CONTINUED FRACTIONS OF CERTAIN SERIES

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Abstract
We study a specific polynomial \( f(T) = T^2g(T) \) where \( g(T) \) is a nonconstant monic polynomial in \( (F(x))[T] \) and \( F \) is a field. Setting \( f_0(T) = f(T) = T \) and \( f_n(T) = f(f_{n-1}(T)) \) \((n \geq 1)\), we then have explicit formulas for the continued fractions of certain series in the form
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{f_{n-1}(T)f_n(T)}.
\]

1. Introduction
The problems involved in the relationship between series and continued fractions have a long history. The area of research in this direction deals with the determination of the closed forms for continued fractions representing some certain series in a particular field, see e.g. \([1],[2], [3], [5], [7], [9], [11], [12], [15], [16] \) and \([17]\). For instance, in 1973, Lehmer \([3]\) showed that the modified Bessel functions can be represented by the simple continued fractions whose partial quotients form arithmetic progression. In 2006, Riyapan, Laohakosol and Chaichana \([9]\) gave another proof of Lehmer’s result and they also found other types of continued fractions with explicit shape. In 1996, Cohn \([1]\) introduced specializable continued fractions over \( \mathbb{Q}(x) \), the continued fractions with the property that each partial quotient has integer coefficients, and gave a complete classification of polynomials \( f(T) \in \mathbb{Z}[T] \) for which the series expansions \( \sum_{n=0}^{\infty} \frac{1}{f_n(T)} \) have specializable continued fractions.
Here we are particularly interested in the result of Tamura in 1991, see [13]. Let $f_0(T) = T$ and $f_n(T) := f(f_{n-1}(T))$ be the $n$th composite iterate of a polynomial $f(T)$. He found the pattern of the simple continued fractions for the series of real numbers in the form
\[
\sum_{n=0}^{\infty} \frac{1}{f_0(T) f_1(T) \ldots f_n(T)}
\]
for
\[
f(T) = f^{(1)}(T) = T(T+2)(T-2)g^{(1)}(T) + T^2 - 2 \quad \text{and}
\]
\[
f(T) = f^{(2)}(T) = T^2(T+2)(T-2)g^{(2)}(T) + T^2 - 2
\]
where $g^{(1)}(T), g^{(2)}(T) \in \mathbb{Z}[T]$ and $T \in \mathbb{Z}$. Later, in [8] Rattanamoong, Laohakosol and Chaichana extended and modified Tamura’s result in the case of formal Laurent series over a field $F$ of characteristic 0. They discovered the specializable continued fractions for the series in the form
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{f_0(T) f_1(T) \ldots f_n(T)},
\]
where $f(T) = T(T+2)(T-2)g(T) - T^2 + 2$ and $T \in F[x] \setminus F$.

Motivated by the above mentioned works, we aim to determine explicit formulas for simple continued fractions of two classes of elements in the field of Laurent series over $F$ having series expansions in the forms
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(T) f_n(T)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{f_{n-1}(T) f_n(T)}.
\]

Let $F((x^{-1}))$ denote the field of formal Laurent series over a field $F$ equipped with a degree valuation $| \cdot |$ defined by $|x^{-1}| = e^{-1}$. It is well known, see [10], that each element $\alpha \in F((x^{-1}))$ can be uniquely expressible as the simple continued fraction in the form
\[
\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}} =: [a_0; a_1, a_2, \ldots],
\]
where $a_0 \in F[x]$ and $a_n \in F[x] \setminus F$ ($n \geq 1$). Note that the continued fraction of $\alpha$ is finite if and only if $\alpha \in F(x)$. Define two sequences $(C_n)$, $(D_n)$ by
\[
C_{-1} = 1, \quad C_0 = a_0, \quad C_{n+1} = a_{n+1}C_n + C_{n-1} \quad (n \geq 0)
\]
\[
D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = a_{n+1}D_n + D_{n-1} \quad (n \geq 0).
\]
The following proposition is easily established by induction so we state it without proof.
Proposition 1. Let \( n \in \mathbb{N} \cup \{0\} \) and \( \beta \in F((x^{-1})) \setminus \{0\} \). With the above notation, we have

