EULER-TYPE RECURRENCE RELATION FOR ARBITRARY ARITHMETICAL FUNCTION

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Abstract

An interplay between the Lambert series and Euler's Pentagonal Number Theorem gives an Euler-type recurrence relation for any given arithmetical function. As consequences of this, we present Euler-type recurrence relations for some well-known arithmetic functions. Furthermore, we derive Euler-type recurrence relations for some partition functions and sum-of-divisors functions using infinite product identities of Jacobi and Gauss.

1. Motivation and Statement of Results

Euler's Pentagonal Number Theorem [4] states that, for complex q with $|q| < 1$,

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{m=0}^{\infty}\omega(m)q^m, \tag{1}$$

where

$$\omega(m) = \begin{cases} 1 & \text{if } m = 0; \\ (-1)^k & \text{if } m = \frac{3k^2 \pm k}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

This result, combined with the generating function for $p(n)$, the number of partitions of n,

$$\prod_{m=1}^{\infty}\frac{1}{1-q^m} = \sum_{n=0}^{\infty}p(n)q^n \tag{2}$$

gives a beautiful recurrence relation for $p(n)$:

$$\sum_{k=0}^{n}\omega(k)p(n-k) = 0 \text{ for each } n \geq 1. \tag{3}$$

When Euler's Pentagonal Number Theorem is combined with the generating function for $q(n)$, the number of distinct partitions (partitions with distinct parts) of n,

$$\prod_{m=1}^{\infty}(1+q^m)=\sum_{n=0}^{\infty}q(n)q^n$$

gives the following recurrence relation for $q(n)$, which is mentioned in [1, p. 826]:

$$\sum_{k=0}^{n}\omega(k)q(n-k)=\begin{cases}\omega(\frac{n}{2}) & \text{if } n \text{ is even;}\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

Using the operator $q\frac{d}{dq}\log$ in (2), one can get

$$\sum_{n=1}^{\infty}n\omega(n)q^n=\left(\sum_{m=0}^{\infty}\omega(m)q^m\right)\left(-\sum_{m=1}^{\infty}m\frac{q^m}{1-q^m}\right).$$

Then, the Lambert series representation gives the following recurrence relation for $\sigma(n)$, the sum of positive divisors of n:

$$\sum_{k=1}^{n}\sigma(k)\omega(n-k)=-n\omega(n). \tag{5}$$

This recurrence relation involving pentagonal numbers was first realized by Euler [5].

Any recurrence relation that assumes the pattern of the recurrence relations mentioned above is termed as Euler-type. For a derivation of the other Euler-type recurrence relations, see [3]. In [3], Euler-type recurrence relation for the function $\tau_A(n)$ (resp. $\sigma_A(n)$), the number of (resp. the sum of) divisors of n with divisors from a set of positive integers A was derived by fusing Euler's Pentagonal Number Theorem and generating function of $\tau_A(n)$ (resp. $\sigma_A(n)$). The derivation technique wielded in [3] is generalized here, and we obtain the first result of this paper.

Theorem 1. *Let g be an arithmetical function. Then we have*

$$\sum_{k=1}^{n}g(k)\omega(n-k)=\sum_{m+k=n;m\geq 1;k\geq 0}f(m)\left(\omega(k)+\omega(k-m)+\omega(k-2m)+\cdots\right), \tag{6}$$

where

$$g(n)=\sum_{d|n}f(d).$$

Remark 1. *The highlighting feature of the above recurrence relation is that though g is a divisor sum of f, if f can be put as an expression without factorization being* *involved, then one can obtain a recurrence relation for g without factoring each k. On the other hand, if g is a known simple expression where factorization is not needed and, though f may depend on factorization, then using (6) one can obtain a recurrence relation for f without factoring each m.*

In Section 2, we present a proof for Theorem 1, and we employ Theorem 1 to obtain recurrence relations of Euler-type for several arithmetical functions including the Euler's phi function $\phi(n)$, the number-of-divisors function $\tau(n)$, the Liouville's function $\lambda(n)$ and the Möbius function $\mu(n)$.

