)}(0,-1) \quad (n \geq 1).$$

The expression $\Gamma^{(n-1)}(0,-1)$ *is meant to be the* $(n-1)$*-th derivative of* $\Gamma(z,-1)$ *at the point* $z = 0$*.*

Proof. It is obvious that $D^{(0)}(0) = D(0) = 1$. Moreover, by the definition of $D(z)$ we have that

$$D^{(n)}(0) = \int_0^\infty e^{-t} \log^n(t-1)\mathrm{d}t.$$

By partial integration, then by a change of the integration variable,

$$\int_0^\infty e^{-t} \log^n(t-1)\mathrm{d}t = [-e^{-t}\log^n(t-1)]_{t=0}^{t=\infty} + n\int_0^\infty e^{-t}\frac{\log^{n-1}(t-1)}{t-1}\mathrm{d}t$$
$$= (i\pi)^n + \frac{n}{e}\int_{-1}^\infty \frac{e^{-t}}{t}\log^{n-1} t\mathrm{d}t.$$

We only need to prove that

$$\int_{-1}^\infty \frac{e^{-t}}{t}\log^{n-1} t\mathrm{d}t = \Gamma^{(n-1)}(0,-1). \tag{11}$$

This, however, is obvious if we consider the integral as a sequence of the parameter n, and look for its generating function:

$$\sum_{n=1}^\infty \left(\int_{-1}^\infty \frac{e^{-t}}{t}\log^n t\mathrm{d}t\right)\frac{z^n}{n!} = \int_{-1}^\infty \frac{e^{-t}}{t}\sum_{n=1}^\infty \log^n t\frac{z^n}{n!}\mathrm{d}t = \int_{-1}^\infty \frac{e^{-t}}{t}t^z\mathrm{d}t = \Gamma(z,-1),$$

by (9). By the standard theory of generating functions we get (11). □

We have seen that $D(z)$ has a zero at $z = 1$. Does the function $D(z)$ have any other zeros? This question might be interesting to study.

5. A Generalization of the r-derangement Numbers

We introduce now a generalization which will permit us to deduce new identities for $D_r(n)$. Among others, we will be able to find the r-version of the famous recursion of Euler:

$$D(n) = nD(n-1) + (-1)^n. \tag{12}$$

For this result, see (17). A combinatorial proof for (12) can be found in [2].

We consider the restriction that every cycle contains at least m elements (cf. [4, 18, 20]). The number of permutations on n letters with the above restriction is denoted by $D_{\geq m}(n)$. It is clear that $D_{\geq 2}(n) = D(n)$. In this section we introduce a subclass of the FPF r-permutations stemming out from the above restriction. (We will always assume that $m \geq 2$.)

Definition 1. An (r,m)*-permutation* on $n+r$ letters is a permutation such that in its cycle decomposition the first r elements are in distinct cycles, with the restriction that every cycle contains at least m elements. The elements $\{1, 2, \ldots, r\}$ will be called *special elements*. A cycle is called *special* if it contains a special element.

For example,

$$(164)(239)(587\,10)$$

is a $(2,3)$-permutation.

Let us denote by $D_{r,\geq m}(n)$ the number of (r,m)-permutations on $n+r$ letters. We shall call the number $D_{r,\geq m}(n)$ *generalized r-derangement numbers*. In particular, $D_{0,\geq m}(n) = D_{\geq m}(n)$ and $D_{r,\geq 2}(n) = D_r(n)$. A basic recursion for these numbers is the following.

Theorem 8. *For any $r > 0$ and $n > m$, we have that*

$$\begin{aligned} D_{r,\geq m}(n) &= r(m-1)(n-1)_{m-2} D_{r-1,\geq m}(n-m+1) \\ &\quad + (n-1)^{\underline{m-1}} D_{r,\geq m}(n-m) + (n+r-1) D_{r,\geq m}(n-1). \end{aligned} \tag{13}$$

Proof. For any (r,m)-permutation on $n+r$ letters we can do the following construction

- the element $n+r$ is in a cycle of length m. There are two such cases: the element $n+r$ is either in a special cycle or not. In the first case, the element $n+r$ can be placed into one of the r cycles, and we can choose the remaining $m-2$ elements from $n-1$. This is done in $r(m-1)!\binom{n-1}{m-2} = r(m-1)(n-1)^{\underline{m-2}}$ ways. Note that we multiply by the factorial $(m-1)!$ because the order of the letters counts. The remaining elements can be choose in $D_{r-1,\geq m}(n-m+1)$ options. Thus, we have $r(m-1)(n-1)^{\underline{m-2}} D_{r-1,\geq m}(n-m+1)$ ways. For the second case the argument is similar. So, we have the second term in (13).