\[
(i) \quad \frac{\beta C_n + C_{n-1}}{\beta D_n + D_{n-1}} = [a_0; a_1, a_2, \ldots, a_n, \beta],
\]

\[
(ii) \quad \frac{C_n}{D_n} = [a_0; a_1, a_2, \ldots, a_n],
\]

\[
(iii) \quad D_nC_{n-1} - C_nD_{n-1} = (-1)^n, \text{ so that } C_n \text{ and } D_n \text{ are relatively prime},
\]

\[
(iv) \quad |D_n| > |D_{n-1}|,
\]

\[
(v) \quad |C_n| > |C_{n-1}| \quad (n \geq 1).
\]

From Proposition 1 (ii), we have

\[
\frac{C_n}{D_n} = [a_0; a_1, a_2, \ldots, a_n] \quad (n \geq 0).
\]

We call \( C_n/D_n \) the \( n \)th convergent of the continued fraction of \( \alpha \).

For each polynomial \( h(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \in F[x], c_n \neq 0 \), we define \( l.c.(h) := c_n \), which is the leading coefficient of \( h(x) \).

Lemma 1. Let \( n \in \mathbb{N} \) and \( n \geq 2 \). If \( C_n/D_n = [0; a_1, a_2, \ldots, a_n] \), then

\[
l.c.(C_n) = l.c.(a_na_{n-1} \cdots a_2) \quad \text{and} \quad l.c.(D_n) = l.c.(a_na_{n-1} \cdots a_2 a_1).
\]

Proof. Note that \( C_1 = 1 \) and \( D_1 = a_1 \). Let \( n \in \mathbb{N} \) and \( n \geq 2 \). By Proposition 1, we have

\[
C_n = a_nC_{n-1} + C_{n-2}
\]

\[
= a_n(a_{n-1}C_{n-2} + C_{n-3}) + C_{n-2}
\]

\[
= a_na_{n-1}C_{n-2} + a_nC_{n-3} + C_{n-2}
\]

\[
\vdots
\]

\[
= a_na_{n-1} \cdots a_2 C_1 + \text{terms of lower degrees}
\]

\[
= a_na_{n-1} \cdots a_2 + \text{terms of lower degrees}.
\]

Similarly, we also have

\[
D_n = a_na_{n-1} \cdots a_2 a_1 + \text{terms of lower degrees}.
\]

Therefore we get \( l.c.(C_n) = l.c.(a_na_{n-1} \cdots a_2) \) and \( l.c.(D_n) = l.c.(a_na_{n-1} \cdots a_2 a_1) \). \( \square \)
The next result which is an important tool for possessing folding symmetry of continued fractions, is classically known as the Folding Lemma, see [6].

**Lemma 2.** Let \( y \in F[x] \setminus \{0\} \), and

\[
\frac{C_n}{D_n} = [0; a_1, a_2, \ldots, a_n] := [0; \overline{X}_n] \quad (n \geq 1).
\]

Then

\[
[0; \overline{X}_n, y, -\overline{X}_n] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2} y.
\]

Here \( \overline{X}_n \) is an abbreviation for the word \( a_1, a_2, \ldots, a_n \), and accordingly, \( -\overline{X}_n \) denotes the word \( -a_n, -a_{n-1}, \ldots, -a_1 \).

2. Main Results

Let

\[
f(T) = T^2 g(T) \in (F[x])[T],
\]

where \( g(T) \) is a nonconstant monic polynomial in \((F[x])[T]\). Let \( f_{-1}(T) = 1, f_0(T) = T \) and \( f_n(T) = f(f_{n-1}(T)) \) for all \( n \geq 1 \), which induces \( f_n = f_1 \circ f_1 \circ \cdots \circ f_1 \) (\( n \) composites).