In Section 3, manipulation of infinite product identities plays a vital role. The following result is established in Section 3 by making use of Jacobi's triple product identity.

Theorem 2. *Let $q(n)$ and $qq(n)$ be the number of distinct partitions of n and the number of distinct partitions of n with odd parts, respectively. We have*

(a)

$$\sum_{k=0}^{n}(-1)^k qq(k)\omega(n-k) = \begin{cases} 1 & \text{if } n=0 \\ 2 & \text{if } \delta_s(n)=1 \text{ and } n\equiv 0(\text{mod } 2) \\ -2 & \text{if } \delta_s(n)=1 \text{ and } n\equiv 1(\text{mod } 2) \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

(b)

$$q(n) + 2\sum_{k\geq 1}(-1)^k \delta_s(k) q(n-k) = \omega(n), \tag{8}$$

where

$$\delta_s(n) = \begin{cases} 1 & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, in Section 3, the following identities due to Gauss [6] were used to establish Theorem 3 and Theorem 4.

1.
$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{\prod_{m=1}^{\infty}(1-q^{2m})}{\prod_{m=1}^{\infty}(1-q^{2m-1})}, \tag{9}$$

2.
$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{m=1}^{\infty}\frac{1-q^m}{1+q^m}. \tag{10}$$

Theorem 3. *We have*

(a)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q(n-2k)\omega(k) = \begin{cases} 1 & \textit{if } \delta_t(n) = 1 \\ 0 & \textit{otherwise,} \end{cases} \tag{11}$$

(b)

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} p(k)\delta_t(n-2k) = q(n), \tag{12}$$

(c)

$$\sum_{k=0}^{n} (-1)^k qq(k)\delta_t(n-k) = \begin{cases} \omega(\frac{n}{2}) & \textit{if } n \textit{ is even} \\ 0 & \textit{otherwise,} \end{cases} \tag{13}$$

where

$$\delta_t(n) = \begin{cases} 1 & \textit{if } n = \frac{m(m+1)}{2} \\ 0 & \textit{otherwise.} \end{cases}$$

Recall that, a *composition* of a positive integer n is an ordered sequence of positive integers, say $\pi = (a_1, a_2, \cdots, a_k)$, such that $a_1 + a_2 + \ldots + a_k = n$. The composition π is said to be *distinct* if $a_i \neq a_j$ for every $i \neq j$. The composition π is said to be *relatively prime* if $\gcd(a_1, a_2, \cdots, a_k) = 1$.

Theorem 4. *Let $s(n)$ and $t(n)$ be the number of compositions of n with square numbers as parts and the number of compositions of n with triangular numbers as parts, respectively. We have*

(a)

$$\sum_{k=0}^{n} (-1)^k s(k) \left(3q(n-k) - \omega(n-k)\right) = 2q(n), \tag{14}$$

(b)

$$\sum_{k=0}^{n} t(k) \left(2(-1)^{n-k} qq(n-k) - \omega'(n-k)\right) = (-1)^n qq(n), \tag{15}$$

where

$$w'(n) = \begin{cases} \omega(\frac{n}{2}) & \textit{if } n \textit{ is even} \\ 0 & \textit{otherwise.} \end{cases}$$

In Section 4, we wield the logarithmic derivative technique, in the way of Euler, to get the following result.

Theorem 5. *Let $\sigma_o(n)$(resp. $\sigma_e(n)$) be the sum of odd (resp. even) positive divisors of n. We have*

(a)

$$\sigma(n)+\sigma_o(n)+2\sum_{k\geq 1}(-1)^k\delta_s(k)(\sigma(n-k)+\sigma_o(n-k)) = 2(-1)^{n+1}n\delta_s(n), \quad (16)$$

(b)

$$\sum_{k=1}^{n}\left(\sigma_o(k)-\sigma_e(k)\right)\delta_t(n-k) = n\delta_t(n), \quad (17)$$

(c) for $n \geq 2$, we have

$$\sigma(n)+\sigma_s(n)+2\sum_{k\geq 1}(-1)^k\delta_s(k)(\sigma(n-k)+\sigma_s(n-k)) = 2(-1)^{n+1}n\delta_s(n), \quad (18)$$

where

$$\sigma_s(n) = \sum_{d|n}(-1)^{d-1}\frac{n}{d}.$$

The meaning of the notation specified in this section stands throughout the article.