- The element $n+r$ is in an existing cycle of longer than m, after constructing a (r,m)-permutation on $n+r-1$ letters. For this we have $(n+r-1)D_{r,\geq m}(n-1)$ possibilities.

We therefore obtain the relation (13). □

By setting $m = 2$ we recover (5). Theorem 9 can be obtained by a slight change in the proof of Theorem 8.

Theorem 9. *For any $r > 0$ and $n > m$ we have that*

$$\begin{aligned} D_{r,\geq m}(n) = r \sum_{j=m-2}^{n-1} (j+1)(n-1)^{\underline{j}} D_{r-1,\geq m}(n-j-1) \\ + \sum_{j=m-1}^{n-1} (n-1)^{\underline{j}} D_{r,\geq m}(n-j-1). \end{aligned}$$

We now give an additional recurrence by analyzing the first special element.

Theorem 10. *For any $r > 0$ and $n \geq m$ we have that*

$$D_{r,\geq m}(n) = \sum_{j=m-1}^{n-(r-1)(m-1)} n^{\underline{j}} D_{r-1,\geq m}(n-j).$$

Proof. For any (r, m)-permutation on $n + r$ letters we have that the first special element can be in a cycle of length j, for $j = m - 1, m, \ldots, n - (r - 1)(m - 1)$. First we have to choose j non-special elements from that of n. This is done in $j!\binom{n}{j} = n^{\underline{j}}$ ways. The remaining cycles can be constructed in $D_{r-1,\geq m}(n-j)$ ways. Altogether, we have $(n)_j D_{r-1,\geq m}(n-j)$ possibilities. Finally, we sum over j. □

5.1. The Generating Function of $D_{r,\geq m}(n)$

In this section we give the exponential generating function for the sequence $(D_{r,\geq m}(n))_n$ and we derive several combinatorial identities.

To make our formulas more comprehensible, we introduce the notation

$$S_m(x) = \sum_{k=1}^{m} \frac{x^k}{k}.$$

Theorem 11. *For $r \geq 0$, the exponential generating function of the sequence of generalized r-derangement numbers is*

$$D_{r,m}(x) := \sum_{n=0}^{\infty} D_{r,\geq m}(n) \frac{x^n}{n!} = \left(\frac{x^{m-1}}{1-x}\right)^r \exp\left(\ln\left(\frac{1}{1-x}\right) - S_{m-1}(x)\right).$$

Proof. Each $(r, m)-$permutation may be uniquely decomposed into cycles as

$$(1\cdots)(2\cdots)\cdots(r\cdots)P,$$

where P is a $(0, m)$-permutation, and the first r cycles are of length at least m. Briggs and Remmel [4] (see also [23]) gave the exponential generating function for the number of $(0, m)$-permutations:

$$\exp\left(\ln\left(\frac{1}{1-x}\right) - S_{m-1}(x)\right).$$

Therefore, from the symbolic method [5] we obtain the following expression

$$D_{r,m}(x) = \left(\frac{x^m}{1-x}\right)^r D_{0,m}(x) = \left(\frac{x^{m-1}}{1-x}\right)^r \exp\left(\ln\left(\frac{1}{1-x}\right) - S_{m-1}(x)\right).$$

□

Theorem 12. *Let $r \geq 0$ and $s \in \{1, \ldots, r\}$. Then for each $n \geq s$ we have*

$$D_{r,\geq m}(n) = \sum_{j=s}^{n-(m-2)s} \binom{j-1}{s-1} \frac{n!}{(n-(m-2)s-j)!} D_{r-s,\geq m}(n-(m-2)s-j). \tag{14}$$

In particular, if $r = s$ then

$$D_{r,\geq m}(n) = \sum_{j=r}^{n-(m-2)r} \binom{j-1}{r-1} \frac{n!}{(n-(m-2)r-j)!} D_{\geq m}(n-(m-2)r-j). \tag{15}$$

Proof. From Theorem 11 we have

$$\begin{aligned} D_{r,m}(x) &= \sum_{n=0}^{\infty} D_{r,\geq m}(n) \frac{x^n}{n!} = \left(\frac{x^{m-1}}{1-x}\right)^r \exp\left(\ln\left(\frac{1}{1-x}\right) - S_{m-1}(x)\right) \\ &= \left(\frac{x^{m-1}}{1-x}\right)^s D_{r-s,m}(x) = [x^{(m-2)s} \left(\sum_{n=s}^{\infty} \binom{n-1}{s-1} x^n\right) \left(\sum_{n=0}^{\infty} D_{r-s,\geq m}(n) \frac{x^n}{n!}\right) \\ &= x^{(m-2)s} \sum_{n=s}^{\infty} \left(\sum_{j=s}^{\infty} \binom{j-1}{s-1} \frac{D_{r-s,\geq m}(n-j)}{(n-j)!}\right) x^n. \end{aligned}$$

