CONTINUED FRACTIONS OF CERTAIN SERIES

Jittinart Rattanamoong
Department of Mathematics, Srinakharinwirot University, Bangkok, Thailand
jittinart@g.swu.ac.th

Tuangrat Chaichana
Department of Mathematics and Computer Science, Chulalongkorn University, Bangkok, Thailand
tuangrat.c@chula.ac.th

Received: 2/4/19, Accepted: 11/6/19, Published: 12/6/19

Abstract

We study a specific polynomial $f(T) = T^2 g(T)$ where $g(T)$ is a nonconstant monic polynomial in $(F(x))[T]$ and F is a field. Setting $f_{-1}(T) = 1, f_0(T) = T$ and $f_n(T) = f(f_{n-1}(T))$ $(n \geq 1)$, we then have explicit formulas for the continued fractions of certain series in the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(T) f_n(T)} \text{ and } \sum_{n=0}^{\infty} \frac{1}{f_{n-1}(T) f_n(T)}.$$

1. Introduction

The problems involved in the relationship between series and continued fractions have a long history. The area of research in this direction deals with the determination of the closed forms for continued fractions representing some certain series in a particular field, see e.g. [1], [2], [3], [5], [7], [9], [11], [12], [15], [16] and [17]. For instance, in 1973, Lehmer [3] showed that the modified Bessel functions can be represented by the simple continued fractions whose partial quotients form arithmetic progression. In 2006, Riyapan, Laohakosol and Chaichana [9] gave another proof of Lehmer's result and they also found other types of continued fractions with explicit shape. In 1996, Cohn [1] introduced specializable continued fractions over $\mathbb{Q}(x)$, the continued fractions with the property that each partial quotient has integer coefficients, and gave a complete classification of polynomials $f(T) \in \mathbb{Z}[T]$ for which the series expansions $\sum_{n=0}^{\infty} \frac{1}{f_n(T)}$ have specializable continued fractions.

Here we are particularly interested in the result of Tamura in 1991, see [13]. Let $f_0(T) = T$ and $f_n(T) := f(f_{n-1}(T))$ be the n^{th} composite iterate of a polynomial $f(T)$. He found the pattern of the simple continued fractions for the series of real numbers in the form

$$\sum_{n=0}^{\infty} \frac{1}{f_0(T) f_1(T) \cdots f_n(T)}$$

for

$$f(T) = f^{(1)}(T) = T(T+2)(T-2) g^{(1)}(T) + T^2 - 2 \quad \text{and}$$
$$f(T) = f^{(2)}(T) = T^2(T+2)(T-2) g^{(2)}(T) + T^2 - 2$$

where $g^{(1)}(T), g^{(2)}(T) \in \mathbb{Z}[T]$ and $T \in \mathbb{Z}$. Later, in [8] Rattanamoong, Laohakosol and Chaichana extended and modified Tamura's result in the case of formal Laurent series over a field F of characteristic 0. They discovered the specializable continued fractions for the series in the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_0(T) f_1(T) \cdots f_n(T)},$$

where $f(T) = T(T+2)(T-2) g(T) - T^2 + 2$ and $T \in F[x] \setminus F$.

Motivated by the above mentioned works, we aim to determine explicit formulas for simple continued fractions of two classes of elements in the field of Laurent series over F having series expansions in the forms

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(T) f_n(T)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{f_{n-1}(T) f_n(T)}.$$

Let $F((x^{-1}))$ denote the field of formal Laurent series over a field F equipped with a degree valuation $|\cdot|$ defined by $|x^{-1}| = e^{-1}$. It is well known, see [10], that each element $\alpha \in F((x^{-1}))$ can be uniquely expressible as the simple continued fraction in the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} =: [a_0; a_1, a_2, \ldots],$$

where $a_0 \in F[x]$ and $a_n \in F[x] \setminus F$ $(n \geq 1)$. Note that the continued fraction of α is finite if and only if $\alpha \in F(x)$. Define two sequences (C_n), (D_n) by

$$C_{-1} = 1, \quad C_0 = a_0, \quad C_{n+1} = a_{n+1} C_n + C_{n-1} \quad (n \geq 0)$$
$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = a_{n+1} D_n + D_{n-1} \quad (n \geq 0).$$

The following proposition is easily established by induction so we state it without proof.