2. Proof and Applications of Theorem 1

The following lemma forms a crucial part of the proof.

Lemma 1. *We have*

$$\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q^k} = \sum_{m=0}^{\infty}\left(\omega(m)+\omega(m-k)+\omega(m-2k)+\cdots\right)q^m \quad (19)$$

with $\omega(m) = 0$ for every $m < 0$.

Proof. Let

$$\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q^k} = \sum_{m=0}^{\infty}\omega_k(m)q^m.$$

Then, by Euler's Pentagonal Number Theorem, we have

$$\omega_k(n)-\omega_k(n-k) = \omega(n).$$

Repeated application of the above relation gives

$$\omega_k(n) = \omega(n)+\omega(n-k)+\omega(n-2k)+\cdots$$

as expected. □

2.1. Main Part of the Proof

From the Lambert series representation, we have

$$\sum_{n=1}^{\infty} g(n)q^n = \sum_{m=1}^{\infty} f(m)\frac{q^m}{1-q^m},$$

where $g(n) = \sum_{d|n} f(d)$.

Then multiplying both sides by the term $\prod_{n=1}^{\infty}(1-q^n)$, and using Euler's Pentagonal Number Theorem with Lemma 1 gives the following equalities:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} g(k)\omega(n-k)\right)q^n &= \sum_{m=1}^{\infty} f(m)\frac{\prod_{n=1}^{\infty}(1-q^n)}{1-q^m}q^m \\ &= \sum_{m=1}^{\infty} f(m)q^m\left(\sum_{k=0}^{\infty}\omega(k)+\omega(k-m)+\omega(k-2m)+\cdots\right)q^k.\end{aligned}$$

Equating the coefficients of q^n gives the desired end.

2.2. Recurrence Relation for ϕ, τ, λ and μ

As the first application of Theorem 1, we present a recurrence relation for the famous Euler's phi function $\phi(n)$.

Theorem 6. *We have*

$$\sum_{k=1}^{n} k\omega(n-k) = \sum_{m+k=n;m\geq 1;k\geq 0} \phi(m)\left(\omega(k)+\omega(k-m)+\omega(k-2m)+\cdots\right). \quad (20)$$

Proof. Applying Gauss identity

$$\sum_{d|n}\phi(d) = n$$

in Theorem 1, we get the above relation. □

Next is a recurrence relation for the number-of-divisors function $\tau(n)$ as an immediate consequence of Theorem 1.

Theorem 7. *We have*

$$\sum_{k=1}^{n}\tau(k)\omega(n-k) = \sum_{m+k=n;m\geq 1;k\geq 0}(\omega(k)+\omega(k-m)+\cdots). \quad (21)$$

The Liouville's function, denoted by $\lambda(n)$, is defined as

$$\lambda(n) = \begin{cases} 1 & \text{if } n=1; \\ (-1)^k & \text{if the number of prime factors of } n \text{ counted with multiplicity is } k. \end{cases} \quad (22)$$

Now we present a recurrence relation for $\lambda(n)$, which is based on the following well-known relation:

$$\sum_{d|n} \lambda(d) = \delta_s(n). \tag{23}$$

Theorem 8. *We have*

$$\sum_{k=1}^{n} \delta_s(k)\omega(n-k) = \sum_{m+k=n; m\geq 1; k\geq 0} \lambda(m)(\omega(k) + \omega(k-m) + \cdots). \tag{24}$$

From the following well-known relation:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

we have the following recurrence relation for the Möbius function $\mu(n)$.