We first establish the specializable continued fraction representing the finite series

\[
\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} \quad (\ell \geq 0)
\]

as follows:

**Theorem 1.** If \( z \in F[x] \setminus F \) is monic, then \( \frac{1}{f_{-1}(z)f_0(z)} = [0; z] \), and for \( \ell \geq 0 \) if

\[
\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z)f_n(z)} = [0; \overline{X}_{k_\ell}]
\]

is a simple continued fraction expansion, then there exists \( \beta_\ell \in F \setminus \{0\} \) such that

\[
\sum_{n=0}^{\ell+1} \frac{(-1)^n}{f_{n-1}(z)f_n(z)} = [0; \overline{X}_{k_\ell}, y_\ell, -\overline{X}_{k_\ell}],
\]

where \( y_\ell := \frac{\beta_\ell^2 (-1)^\ell f_{\ell+1}(z)}{f_{\ell-1}(z)f_\ell(z)} \).
Proof. Let \( z \in F[x] \setminus F \) be monic. Clearly \( \frac{1}{f_{-1}(z)f_0(z)} = \frac{1}{z} = [0; z] \).

Let \( \ell \geq 0 \) and assume that the series \( \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} \) has the continued fraction expansion of the form
\[
\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} = [0; \overline{X_{k_\ell}}].
\]

Denote the \( k_\ell \)th convergent of the continued fraction \( [0; \overline{X_{k_\ell}}] \) by \( \frac{C_{k_\ell}}{D_{k_\ell}} \). By direct computation, one can verify that
\[
\frac{C_{k_\ell}}{D_{k_\ell}} = \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z)f_n(z)} := \frac{P_{k_\ell}}{Q_{k_\ell}},
\]
where \( P_{k_\ell} = \frac{f_{\ell-1}(z)f_\ell(z)}{z} - \frac{f_{\ell-2}(z)f_\ell(z)}{zf_1(z)} + \cdots + (-1)^{\ell-1} \frac{f_{\ell-1}(z)f_\ell(z)}{f_{\ell-2}(z)f_{\ell-1}(z)} + (-1)^\ell \) and \( Q_{k_\ell} = f_{\ell-1}(z)f_\ell(z) \). Observe that \( Q_{k_\ell} \) is monic and
\[
Q_{k_{\ell+1}} = f_\ell(z)f_{\ell+1}(z) = (f_{\ell-1}(z)f_\ell(z))g(f_\ell(z)) = Q_{k_\ell}g(f_{\ell-1}(z))g(f_\ell(z)),
\]
so \( Q_{k_{\ell+1}}Q_{k_\ell}^{-1} \in F[x] \setminus F \). Thus
\[
P_{k_\ell} = Q_{k_\ell}Q_{k_0}^{-1} - Q_{k_\ell}Q_{k_1}^{-1} + \cdots + (-1)^{\ell-1} Q_{k_\ell} Q_{k_{\ell-1}}^{-1} + (-1)^\ell \in F[x]
\]
is also monic. Clearly \( P_{k_\ell} \) and \( Q_{k_\ell} \) are relatively prime, then there exists \( \beta_\ell \in F \setminus \{0\} \) such that
\[
P_{k_\ell} = \beta_\ell C_{k_\ell} \quad \text{and} \quad Q_{k_\ell} = \beta_\ell D_{k_\ell}.
\]

Next, we claim that if \( z \in F[x] \setminus F \), then \( \frac{f_{\ell+1}(z)}{f_{\ell-1}(z)f_\ell(z)} \in F[x] \setminus F \) for all \( \ell \geq 0 \). To this end, let \( z \in F[x] \setminus F \). We get
\[
\frac{f_{\ell+1}(z)}{f_{\ell-1}(z)f_\ell(z)} = \frac{f_\ell(z)g(f_\ell(z))}{f_{\ell-1}(z)} = \frac{f_{\ell-1}(z)g(f_\ell(z))g(f_{\ell-1}(z))}{f_{\ell-1}(z)} = g(f_\ell(z))g(f_{\ell-1}(z)) \in F[x] \setminus F.
\]