Proposition 1. *Let $n \in \mathbb{N} \cup \{0\}$ and $\beta \in F((x^{-1})) \setminus \{0\}$. With the above notation, we have*

(i) $\dfrac{\beta C_n + C_{n-1}}{\beta D_n + D_{n-1}} = [a_0; a_1, a_2, \ldots, a_n, \beta]$,

(ii) $\dfrac{C_n}{D_n} = [a_0; a_1, a_2, \ldots, a_n]$,

(iii) $D_n C_{n-1} - C_n D_{n-1} = (-1)^n$, *so that* C_n *and* D_n *are relatively prime,*

(iv) $|D_n| > |D_{n-1}|$,

(v) $|C_n| > |C_{n-1}| \quad (n \geq 1)$.

From Proposition 1 (ii), we have

$$\frac{C_n}{D_n} = [a_0; a_1, a_2, \ldots, a_n] \qquad (n \geq 0).$$

We call C_n/D_n the n^{th} convergent of the continued fraction of α.

For each polynomial $h(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in F[x], c_n \neq 0$, we define $l.c.(h) := c_n$, which is the leading coefficient of $h(x)$.

Lemma 1. *Let $n \in \mathbb{N}$ and $n \geq 2$. If $C_n/D_n = [0; a_1, a_2, \ldots, a_n]$, then*

$$l.c.(C_n) = l.c.(a_n a_{n-1} \cdots a_2) \quad \text{and} \quad l.c.(D_n) = l.c.(a_n a_{n-1} \cdots a_2 a_1).$$

Proof. Note that $C_1 = 1$ and $D_1 = a_1$. Let $n \in \mathbb{N}$ and $n \geq 2$. By Proposition 1, we have

$$\begin{aligned} C_n &= a_n C_{n-1} + C_{n-2} \\ &= a_n (a_{n-1} C_{n-2} + C_{n-3}) + C_{n-2} \\ &= a_n a_{n-1} C_{n-2} + a_n C_{n-3} + C_{n-2} \\ &\vdots \\ &= a_n a_{n-1} \cdots a_2 C_1 + \text{terms of lower degrees} \\ &= a_n a_{n-1} \cdots a_2 + \text{terms of lower degrees.} \end{aligned}$$

Similarly, we also have

$$D_n = a_n a_{n-1} \cdots a_2 a_1 + \text{terms of lower degrees.}$$

Therefore we get $l.c.(C_n) = l.c.(a_n a_{n-1} \cdots a_2)$ and $l.c.(D_n) = l.c.(a_n a_{n-1} \cdots a_2 a_1)$. □

The next result which is an important tool for possessing folding symmetry of continued fractions, is classically known as the Folding Lemma, see [6].

Lemma 2. *Let* $y \in F[x] \setminus \{0\}$*, and*

$$\frac{C_n}{D_n} = [0; a_1, a_2, \ldots, a_n] := \left[0; \overrightarrow{X}_n\right] \quad (n \geq 1).$$

Then

$$\left[0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n\right] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y}.$$

Here $\overrightarrow{X}_n$ *is an abbreviation for the word* $a_1, a_2, \ldots, a_n$*, and accordingly,* $-\overleftarrow{X}_n$ *denotes the word* $-a_n, -a_{n-1}, \ldots, -a_1$*.*

2. Main Results

Let

$$f(T) = T^2 g(T) \in (F[x])[T],$$

where $g(T)$ is a nonconstant monic polynomial in $(F[x])[T]$. Let $f_{-1}(T) = 1, f_0(T) = T$ and $f_n(T) = f(f_{n-1}(T))$ for all $n \geq 1$, which induces $f_n = f_1 \circ f_1 \circ \cdots \circ f_1$ (n composites).