Theorem 9. *For $n > 1$, we have*

$$\omega(n-1) = \sum_{m+k=n; m\geq 1; k\geq 0} \mu(m)(\omega(k) + \omega(k-m) + \cdots). \tag{26}$$

2.3. Relatively Prime (Distinct) Partitions and Compositions

Theorem 10. *Let $p_\psi(n)$ be the number of relatively prime partitions of n. We have*

$$-\omega(n) = \sum_{m+k=n; m\geq 1; k\geq 0} p_\psi(m)(\omega(k) + \omega(k-m) + \cdots). \tag{27}$$

Proof. Mohamed El Bachraoui [12] noted that

$$p(n) = \sum_{d|n} p_\psi(d).$$

Keeping this observation, the result follows as a consequence of Theorem 1 and Recurrence relation (3). □

Theorem 11. *Let $q_\psi(n)$ be the number of relatively prime distinct partitions of n. We have*

$$\sum_{m+k=n; m\geq 1; k\geq 0} q_\psi(m)(\omega(k) + \omega(k-m) + \cdots) = \begin{cases} -\omega(n) & \text{if } n \text{ is odd;} \\ -\omega(n) + \omega(\frac{n}{2}) & \text{if } n \text{ is even.} \end{cases} \tag{28}$$

Proof. Using the Recurrence relation (4) in Theorem 1, together with the following observation:

$$q(n) = \sum_{d|n} q_\psi(d),$$

completes the proof. □

Theorem 12. *Let $c_\psi(n)$ be the number of relatively prime compositions of n. We have*

$$\sum_{k=1}^{n} 2^{k-1}\omega(n-k) = \sum_{m+k=n;m\geq 1;k\geq 0} c_\psi(m)(\omega(k)+\omega(k-m)+\cdots). \tag{29}$$

Proof. Let $c(n)$ be the number of compositions of n. Gould [7] noted that

$$c(n) = \sum_{d|n} c_\psi(d) \tag{30}$$

and

$$c(n) = 2^{n-1}. \tag{31}$$

Now the result follows while presenting the above relations in Theorem 1. □

Theorem 13. *Let $c_\psi(n,r)$ be the number of relatively prime compositions of n with exactly r parts. We have*

$$\sum_{k=1}^{n} \binom{k-1}{r-1}\omega(n-k) = \sum_{m+k=n;m\geq 1;k\geq 0} c_\psi(m,r)(\omega(k)+\omega(k-m)+\cdots). \tag{32}$$

Proof. Let $c(n,r)$ be the number of compositions of n with exactly r parts. Gould [7] observed that,

$$c(n,r) = \sum_{d|n} c_\psi(d,r). \tag{33}$$

Catalan [2] counted that,

$$c(n,r) = \binom{n-1}{r-1}. \tag{34}$$

Now the result follows from Theorem 1. □

2.4. Representations as Sum of Squares

Let $r_k(n)$ be the number of ways integer n can be represented as a sum of k squares. The following result due to Jacobi [10] puts $r_2(n)$, $r_4(n)$ and $r_8(n)$ separately as a divisor sum of simple functions. This representation paves the way for deriving Euler-type recurrence relations for $r_i(n)$ $(i = 2, 4, 8)$ with the aid of Theorem 1.

Lemma 2 (Jacobi). *We have*

(a)

$$\frac{r_2(n)}{4} = \sum_{d|n} \eta_1(d),$$

where

$$\eta_1(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \end{cases}$$

(b)

$$r_4(n) = 8\sum_{d|n}\eta_2(d),$$

where

$$\eta_2(n) = \begin{cases} 0 & \text{if } 4|n \\ n & \text{otherwise,} \end{cases}$$

(c)

$$r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3.$$

Now, in accordance with Theorem 1, we have the following result.