The comparison of the coefficients of x^n yields the identity (14). $\square$

Theorem 13. *For any $r > 0$ and $n > m$ we have that*

$$D_{\geq m}(n) = n! \sum_{s=0}^{n} \frac{1}{s!} \sum_{(\ell_1,\ldots,\ell_{m-1}) \in \Omega(s,m-1)} (-1)^{\ell_1+\ell_2+\cdots+\ell_{m-1}} \times \binom{s}{\ell_1, 2\ell_2, \ldots, (m-1)\ell_{m-1}} \prod_{j=1}^{m-1} \frac{(j\ell_j)!}{j^{\ell_j}(\ell_j!)},$$

where

$$\Omega(n, m-1) = \left\{(\ell_1, \ell_2, \ldots, \ell_{m-1}) \in \mathbb{Z}_{n\geq 0}^{m-1} : \ell_1 + 2\ell_2 + \cdots + (m-1)\ell_{m-1} = n\right\}.$$

Proof. The generating function (with $r = 0$) in Theorem 11 can be written as

$$\sum_{n=0}^{\infty} D_{\geq m}(n)\frac{x^n}{n!} = \frac{1}{1-x}\exp\left(-x - \frac{x^2}{2} - \cdots - \frac{x^{m-1}}{m-1}\right)$$
$$= \frac{1}{1-x}\prod_{j=1}^{m-1}\sum_{\ell_j=0}^{\infty}\frac{(-1)^{\ell_j}x^{j\ell_j}}{j^{\ell_j}(\ell_j)!}$$
$$= \frac{1}{1-x}\sum_{n=0}^{\infty}\left(\sum_{(\ell_1,\ldots,\ell_{m-1})\in\Omega(n,m-1)}(-1)^{\ell_1+\ell_2+\cdots+\ell_{m-1}}\right.$$
$$\left.\binom{n}{\ell_1, 2\ell_2, \ldots, (m-1)\ell_{m-1}}\prod_{j=1}^{m-1}\frac{(j\ell_j)!}{j^{\ell_j}(\ell_j!)}\right)\frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(n!\sum_{s=0}^{n}\frac{1}{s!}\sum_{(\ell_1,\ldots,\ell_{m-1})\in\Omega(s,m-1)}(-1)^{\ell_1+\ell_2+\cdots+\ell_{m-1}}\right.$$
$$\left.\binom{n}{\ell_1, 2\ell_2, \ldots, (m-1)\ell_{m-1}}\prod_{j=1}^{m-1}\frac{(j\ell_j)!}{j^{\ell_j}(\ell_j!)}\right)\frac{x^n}{n!}.$$

The result now comes by comparing the coefficients of $x^n/n!$. □

In particular, if $m = 2$, we recover a basic identity for the derangement numbers:

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right). \tag{16}$$

If $m = 3$, we obtain a relation with the Hermite polynomials $H_n(x)$. Remember that the Hermite polynomials can be defined (cf. [19] or [12, Section F]) by the following generating function:

$$\sum_{n=0}^{\infty} H_n(x)\frac{z^n}{n!} = e^{2xz-z^2}.$$

Therefore we have the expression

$$\frac{D_{\geq 3}(n)}{n!} = \sum_{\ell=0}^{n}\frac{(-1)^{\ell}H_{\ell}(\sqrt{2}/2)}{(\sqrt{2})^{\ell}\ell!}.$$

Theorem 14. *The generalized r-derangement numbers have the below representa-*

tion:

$$D_{r,\geq m}(n) = \sum_{s=r}^{n-r(m-2)} \binom{s}{r} \frac{n!}{(n-r(m-2)-s)!} \sum_{(\ell_1,\dots,\ell_{m-1})\in\Omega(n-r(m-2)-s,m-1)} (-1)^{\ell_1+\ell_2+\cdots+\ell_{m-1}} \binom{n-r(m-2)-s}{\ell_1, 2\ell_2, \dots, (m-1)\ell_{m-1}} \prod_{j=1}^{m-1} \frac{(j\ell_j)!}{j^{\ell_j}(\ell_j!)}.$$

Proof. The proof is a corollary of (15) and Theorem 13. □

If $m = 2$, we have a closed formula for the r-derangement numbers (see also Theorem 4 in [21])

$$D_r(n) = \sum_{s=r}^{n} \binom{s}{r} \frac{n!}{(n-s)!} (-1)^{n-s}, \quad n \geq r.$$

In particular, if $r = 0$ and $m = 2$ we obtain (16) again.