We first establish the specializable continued fraction representing the finite series

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(T) f_n(T)} \quad (\ell \geq 0)$$

as follows:

Theorem 1. *If* $z \in F[x] \setminus F$ *is monic, then* $\dfrac{1}{f_{-1}(z) f_0(z)} = [0; z]$*, and for* $\ell \geq 0$ *if*

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z) f_n(z)} = [0; \overrightarrow{X}_{k_\ell}]$$

is a simple continued fraction expansion, then there exists $\beta_\ell \in F \setminus \{0\}$ *such that*

$$\sum_{n=0}^{\ell+1} \frac{(-1)^n}{f_{n-1}(z) f_n(z)} = [0; \overrightarrow{X}_{k_\ell}, y_\ell, -\overleftarrow{X}_{k_\ell}],$$

where $y_\ell := \dfrac{\beta_\ell^2 (-1)^\ell f_{\ell+1}(z)}{f_{\ell-1}^2(z) f_\ell(z)}$.

Proof. Let $z \in F[x] \setminus F$ be monic. Clearly $\dfrac{1}{f_{-1}(z)f_0(z)} = \dfrac{1}{z} = [0; z]$.

Let $\ell \geq 0$ and assume that the series $\displaystyle\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(T)f_n(T)}$ has the continued fraction expansion of the form

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} = [0; \overrightarrow{X}_{k_\ell}].$$

Denote the $k_\ell{}^{\text{th}}$ convergent of the continued fraction $[0; \overrightarrow{X}_{k_\ell}]$ by $\dfrac{C_{k_\ell}}{D_{k_\ell}}$. By direct computation, one can verify that

$$\frac{C_{k_\ell}}{D_{k_\ell}} = \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z)f_n(z)} := \frac{P_{k_\ell}}{Q_{k_\ell}},$$

where $P_{k_\ell} = \dfrac{f_{\ell-1}(z)f_\ell(z)}{z} - \dfrac{f_{\ell-1}(z)f_\ell(z)}{zf_1(z)} + \cdots + \dfrac{(-1)^{\ell-1} f_{\ell-1}(z)f_\ell(z)}{f_{\ell-2}(z)f_{\ell-1}(z)} + (-1)^\ell$ and $Q_{k_\ell} = f_{\ell-1}(z)f_\ell(z)$. Observe that Q_{k_ℓ} is monic and

$$Q_{k_{\ell+1}} = f_\ell(z)f_{\ell+1}(z) = (f_{\ell-1}(z)f_\ell(z))^2 g(f_{\ell-1}(z))\, g(f_\ell(z)) = Q_{k_\ell}^2 g(f_{\ell-1}(z))\, g(f_\ell(z)),$$

so $Q_{k_{\ell+1}} Q_{k_\ell}^{-1} \in F[x] \setminus F$. Thus

$$P_{k_\ell} = Q_{k_\ell} Q_{k_0}^{-1} - Q_{k_\ell} Q_{k_1}^{-1} + \cdots + (-1)^{\ell-1} Q_{k_\ell} Q_{k_{\ell-1}}^{-1} + (-1)^\ell \in F[x]$$

is also monic. Clearly P_{k_ℓ} and Q_{k_ℓ} are relatively prime, then there exists $\beta_\ell \in F \setminus \{0\}$ such that

$$P_{k_\ell} = \beta_\ell C_{k_\ell} \quad \text{and} \quad Q_{k_\ell} = \beta_\ell D_{k_\ell}.$$

Next, we claim that if $z \in F[x] \setminus F$, then $\dfrac{f_{\ell+1}(z)}{f_{\ell-1}^2(z)f_\ell(z)} \in F[x] \setminus F$ for all $\ell \geq 0$. To this end, let $z \in F[x] \setminus F$. We get

$$\begin{aligned}
\frac{f_{\ell+1}(z)}{f_{\ell-1}^2(z)f_\ell(z)} &= \frac{f_\ell(z)g(f_\ell(z))}{f_{\ell-1}^2(z)} \\
&= \frac{f_{\ell-1}^2(z)g(f_\ell(z))g(f_{\ell-1}(z))}{f_{\ell-1}^2(z)} \\
&= g(f_\ell(z))g(f_{\ell-1}(z)) \in F[x] \setminus F.
\end{aligned}$$