Theorem 14. *We have*

(a)

$$\sum_{k=1}^{n} r_2(k)\omega(n-k) = 4\left(\sum_{\substack{m+k=n;\\ m\geq 1;k\geq 0}}\eta_1(m)(\omega(k)+\omega(k-m)+\cdots)\right), \tag{35}$$

(b)

$$\sum_{k=1}^{n} r_4(k)\omega(n-k) = 8\left(\sum_{\substack{m+k=n;\\ m\geq 1;k\geq 0}}\eta_2(m)(\omega(k)+\omega(k-m)+\cdots)\right), \tag{36}$$

(c)

$$\sum_{k=1}^{n} \frac{r_8(k)}{(-1)^k}\omega(n-k) = 16\left(\sum_{\substack{m+k=n;\\ m\geq 1;k\geq 0}}(-1)^m m^3(\omega(k)+\omega(k-m)+\cdots)\right). \tag{37}$$

2.5. Subsets Relatively Prime to a Positive Integer

Melvyn B. Nathanson [11] denoted the number of nonempty subsets and the number of subsets of cardinality r of $\{1, 2, \ldots, n\}$ such that the greatest common divisor of each subset is relatively prime to n, respectively, by $\Phi(n)$ and $\Phi_r(n)$; and he showed that

$$\sum_{d|n}\Phi(d) = 2^n - 1 \tag{38}$$

and

$$\sum_{d|n}\Phi_r(d) = \binom{n}{r}. \tag{39}$$

Presenting these identities appropriately in Theorem 1 lead to the following result.

Theorem 15. *We have*

(a)

$$\sum_{k=1}^{n}(2^k-1)\omega(n-k)=\sum_{m+k=n;m\geq 1;k\geq 0}\Phi(m)(\omega(k)+\omega(k-m)+\cdots), \quad (40)$$

(b)

$$\sum_{k=1}^{n}\binom{k}{r}\omega(n-k)=\sum_{m+k=n;m\geq 1;k\geq 0}\Phi_r(m)(\omega(k)+\omega(k-m)+\cdots). \quad (41)$$

Now we replace the set $\{1,2,\ldots,n\}$ in the above definition with the set of positive divisors of n. Here too the pattern of the above identities retains.

Definition 1. Let n be a positive integer. Denote by $\Phi^\tau(n)$ *(*resp. $\Phi_r^\tau(n)$*)*, the number of nonempty subsets *(*resp. the number of subsets of cardinality r*)* of the set of positive divisors of n such that the greatest common divisor of each subset is relatively prime to n.

Lemma 3. *We have*

(a)

$$\sum_{d|n}\Phi^\tau(d)=2^{\tau(n)}-1, \quad (42)$$

(b)

$$\sum_{d|n}\Phi_r^\tau(d)=\binom{\tau(n)}{r}. \quad (43)$$

Theorem 16. *We have*

(a)

$$\sum_{k=1}^{n}(2^{\tau(k)}-1)\omega(n-k)=\sum_{m+k=n;m\geq 1;k\geq 0}\Phi^\tau(m)(\omega(k)+\omega(k-m)+\cdots), \quad (44)$$

(b)

$$\sum_{k=1}^{n}\binom{\tau(k)}{r}\omega(n-k)=\sum_{m+k=n;m\geq 1;k\geq 0}\Phi_r^\tau(m)(\omega(k)+\omega(k-m)+\cdots). \quad (45)$$

Note 1. Comparing Equation (38) and Equation (31), we get

$$\Phi(n)=\begin{cases}2c_\psi(n)-1 & \text{if } n=1;\\ 2c_\psi(n) & \text{if } n\geq 2.\end{cases} \quad (46)$$

Comparing Equation (39) and Equation (34) in light of Pascal's identity, one can get that

$$\Phi_r(n)=c_\psi(n,r)+c_\psi(n,r+1). \quad (47)$$

3. Another Way to Euler-type Recurrence Relations

3.1. Proof of Theorem 2

Now we look at some unnoticed things in a recent paper by Yuriy Choliy, Louis W. Kolitsch and Andrew V. Sills [13] that lead to an Euler-type recurrence relation for $qq(n)$ and a similar type of recurrence relation for $q(n)$ involving square numbers in place of pentagonal numbers, which is the crux of Theorem 2.