We next show that \( D_{k_\ell}^2y_\ell = (-1)^\ell f_\ell(z)f_{\ell+1}(z) \). Consider
\[
D_{k_\ell}^2y_\ell = D_{k_\ell}^2 \left( \frac{\beta_\ell^2}{f_{\ell-1}(z)f_\ell(z)} \right).
\]
We observe that \( \{k_\ell\}_{\ell \geq 0} \) obtained by this process is a sequence of odd positive integers. Using Lemma 2, we get

\[
[0; \overrightarrow{X}_{k_\ell}, y_\ell, -\overrightarrow{X}_{k_\ell}] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell} y_\ell} \\
= \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z) f_n(z)} + \frac{(-1)^{\ell+1}}{f_\ell(z) f_{\ell+1}(z)} \\
= \sum_{n=0}^{\ell+1} \frac{(-1)^n}{f_{n-1}(z) f_n(z)},
\]

as desired. 

Note from the proof of the previous theorem that l.c. \((y_\ell) = (-1)^\ell\) for all \(\ell \geq 0\).

**Corollary 1.** Let the notation be as in Theorem 1. For \(\ell \geq 0\), we get

\[
\beta_\ell = \begin{cases} 
1 & \text{if } \ell \text{ is even,} \\
-1 & \text{if } \ell \text{ is odd.}
\end{cases}
\]

**Proof.** For \(\ell \geq 0\), we have \(P_{k_\ell} = \beta_\ell C_{k_\ell}\) and \(Q_{k_\ell} = \beta_\ell D_{k_\ell}\) for some \(\beta_\ell \in F \setminus \{0\}\).

Since \(\frac{C_{k_0}}{D_{k_0}} = [0; z]\), by Lemma 1,

\[
1 = \text{l.c.}(Q_{k_0}) = \beta_0 \text{l.c.}(D_{k_0}) = \beta_0 \text{l.c.}(z) = \beta_0.
\]

From the process in Theorem 1, we get

\[
\frac{C_{k_1}}{D_{k_1}} = [0; z, y_0, -z].
\]

By Lemma 1, we have \(1 = \text{l.c.}(Q_{k_1}) = \beta_1 \text{l.c.}(D_{k_1}) = -\beta_1 \text{l.c.}(z) = -\beta_1\), so \(\beta_1 = -1\).

Assume that

\[
\beta_t = \begin{cases} 
1 & \text{if } t \text{ is even,} \\
-1 & \text{if } t \text{ is odd.}
\end{cases}
\]
for \( t \geq 0 \). Let
\[
\frac{C_{k_t}}{D_{k_t}} = [0; X_{k_t}].
\]

By Lemma 1, we have \( 1 = \text{l.c.} (Q_{k_t}) = \beta_t \text{l.c.} (D_{k_t}) = \beta_t \text{l.c.} (X_{k_t}) \), so
\[
\text{l.c.} (X_{k_t}) = \beta_t^{-1}.
\]
(1)

Here \( \text{l.c.} (X_{k_t}) \) denotes \( \text{l.c.} (a_1 a_2 \cdots a_{k_t}) \) if \([0; X_{k_t}] = [0; a_1, a_2, \ldots, a_{k_t}]\). From the process in Theorem 1, we get
\[
\frac{C_{k_{t+1}}}{D_{k_{t+1}}} = [0; X_{k_t}, y_t, \overline{X}_{k_t}].
\]