We next show that $D_{k_\ell}^2 y_\ell = (-1)^\ell f_\ell(z)f_{\ell+1}(z)$. Consider

$$D_{k_\ell}^2 y_\ell = D_{k_\ell}^2 \left(\frac{\beta_\ell^2 (-1)^\ell f_{\ell+1}(z)}{f_{\ell-1}^2(z)f_\ell(z)} \right)$$

$$= \left(\frac{Q_{k_\ell}^2}{\beta_\ell^2}\right)\left(\frac{\beta_\ell^2\,(-1)^\ell\, f_{\ell+1}(z)}{f_{\ell-1}^2(z) f_\ell(z)}\right)$$
$$= f_{\ell-1}^2(z) f_\ell^2(z)\left(\frac{(-1)^\ell\, f_{\ell+1}(z)}{f_{\ell-1}^2(z) f_\ell(z)}\right)$$
$$= (-1)^\ell\, f_\ell(z) f_{\ell+1}(z).$$

We observe that $\{k_\ell\}_{\ell\geq 0}$ obtained by this process is a sequence of odd positive integers. Using Lemma 2, we get

$$\begin{aligned}[0; \overrightarrow{X}_{k_\ell}, y_\ell, -\overleftarrow{X}_{k_\ell}] &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2 y_\ell} \\ &= \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z) f_n(z)} + \frac{(-1)^{\ell+1}}{f_\ell(z) f_{\ell+1}(z)} \\ &= \sum_{n=0}^{\ell+1} \frac{(-1)^n}{f_{n-1}(z) f_n(z)},\end{aligned}$$

as desired. □

Note from the proof of the previous theorem that $l.c.\,(y_\ell) = (-1)^\ell$ for all $\ell \geq 0$.

Corollary 1. *Let the notation be as in Theorem 1. For $\ell \geq 0$, we get*

$$\beta_\ell = \begin{cases} 1 & ; \textit{if} \;\; \ell \;\; \textit{is even}, \\ -1 & ; \textit{if} \;\; \ell \;\; \textit{is odd}. \end{cases}$$

Proof. For $\ell \geq 0$, we have $P_{k_\ell} = \beta_\ell C_{k_\ell}$ and $Q_{k_\ell} = \beta_\ell D_{k_\ell}$ for some $\beta_\ell \in F \setminus \{0\}$. Since $\dfrac{C_{k_0}}{D_{k_0}} = [0; z]$, by Lemma 1,

$$1 = l.c.\,(Q_{k_0}) = \beta_0 l.c.\,(D_{k_0}) = \beta_0 l.c.\,(z) = \beta_0.$$

From the process in Theorem 1, we get

$$\frac{C_{k_1}}{D_{k_1}} = [0; z, y_0, -z].$$

By Lemma 1, we have $1 = l.c.\,(Q_{k_1}) = \beta_1 l.c.\,(D_{k_1}) = -\beta_1 \;\, l.c.\,(z^2) = -\beta_1$, so $\beta_1 = -1$.

Assume that

$$\beta_t = \begin{cases} 1 & ; \text{if} \;\; t \;\; \text{is even}, \\ -1 & ; \text{if} \;\; t \;\; \text{is odd} \end{cases}$$

for $t \geq 0$. Let

$$\frac{C_{k_t}}{D_{k_t}} = [0; \overrightarrow{X}_{k_t}].$$

By Lemma 1, we have $1 = l.c.\,(Q_{k_t}) = \beta_t l.c.\,(D_{k_t}) = \beta_t l.c.\left(\overrightarrow{X}_{k_t}\right)$, so

$$l.c.\left(\overrightarrow{X}_{k_t}\right) = \beta_t^{-1}. \tag{1}$$

Here $l.c.\left(\overrightarrow{X}_{k_t}\right)$ denotes $l.c.\,(a_1 a_2 \cdots a_{k_t})$ if $[0; \overrightarrow{X}_{k_t}] = [0; a_1, a_2, \ldots, a_{k_t}]$. From the process in Theorem 1, we get

$$\frac{C_{k_{t+1}}}{D_{k_{t+1}}} = [0; \overrightarrow{X}_{k_t}, y_t, -\overleftarrow{X}_{k_t}].$$

By Lemma 1 and (1), we get

$$\begin{aligned} 1 &= l.c.\left(Q_{k_{t+1}}\right) \\ &= \beta_{t+1} l.c.\left(D_{k_{t+1}}\right) \\ &= \beta_{t+1} l.c.\left(\overrightarrow{X}_{k_t}\right) (-1)^t (-1)^{k_t}\, l.c.\left(\overrightarrow{X}_{k_t}\right) \\ &= \beta_{t+1} (-1)^t (-1)^{k_t} \beta_t^{-2} \\ &= (-1)^{t+1} \beta_{t+1} \end{aligned}$$

since k_t is odd. Therefore $\beta_{t+1} = (-1)^{t+1}$ and the proof of Corollary 1 is now complete. □