Lemma 4 (Euler's partition theorem [4]). *The following equality holds:*

$$\sum_{n=0}^{\infty} q(n)q^n = \prod_{m=1}^{\infty}(1+q^m) = \prod_{m=1}^{\infty}\frac{1}{1-q^{2m-1}}.$$

Consider the following identity due to Jacobi [9]:

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})(1+q^{2n-1}) = 1 + 2\sum_{m=1}^{\infty}\delta_s(m)q^m. \tag{48}$$

Replacing q with $-q$ gives

$$\prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^{2n-1}) = 1 + 2\sum_{m=1}^{\infty}(-1)^m\delta_s(m)q^m. \tag{49}$$

When products in the left extreme were expanded then one can get (a) of Theorem 2.

Equation (49) can be put as

$$\prod_{n=1}^{\infty}(1-q^n) = \frac{1+2\sum_{m=1}^{\infty}(-1)^m\delta_s(m)q^m}{\prod_{n=1}^{\infty}(1-q^{2n-1})}.$$

Then by Euler's partition theorem, we have

$$\prod_{n=1}^{\infty}(1-q^n) = \left(1+2\sum_{m=1}^{\infty}(-1)^m\delta_s(m)q^m\right)\left(\sum_{m=0}^{\infty}q(m)q^m\right).$$

Now (b) of Theorem 2 follows by Euler's Pentagonal Number Theorem.

3.2. Proof of Theorem 3

Euler's partition theorem and Gauss identity (9) implies

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \left(\prod_{m=1}^{\infty}(1-q^{2m})\right)\left(\sum_{m=0}^{\infty}q(m)q^m\right). \tag{50}$$

Since

$$\prod_{m=1}^{\infty}(1-q^{2m}) = 1+\omega(1)q^2+\omega(2)q^4+\omega(3)q^6+\cdots,$$

(a) of Theorem 3 follows.

Another form of Gauss identity (9), namely,

$$\left(\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\right)\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right) = \sum_{m=0}^{\infty}q(m)q^m$$

gives (b) of Theorem 3, since

$$\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}} = p(0)+p(1)q^2+p(2)q^4+p(3)q^6+\cdots.$$

Gauss identity (9) can also be put as

$$\left(\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\right)\left(\prod_{n=1}^{\infty}(1-q^{2n-1})\right) = \prod_{m=1}^{\infty}(1-q^{2m}).$$

Since

$$\prod_{m=1}^{\infty}(1-q^{2m-1}) = \sum_{n=0}^{\infty}(-1)^n qq(n)q^n,$$

(c) of Theorem 3 follows.

3.3. Proof of Theorem 4

Let A be a set of positive integers. We define

$$\chi_A(q) = \sum_{a\in A}q^a.$$

Let $c_A(n)$ be the number of compositions of n with parts from set A. Heubach and Mansour [8] documented that

$$\sum_{n=0}^{\infty}c_A(n)q^n = \frac{1}{1-\chi_A(q)}. \tag{51}$$

Now, in accordance with the Gauss identity (10), we can write

$$\begin{aligned}1+\sum_{n=1}^{\infty}(-1)^n s(n)q^n &= \frac{1}{1-\sum_{n=1}^{\infty}(-1)^{n^2}q^{n^2}}\\ &= \frac{1}{1-\left(\frac{\prod_{m=1}^{\infty}\frac{1-q^m}{1+q^m}-1}{2}\right)}\\ &= \frac{2}{3-\prod_{m=1}^{\infty}\frac{1-q^m}{1+q^m}}\\ &= \frac{2\prod_{m=1}^{\infty}(1+q^m)}{3\prod_{m=1}^{\infty}(1+q^m)-\prod_{m=1}^{\infty}(1-q^m)}.\end{aligned}$$

This gives

$$\left(1+\sum_{n=1}^{\infty}(-1)^n s(n)q^n\right)\left(3\prod_{m=1}^{\infty}(1+q^m)-\prod_{m=1}^{\infty}(1-q^m)\right)=2\prod_{m=1}^{\infty}(1+q^m).$$

Equating the coefficients of like powers of q in both extremes gives (a) of Theorem 4. A similar application of Gauss identity (9) gives (b) of Theorem 4.