Theorem 15. *The generalized r-derangement numbers satisfy the relation*

$$D_{r,\geq m}(n) = \sum_{j=1}^{r+1} n^{\underline{j}} \binom{r+1}{j} (-1)^{j+1} D_{r,\geq m}(n-j) + n^{\underline{r(m-1)}} \sum_{(\ell_1,\dots,\ell_{m-1})\in\Omega(n-r(m-1),m-1)} (-1)^{\ell_1+\ell_2+\cdots+\ell_{m-1}} \times \binom{n-r(m-1)}{\ell_1, 2\ell_2, \dots, (m-1)\ell_{m-1}} \prod_{j=1}^{m-1} \frac{(j\ell_j)!}{j^{\ell_j}(\ell_j!)}.$$

Proof. The proof follows from the equality

$$(1-x)^{r+1} \sum_{n=0}^{\infty} D_{r,\geq m}(n) \frac{x^n}{n!} = x^{(m-1)r} \exp\left(-S_{m-1}(x)\right).$$

□

If $m = 2$ we have the relation for the r-derangement numbers

$$D_r(n) = \sum_{j=1}^{r+1} n^{\underline{j}} \binom{r+1}{j} (-1)^{j+1} D_r(n-j) + n^{\underline{r}} (-1)^{n-r}. \tag{17}$$

Remarkably, if $r = 0$ and $m = 2$ we recover the Euler recursion (12), again.

5.2. A Connection With the Associated Lah Numbers

The Lah numbers $L(n, k)$ count the number of set partitions of a set with n elements into k non-empty ordered lists. This sequence satisfies the following recurrence

$$L(n, k) = L(n-1, k-1) + (n+k-1)L(n-1, k)$$

with the initial values $L(0, 0) = 1$, $L(n, 0) = 0$ if $n > 0$, and $L(n, 1) = n!$.

They can be computed by the following explicit formula

$$L(n, k) = \frac{n!}{k!}\binom{n-1}{k-1},$$

or by the generating function

$$\sum_{n\geq k} L(n, k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k.$$

The associated Lah numbers $L_{\geq m}(n, k)$ are defined as the number of set partitions of a set with n elements into k non-empty ordered lists such that every list contains at least m elements. It is not difficult to show that the exponential generating function of this sequence is given by

$$\sum_{n\geq k} L_{\geq m}(n, k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x^m}{1-x}\right)^k. \tag{18}$$

Theorem 16. *For $r \geq 0$ and $n \geq 1$, such that $n \geq r$, we have*

$$(r+1)!L_{\geq m-1}(n, r-1) = \sum_{s=0}^{n-m+1}\binom{n}{s}(n-s)^{\underline{m-1}}D_{r,\geq m}(s)$$
$$\sum_{(\ell_1,\ldots,\ell_{m-1})\in\Omega(n-m-s+1,m-1)}\binom{n-m-s+1}{\ell_1, 2\ell_2, \ldots, (m-1)\ell_{m-1}}\prod_{j=1}^{m-1}\frac{(j\ell_j)!}{j_j^{\ell}(\ell_j!)}.$$

Proof. Let us first introduce the shorthand notation

$$B(n, m-1) := \sum_{(\ell_1,\ldots,\ell_{m-1})\in\Omega(n,m-1)}\binom{n}{\ell_1, 2\ell_2, \ldots, (m-1)\ell_{m-1}}\prod_{j=1}^{m-1}\frac{(j\ell_j)!}{j^{\ell_j}(\ell_j!)}.$$

From the generating functions given in (18) and Theorem 11 we have

$$(r+1)!\sum_{n=0}^{\infty} L_{\geq m}(n, r+1)\frac{x^n}{n!} = \left(\frac{x^m}{1-x}\right)^{r+1} =$$

$$x^{m-1}\exp\left(S_{m-1}(x)\right)\left(\sum_{n=0}^{\infty}D_{r,\geq m}(n)\frac{x^n}{n!}\right)=$$
$$x^{m-1}\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{s=0}^{n}\binom{n}{s}D_{r,\geq m}(s)B(n-s,m-1).$$

The conclusion follows by comparing coefficients of $x^n/n!$. □

If $m=2$, we obtain the relation for the r-derangement numbers and Lah numbers (see Theorem 9 of [21] and [16]):

$$(r+1)!L(n,r-1)=\sum_{s=0}^{n-1}\binom{n}{s}(n-s)D_r(s)=\sum_{s=0}^{n}\binom{n}{s}sD_r(n-s).$$

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