Corollary 2. *The continued fraction representing the infinite sum* $\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(z) f_n(z)}$ *takes the form*

$$[0; \underbrace{\underbrace{z}, \frac{f_1(z)}{z}, \underbrace{-z}}, -\frac{f_2(z)}{z^2 f_1(z)}, \underbrace{z, -\frac{f_1(z)}{z}, -z}, \frac{f_3(z)}{f_1^2(z) f_2(z)}, \ldots].$$

Example 1. Let $h \in \mathbb{N}$ and $f(T) = T^{h+2} \in (F[x])[T]$, $\left(g(T) = T^h\right)$. Let $z := z(x) \in F[x] \setminus F$ be monic. Then

$$\begin{aligned} f_0(z) &= z, \\ f_1(z) &= f(z) = z^{h+2}, \\ f_2(z) &= f(f_1(z)) = f\left(z^{h+2}\right) = \left(z^{h+2}\right)^{h+2} = z^{(h+2)^2}, \\ &\vdots \\ f_n(z) &= z^{(h+2)^n} \quad (n \geq 0). \end{aligned}$$

Applying Theorem 1 and Corollary 2, we get

$$\frac{1}{z} - \frac{1}{z\ z^{h+2}} = [0; z, z^{(h+2)-1}, -z],$$

$$\frac{1}{z} - \frac{1}{z\ z^{h+2}} + \frac{1}{z^{h+2}\ z^{(h+2)^2}} = [0; z, z^{(h+2)-1}, -z, -z^{(h+2)^2-(h+2)-2}, z, -z^{(h+2)-1}, -z],$$

$$\vdots$$

and

$$\frac{1}{z} - \sum_{n=1}^{\infty} \frac{(-1)^n}{z^{(h+3)\cdot(h+2)^{n-1}}} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(-1)^n}{z^{(h+2)^{n-1}} z^{(h+2)^n}}$$

$$= [0; z, z^{(h+2)-1}, -z, -z^{(h+2)^2-(h+2)-2}, z, -z^{(h+2)-1}, -z, z^{(h+2)^3-(h+2)^2-(h+2)}, \dots].$$

Next, the specializable continued fraction representing the finite series

$$\sum_{n=0}^{\ell} \frac{1}{f_{n-1}(T) f_n(T)} \quad (\ell \geq 0)$$

is shown below. Since the proof of Theorem 1 and Theorem 2 are mostly the same, we give here only the proof of the first theorem.

Theorem 2. *If $z \in F[x] \setminus F$ is monic, then $\frac{1}{f_{-1}(z) f_0(z)} = [0; z]$, and for $\ell \geq 0$ if*

$$\sum_{n=0}^{\ell} \frac{1}{f_{n-1}(z) f_n(z)} = [0; \overrightarrow{X}_{k_\ell}]$$

is a simple continued fraction expansion, then there exists $\beta_\ell \in F \setminus \{0\}$ such that

$$\sum_{n=0}^{\ell+1} \frac{1}{f_{n-1}(z) f_n(z)} = [0; \overrightarrow{X}_{k_\ell}, y_\ell, -\overleftarrow{X}_{k_\ell}],$$

where $y_\ell := -\frac{\beta_\ell^2 f_{\ell+1}(z)}{f_{\ell-1}^2(z) f_\ell(z)}$.

Corollary 3. *Let the notation be as in Theorem 2. Then $\beta_\ell = 1$ for all $\ell \geq 0$.*

Corollary 4. *The continued fraction representing the infinite sum $\sum_{n=0}^{\infty} \frac{1}{f_{n-1}(z) f_n(z)}$ takes the form*

$$[0; \underbrace{\underbrace{z}, -\frac{f_1(z)}{z}, \underbrace{-z}}, -\frac{f_2(z)}{z^2 f_1(z)}, \underbrace{z, \frac{f_1(z)}{z}, -z}, -\frac{f_3(z)}{f_1^2(z) f_2(z)}, \dots].$$