4. In the Way of Euler's Logarithmic Derivative

In this section, we wield the logarithmic derivative technique of Euler [5] to conclude Theorem 5.

4.1. Proof of Theorem 5

Taking log on both sides of Equation (48), we have

$$\sum_{n=1}^{\infty}\log(1-q^{2n})+2\sum_{n=1}^{\infty}\log(1+q^{2n-1})=\log\left(1+2\sum_{m=1}^{\infty}\delta_s(m)q^m\right).$$

Differentiating and then multiplying by q, we get

$$-\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}+2\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}=\frac{2\sum_{m=1}^{\infty}m\delta_s(m)q^m}{1+2\sum_{m=1}^{\infty}\delta_s(m)q^m}.$$

Replacing q with $-q$, we have then by the Lambert series representation that

$$\left(1+2\sum_{n=1}^{\infty}(-1)^n\delta_s(n)q^n\right)\left(\sum_{n=1}^{\infty}(\sigma(n)+\sigma_o(n))q^n\right)=2\sum_{m=1}^{\infty}(-1)^{m+1}m\delta_s(m)q^m.$$

Now (a) of Theorem 5 follows. One can apply the same method in Gauss identity (9) to obtain (b) of Theorem 5. To realize (c) of Theorem 5, the following lemma is an essential one.

Lemma 5. *We have*

$$\sum_{n=1}^{\infty} n\frac{q^n}{1+q^n} = \sum_{n=1}^{\infty} \sigma_s(n)q^n. \tag{52}$$

Proof. Suppose that $n = dk$ for some positive integers d and k. Then the coefficient of q^n in the binomial expansion of $k\frac{q^k}{1+q^k}$ is $(-1)^{d-1}\frac{n}{d}$. Therefore the coefficient of q^n in the sum $\sum_{k=1}^{\infty} k\frac{q^k}{1+q^k}$ is $\sum_{d|n}(-1)^{d-1}\frac{n}{d}$ as expected. □

Apply the logarithmic derivative technique in Gauss identity (10), then (c) of Theorem 5 follows in light of Lemma 5.

4.2. An Extension of Theorem 5

Again consider Jacobi's identity [9]

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})(1+q^{2n-1}) = \sum_{m=-\infty}^{\infty} q^{m^2}.$$

Then for every integer $k \geq 2$, we have

$$\left(\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})(1+q^{2n-1})\right)^k = 1 + \sum_{m=1}^{\infty} r_k(m)q^m,$$

where $r_k(m)$, as denoted earlier, represents the number of ways in which m can be written as a sum of k squares.

The transformation $q \to -q$ gives

$$\left(\prod_{n=1}^{\infty}(1-q^n)\right)^k \left(\prod_{n=1}^{\infty}(1-q^{2n-1})\right)^k = 1 + \sum_{m=1}^{\infty}(-1)^m r_k(m)q^m.$$

Applying the operator $q\frac{d}{dq}$ and using the Lambert series representation as before, we have the following recurrence relation for $r_k(n)$.

Theorem 17. *We have*

$$k\left(\sum_{i=1}^{n}(\sigma(i)+\sigma_o(i))((-1)^{n-i}r_k(n-i))\right) = (-1)^{n+1}nr_k(n). \tag{53}$$

As an immediate consequence, we get the following congruence property of $r_k(n)$.

Corollary 1. *Let n and k be positive integers such that* $\gcd(k,n) = 1$. *We have*

$$r_k(n) \equiv 0 (mod\ k). \tag{54}$$

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