By Lemma 1 and (1), we get
\[
1 = \text{l.c.} (Q_{k_{t+1}})
= \beta_{t+1} \text{l.c.} (D_{k_{t+1}})
= \beta_{t+1} \text{l.c.} (X_{k_t}) (-1)^t \beta_t^{-2}
= (-1)^{t+1} \beta_{t+1}
\]
since \( k_t \) is odd. Therefore \( \beta_{t+1} = (-1)^{t+1} \) and the proof of Corollary 1 is now complete.

\[\square\]

**Corollary 2.** The continued fraction representing the infinite sum \( \sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(z)f_n(z)} \) takes the form
\[
[0; \underbrace{\frac{f_1(z)}{z}, -z, \frac{f_2(z)}{z^2 f_1(z)^2}, z, \frac{f_1(z)}{z}, -z, \frac{f_3(z)}{f_1^2(z)f_2(z)}, \ldots}].
\]

**Example 1.** Let \( h \in \mathbb{N} \) and \( f (T) = T^{h+2} \in (F[x])[T] \), \( g (T) = T^h \). Let \( z := z (x) \in F[x] \setminus F \) be monic. Then
\[
\begin{align*}
  f_0 (z) &= z, \\
  f_1 (z) &= f (z) = z^{h+2}, \\
  f_2 (z) &= f (f_1 (z)) = f (z^{h+2}) = (z^{h+2})^{h+2} = z^{(h+2)^2}, \\
  & \vdots \\
  f_n (z) &= z^{(h+2)^n} \quad (n \geq 0).
\end{align*}
\]
Applying Theorem 1 and Corollary 2, we get
\[
\frac{1}{z} - \frac{1}{z^z} \frac{1}{z^{h+2}} = [0; z, z^{(h+2)-1}, -z],
\]
\[
\frac{1}{z} - \frac{1}{z^{h+2}} + \frac{1}{z^{h+2}} = [0; z, z^{(h+2)-1}, -z, z^{(h+2)^2-(h+2)^2}, z, -z^{(h+2)^2-(h+2)^2}, -z],
\]
\[
\vdots
\]
and
\[
\frac{1}{z} - \sum_{n=1}^{\infty} \frac{(-1)^n}{z^{(h+3)z^{(h+2)^n}}} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(-1)^n}{z^{(h+2)^n}},
\]
\[
= [0; z, z^{(h+2)-1}, -z, -z^{(h+2)^2-(h+2)^2}, z, -z^{(h+2)^2-(h+2)^2}, -z, z^{(h+2)^3-(h+2)^2-(h+2)^2}, \ldots].
\]

Next, the specializable continued fraction representing the finite series
\[
\sum_{n=0}^{\ell} \frac{1}{f_{n-1}(T)f_n(T)} \quad (\ell \geq 0)
\]
is shown below. Since the proof of Theorem 1 and Theorem 2 are mostly the same, we give here only the proof of the first theorem.

**Theorem 2.** If \(z \in F[x] \setminus F\) is monic, then \(\frac{1}{f_{-1}(z)f_0(z)} = [0; z]\), and for \(\ell \geq 0\) if
\[
\sum_{n=0}^{\ell} \frac{1}{f_{n-1}(z)f_n(z)} = [0; X_{k_1}]
\]
is a simple continued fraction expansion, then there exists \(\beta \in F \setminus \{0\}\) such that
\[
\bigg(\sum_{n=0}^{\ell+1} \frac{1}{f_{n-1}(z)f_n(z)} = [0; X_{k_1}, y, -X_{k_1}],
\]
where \(y := -\frac{\beta^2 f_{\ell+1}(z)}{f_{\ell-1}(z)f_\ell(z)}\).