Example 2. Let $h \in \mathbb{N}$ and $f(T) = T^{h+2} \in (F[x])[T]$, $\left(g(T) = T^h\right)$. Let $z := z(x) \in F[x] \setminus F$ be monic. Then

$$f_n(z) = z^{(h+2)^n} \quad (n \geq 0).$$

Applying Theorem 2 and Corollary 4, we get

$$\frac{1}{z} + \frac{1}{z\ z^{h+2}} = [0; z, -z^{(h+2)-1}, -z],$$

$$\frac{1}{z} + \frac{1}{z\ z^{h+2}} + \frac{1}{z^{h+2}\ z^{(h+2)^2}} = [0; z, -z^{(h+2)-1}, -z, -z^{(h+2)^2-(h+2)-2}, z, z^{(h+2)-1}, -z]$$

$$\vdots$$

and

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{(h+3)\cdot(h+2)^{n-1}}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{(h+2)^{n-1}}\ z^{(h+2)^n}}$$

$$= [0; z, -z^{(h+2)-1}, -z, -z^{(h+2)^2-(h+2)-2}, z, z^{(h+2)-1}, -z, -z^{(h+2)^3-(h+2)^2-(h+2)}, \ldots].$$

3. Another Proof of Theorem 2

In this part, we give another proof of Theorem 2. In order to do so, we prove the following theorem extending the idea of Rattanamoong et al. [8] and use it as our tool.

Theorem 3. *Let $\{W_i\}_{i\geq 1}$ be a sequence of nonconstant monic polynomials over the field F. Assume that there exists $N \in \mathbb{N} \cup \{0\}$ such that*

$$W_1 W_2 \cdots W_{j+1} \mid W_{j+2} \quad (j \geq N) \tag{2}$$

and that if $N \geq 1$, then

$$\gcd\left((W_2 \cdots W_{N+1}) + (W_3 \cdots W_{N+1}) + \cdots + W_{N+1} + 1, W_1 W_2 \cdots W_{N+1}\right) = 1. \tag{3}$$

If $\displaystyle\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} = \left[0; \overrightarrow{X}_{k_\ell}\right] \quad (\ell \geq 1)$, *then*

$$\sum_{i=1}^{N+\ell+1} \frac{1}{W_1 W_2 \cdots W_i} = [0; \overrightarrow{X}_{k_\ell}, \frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}, -\overleftarrow{X}_{k_\ell}].$$

Proof. Let $\ell \geq 1$ and assume that the series $\displaystyle\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i}$ has the continued fraction expansion of the form

$$\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} = \left[0; \overrightarrow{X}_{k_\ell}\right].$$

Denote the k_ℓ^{th} convergent of the continued fraction $[0; \overrightarrow{X}_{k_\ell}]$ by $\dfrac{C_{k_\ell}}{D_{k_\ell}}$. We observe that both C_{k_ℓ} and D_{k_ℓ} are monic. By direct computation, we have

$$\sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} = \frac{(W_2 \cdots W_{N+\ell}) + (W_3 \cdots W_{N+\ell}) + \cdots + W_{N+\ell} + 1}{W_1 W_2 \cdots W_{N+\ell}}.$$

We first show that the right-hand side is the reduced fraction. Note that the case $N \geq 1$ and $\ell = 1$ is obvious by the assumption (3), so we treat the other two cases. We suppose for contradiction that there exists a prime $p \in F[x]$ such that p is a common divisor of both

$$(W_2 \cdots W_{N+\ell}) + (W_3 \cdots W_{N+\ell}) + \cdots + W_{N+\ell} + 1 \text{ and } W_1 W_2 \cdots W_{N+\ell}.$$

`Case` $N = 0$: By (2), we have $W_i \mid W_{i+1}$ for all $i \in \mathbb{N}$.
Since $p \mid W_1 W_2 \cdots W_\ell$, then $p \mid W_k$ for some $1 \leq k \leq \ell$. Therefore $p \mid W_j \cdots W_\ell$ for all $2 \leq j \leq \ell$. Since

$$p \mid ((W_2 \cdots W_\ell) + (W_3 \cdots W_\ell) + \cdots + W_\ell + 1),$$

then $p \mid 1$, which is a contradiction.