**Corollary 3.** Let the notation be as in Theorem 2. Then \(\beta = 1\) for all \(\ell \geq 0\).

**Corollary 4.** The continued fraction representing the infinite sum \(\sum_{n=0}^{\infty} \frac{1}{f_{n-1}(z)f_n(z)}\) takes the form
\[
[0; z, -f_1(z), z, z, z, f_2(z), z, f_3(z), z, \ldots].
\]
Example 2. Let \( h \in \mathbb{N} \) and \( f(T) = T^{h+2} \in (F[x])[T] \), \( (g(T) = T^h) \). Let \( z := z(x) \in F[x] \setminus F \) be monic. Then
\[
\frac{1}{z} + \frac{1}{z^{h+2}} = [0; z, -z^{(h+2)^{-1}}, -z, \ldots],
\]
\[
\frac{1}{z} + \frac{1}{z^{h+2}} + \frac{1}{z^{2h+4}} z^{(h+2)^2} = [0; z, -z^{(h+2)^{-1}}, -z, -z^{(h+2)^2} - (h+2)^{-2}, z, z^{(h+2)^{-1}}, -z, \ldots]
\]
\[
\vdots
\]
and
\[
\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{(n+1)(h+2)^{n-1}}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{(h+2)^n}} = [0; z, -z^{(h+2)^{-1}}, -z, -z^{(h+2)^2} - (h+2)^{-2}, z, z^{(h+2)^{-1}}, -z, -z^{(h+2)^3} - (h+2)^2, \ldots].
\]

3. Another Proof of Theorem 2

In this part, we give another proof of Theorem 2. In order to do so, we prove the following theorem extending the idea of Rattanamoong et al. [8] and use it as our tool.

Theorem 3. Let \( \{W_i\}_{i=1}^\infty \) be a sequence of nonconstant monic polynomials over the field \( F \). Assume that there exists \( N \in \mathbb{N} \setminus \{0\} \) such that
\[
W_1 W_2 \cdots W_{j+1} \mid W_{j+2} \quad (j \geq N)
\] (2)
and that if \( N \geq 1 \), then
\[
\gcd((W_2 \cdots W_{N+1}) + (W_3 \cdots W_{N+1}) + \cdots + W_{N+1} + 1, W_1 W_2 \cdots W_{N+1}) = 1.
\] (3)

If
\[
\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} = [0; k_1, \ldots, k_1] \quad (\ell \geq 1),
\]
then
\[
\sum_{i=1}^{N+\ell+1} \frac{1}{W_1 W_2 \cdots W_i} = [0; k_1, \ldots, k_1, \frac{(-1)^{k_1} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}} - k_1].
\]

Proof. Let \( \ell \geq 1 \) and assume that the series \( \sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} \) has the continued fraction expansion of the form
\[
\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} = [0; k_1, \ldots, k_1].
\]
Denote the $k^\text{th}$ convergent of the continued fraction $[0; \overline{X}_{k_1}]$ by $\frac{C_{k_\ell}}{D_{k_\ell}}$. We observe that both $C_{k_\ell}$ and $D_{k_\ell}$ are monic. By direct computation, we have

$$\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} = \frac{(W_2 \cdots W_{N+\ell}) + (W_3 \cdots W_{N+\ell}) + \cdots + W_{N+\ell} + 1}{W_1 W_2 \cdots W_{N+\ell}}.$$ 

We first show that the right-hand side is the reduced fraction. Note that the case $N \geq 1$ and $\ell = 1$ is obvious by the assumption (3), so we treat the other two cases. We suppose for contradiction that there exists a prime $p \in F[x]$ such that $p$ is a common divisor of both

$$(W_2 \cdots W_{N+\ell}) + (W_3 \cdots W_{N+\ell}) + \cdots + W_{N+\ell} + 1 \text{ and } W_1 W_2 \cdots W_{N+\ell}.$$ 

**Case** $N = 0$ : By (2), we have $W_i \mid W_{i+1}$ for all $i \in \mathbb{N}$.

Since $p \mid W_1 W_2 \cdots W_\ell$, then $p \mid W_k$ for some $1 \leq k \leq \ell$. Therefore $p \mid W_j \cdots W_\ell$ for all $2 \leq j \leq \ell$. Since

$$p \mid ((W_2 \cdots W_\ell) + (W_3 \cdots W_\ell) + \cdots + W_\ell + 1),$$

then $p \mid 1$, which is a contradiction.