`Case` $N \geq 1$ `and` $\ell \geq 2$: Since $p \mid W_1 W_2 \cdots W_{N+\ell}$, $p \mid W_k$ for some $1 \leq k \leq N + \ell$. If $p \mid W_{N+\ell}$, since $p \mid ((W_2 \cdots W_{N+\ell}) + (W_3 \cdots W_{N+\ell}) + \cdots + W_{N+\ell} + 1)$, then $p \mid 1$ which is a contradiction. Assume that $p \mid W_k$ for some $1 \leq k \leq N + \ell - 1$. Using (2) when $j = N + \ell - 2 \geq N$, we get $W_1 W_2 \cdots W_{N+\ell-1} \mid W_{N+\ell}$, which implies that $p \mid W_{N+\ell}$. Again we have a contradiction.

Now we have $D_{k_\ell} = W_1 W_2 \cdots W_{N+\ell}$ since C_{k_ℓ} and D_{k_ℓ} are relatively prime and all W_i are monic. Using (2) when $j = N + \ell - 1 \geq N$, we get

$$\frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}} \in F[x] \setminus \{0\}.$$

Applying Lemma 2, we get

$$\begin{aligned}
[0; \overrightarrow{X}_{k_\ell}, \frac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}, -\overleftarrow{X}_{k_\ell}] &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2 \dfrac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{(W_1 W_2 \cdots W_{N+\ell})^2 \dfrac{(-1)^{k_\ell} W_{N+\ell+1}}{W_1 W_2 \cdots W_{N+\ell}}} \\
&= \sum_{i=1}^{N+\ell} \frac{1}{W_1 W_2 \cdots W_i} + \frac{1}{W_1 W_2 \cdots W_{N+\ell} W_{N+\ell+1}} \\
&= \sum_{i=1}^{N+\ell+1} \frac{1}{W_1 W_2 \cdots W_i},
\end{aligned}$$

and the proof is complete. □

Now we are ready to present a proof of Theorem 2. Taking $W_1 = f_0(z) = z, W_2 = f_1(z)$,

$$W_i = \frac{f_{i-1}(z)}{f_{i-3}(z)} = f_{i-3}(z)f_{i-2}(z)g\left(f_{i-3}(z)\right)g\left(f_{i-2}(z)\right) \in F[x] \setminus F \quad (i \geq 3)$$

and $N = 0$ in Theorem 3, we get $W_{j+2} = f_{j-1}(z)f_j(z)g\left(f_{j-1}(z)\right)g\left(f_j(z)\right)$ and

$$W_1W_2 \cdots W_{j+1} = f_{j-1}(z)f_j(z)$$

for $j \geq 0$. Therefore (2) is true. For $\ell \geq 1$, let $[0; \overrightarrow{X}_{k_\ell}]$ be the k_ℓ^{th} convergent of the continued fraction of

$$\sum_{i=1}^{\ell} \frac{1}{W_1W_2 \cdots W_i} = \sum_{i=1}^{\ell} \frac{1}{f_{i-2}(z)f_{i-1}(z)} = \sum_{i=0}^{\ell-1} \frac{1}{f_{i-1}(z)f_i(z)}.$$

Clearly

$$\sum_{i=1}^{1} \frac{1}{W_1W_2 \cdots W_i} = \sum_{i=0}^{0} \frac{1}{f_{i-1}(z)f_i(z)} = \frac{1}{f_{-1}(z)f_0(z)} = \frac{1}{z} = [0; z].$$

By the process in Theorem 3, we have $\{k_\ell\}_{\ell \geq 1}$ is a sequence of odd positive integers. Consider

$$\frac{(-1)^{k_\ell}W_{\ell+1}}{W_1W_2 \cdots W_\ell} = -g\left(f_{\ell-2}(z)\right)g\left(f_{\ell-1}(z)\right) = -\frac{f_\ell(z)}{f_{\ell-2}^2(z)f_{\ell-1}(z)}.$$

Therefore

$$\sum_{i=0}^{\ell} \frac{1}{f_{i-1}(z)f_i(z)} = \sum_{i=1}^{\ell+1} \frac{1}{W_1W_2 \cdots W_i} = [0; \overrightarrow{X}_{k_\ell}, -\frac{f_\ell(z)}{f_{\ell-2}^2(z)f_{\ell-1}(z)}, -\overleftarrow{X}_{k_\ell}].$$

4. Remarks and Explanations

We observe that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(T)f_n(T)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{f_{n-1}(T)f_n(T)},$$

where T is nonconstant monic polynomial in $F\left[x\right]$, are irrational since their continued fractions are infinite. Although it is difficult to determine whether they are algebraic or transcendental, we exhibit the examples showing that the series can not be algebraic of some certain degrees. To do so, we recall the theorem of Mahler [4], stated as follows.