**Case** $N \geq 1$ and $\ell \geq 2$ : Since $p \mid W_1 W_2 \cdots W_{N+\ell}$, $p \mid W_k$ for some $1 \leq k \leq N + \ell$. If $p \mid W_{N+\ell}$, since $p \mid ((W_2 \cdots W_{N+\ell}) + (W_3 \cdots W_{N+\ell}) + \cdots + W_{N+\ell} + 1)$, then $p \mid 1$ which is a contradiction. Assume that $p \mid W_k$ for some $1 \leq k \leq N + \ell - 1$.

Using (2) when $j = N + \ell - 2 \geq N$, we get $W_1 W_2 \cdots W_{N+\ell-1} \mid W_{N+\ell}$, which implies that $p \mid W_{N+\ell}$. Again we have a contradiction.

Now we have $D_{k_\ell} = W_1 W_2 \cdots W_{N+\ell}$ since $C_{k_\ell}$ and $D_{k_\ell}$ are relatively prime and all $W_i$ are monic. Using (2) when $j = N + \ell - 1 \geq N$, we get

$$\frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}} \in F[x] \setminus \{0\}.$$ 

Applying Lemma 2, we get

$$[0; \overline{X}_{k_1}, \frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}, -R_{k_1}] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2} \frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}$$ 

$$= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{(W_1 W_2 \cdots W_{N+\ell})^2} \frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}$$ 

$$= \sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} \frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}$$ 

$$= \sum_{i=1}^{N+\ell+1} \frac{1}{W_1 W_2 \cdots W_i}.$$
and the proof is complete.

Now we are ready to present a proof of Theorem 2. Taking $W_1 = f_0(z) = z, W_2 = f_1(z),\]

\[
W_i = \frac{f_{i-1}(z)}{f_{i-3}(z)} = f_{i-3}(z)f_{i-2}(z)g(f_{i-3}(z))g(f_{i-2}(z)) \in F[x] \setminus F \quad (i \geq 3)
\]

and $N = 0$ in Theorem 3, we get $W_{j+2} = f_{j-1}(z)f_j(z)g(f_{j-1}(z))g(f_j(z))$ and

\[
W_1W_2 \cdots W_{j+1} = f_{j-1}(z)f_j(z)
\]

for $j \geq 0$. Therefore (2) is true. For $\ell \geq 1$, let $[0; \overline{X}_{k_{\ell}}]$ be the $k_{\ell}^{th}$ convergent of the continued fraction of

\[
\sum_{i=1}^{\ell} \frac{1}{W_1W_2 \cdots W_i} = \sum_{i=1}^{\ell} \frac{1}{f_{i-2}(z)f_{i-1}(z)} = \sum_{i=0}^{\ell-1} \frac{1}{f_{i-1}(z)f_i(z)}.
\]

Clearly

\[
\sum_{i=1}^{1} \frac{1}{W_1W_2 \cdots W_i} = \sum_{i=0}^{0} \frac{1}{f_{i-1}(z)f_i(z)} = \frac{1}{f_0(z)f_1(z)} = \frac{1}{z} = [0; z].
\]

By the process in Theorem 3, we have $\{k_{\ell}\}_{\ell \geq 1}$ is a sequence of odd positive integers. Consider

\[
\frac{(-1)^{k_{\ell}}W_{\ell+1}}{W_1W_2 \cdots W_{\ell}} = -g(f_{\ell-2}(z))g(f_{\ell-1}(z)) = -\frac{f_{\ell}(z)}{f_{\ell-2}(z)f_{\ell-1}(z)}.
\]

Therefore

\[
\sum_{i=0}^{\ell} \frac{1}{f_{i-1}(z)f_i(z)} = \sum_{i=1}^{\ell+1} \frac{1}{W_1W_2 \cdots W_i} = [0; \overline{X}_{k_{\ell}}, -\frac{f_{\ell}(z)}{f_{\ell-2}(z)f_{\ell-1}(z)}, \overline{X}_{k_{\ell}}].
\]

4. Remarks and Explanations

We observe that the series

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{f_{n-1}(T)f_n(T)},
\]

where $T$ is nonconstant monic polynomial in $F[x]$, are irrational since their continued fractions are infinite. Although it is difficult to determine whether they are algebraic or transcendental, we exhibit the examples showing that the series can not be algebraic of some certain degrees. To do so, we recall the theorem of Mahler [4], stated as follows.
Theorem 4. If an element $\alpha$ of $F \left( \left( x^{-1} \right) \right)$ is algebraic of degree $n \geq 2$ over $F \left( x \right)$, then there exists a constant $c > 0$ such that

$$\left| \alpha - \frac{a}{b} \right| \geq \frac{c}{|b|^n}$$

for all pairs of elements $a$ and $b \neq 0$ of $F \left[ x \right]$.

Throughout write $g \left( T \right) = T^n + a_{h-1}T^{h-1} + \cdots + a_1T + a_0$. Assume that

$$\max_{0 \leq i \leq h-1} |a_i| < |T|.$$ 

Therefore $|g \left( T \right)| = |T|^h$.

Example 3. Let $z := z \left( x \right) \in F \left[ x \right] \setminus F$ be monic and $\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1} \left( z \right) f_n \left( z \right)}.$

For each $\ell \in \mathbb{N}$, let

$$C_{k\ell} = \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{\ell} \left( z \right) f_{\ell+1} \left( z \right)}.$$ 

By the proof of Theorem 1, we have $|D_{k\ell}| = |f_{\ell-1} \left( z \right) f_\ell \left( z \right)|$ ($\ell \geq 1$). From Proposition 1 (iv), for $\ell \geq 1$, we get

$$\left| \alpha - \frac{C_{k\ell}}{D_{k\ell}} \right| = \left| \frac{(-1)^{\ell+1}}{f_\ell \left( z \right) f_{\ell+1} \left( z \right)} + \frac{(-1)^{\ell+2}}{f_{\ell+1} \left( z \right) f_{\ell+2} \left( z \right)} + \cdots \right|$$

$$= \frac{1}{|D_{k\ell+1}|}$$

$$= \frac{1}{|f_\ell \left( z \right) f_{\ell+1} \left( z \right)|}$$

$$= \frac{1}{|f_{\ell-1}^2 \left( z \right) f_\ell^2 \left( z \right) g \left( f_{\ell-1} \left( z \right) \right) g \left( f_\ell \left( z \right) \right)|}$$

$$= \frac{1}{|D_{k\ell}|^3}.$$ 

If $\alpha$ is algebraic of degree $d \geq 2$, then, by Theorem 4, there is a constant $c > 0$ such that

$$\frac{c}{|D_{k\ell}|^d} \leq \left| \alpha - \frac{C_{k\ell}}{D_{k\ell}} \right| = \frac{1}{|D_{k\ell}|^3},$$

which is a contradiction if $2 \leq d \leq h + 1$ and $\ell$ is sufficiently large. Hence $\alpha$ can not be algebraic of degree at most $h + 1$.

Example 4. Let $z := z \left( x \right) \in F \left[ x \right] \setminus F$ be monic and $\alpha = \sum_{n=0}^{\infty} \frac{1}{f_{n-1} \left( z \right) f_n \left( z \right)}$.

For each $\ell \in \mathbb{N}$, let

$$C_{k\ell} = \sum_{n=0}^{\ell} \frac{1}{f_{n-1} \left( z \right) f_n \left( z \right)}.$$
With similar arguments as in Example 3, $\alpha$ cannot be algebraic of degree at most $h + 1$.

References