Theorem 4. *If an element α of $F\left(\left(x^{-1}\right)\right)$ is algebraic of degree $n \geq 2$ over $F(x)$, then there exists a constant $c > 0$ such that*

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{c}{|b|^n}$$

for all pairs of elements a and $b \neq 0$ of $F[x]$.

Throughout write $g(T) = T^h + a_{h-1}T^{h-1} + \cdots + a_1 T + a_0$. Assume that

$$\max_{0 \leq i \leq h-1} |a_i| < |T|.$$

Therefore $|g(T)| = |T|^h$.

Example 3. Let $z := z(x) \in F[x] \setminus F$ be monic and $\alpha = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{f_{n-1}(z) f_n(z)}$.

For each $\ell \in \mathbb{N}$, let

$$\frac{C_{k_\ell}}{D_{k_\ell}} = \sum_{n=0}^{\ell} \frac{(-1)^n}{f_{n-1}(z) f_n(z)}.$$

By the proof of Theorem 1, we have $|D_{k_\ell}| = |f_{\ell-1}(z) f_\ell(z)|$ $(\ell \geq 1)$. From Proposition 1 (iv), for $\ell \geq 1$, we get

$$\begin{aligned}
\left|\alpha - \frac{C_{k_\ell}}{D_{k_\ell}}\right| &= \left|\frac{(-1)^{\ell+1}}{f_\ell(z) f_{\ell+1}(z)} + \frac{(-1)^{\ell+2}}{f_{\ell+1}(z) f_{\ell+2}(z)} + \cdots\right| \\
&= \frac{1}{|D_{k_{\ell+1}}|} \\
&= \frac{1}{|f_\ell(z) f_{\ell+1}(z)|} \\
&= \frac{1}{\left|f_{\ell-1}^2(z) f_\ell^2(z) g(f_{\ell-1}(z)) g(f_\ell(z))\right|} \\
&= \frac{1}{|D_{k_\ell}|^{h+2}}.
\end{aligned}$$

If α is algebraic of degree $d \geq 2$, then, by Theorem 4, there is a constant $c > 0$ such that

$$\frac{c}{|D_{k_\ell}|^d} \leq \left|\alpha - \frac{C_{k_\ell}}{D_{k_\ell}}\right| = \frac{1}{|D_{k_\ell}|^{h+2}},$$

which is a contradiction if $2 \leq d \leq h+1$ and ℓ is sufficiently large. Hence α can not be algebraic of degree at most $h+1$.

Example 4. Let $z := z(x) \in F[x] \setminus F$ be monic and $\alpha = \displaystyle\sum_{n=0}^{\infty} \frac{1}{f_{n-1}(z) f_n(z)}$.

For each $\ell \in \mathbb{N}$, let

$$\frac{C_{k_\ell}}{D_{k_\ell}} = \sum_{n=0}^{\ell} \frac{1}{f_{n-1}(z) f_n(z)}.$$

With similar arguments as in Example 3, α can not be algebraic of degree at most $h+1$.

References

[1] H. Cohn, Symmetry and specializability in continued fractions, *Acta Arith.* **75** (1996), 297–320.

[2] G. Köhler, Some more predictable continued fractions, *Monatsh. Math.* **89** (1980), 95–100.

[3] D. H. Lehmer, Continued fractions containing arithmetic progressions, *Manuscripta Math.* **29** (1974), 17–24.

[4] K. Mahler, On a theorem of Liouville in fields of positive characteristic, *Canad. J. Math.* **1** (1949), 397–400.

[5] J. Mc Laughlin, Symmetry and specializability in the continued fraction expansions of some infinite products, *J. Number Theory* **127** (2007), 184–219.

[6] M. Mendés France, Sur les fractions continues limitées, *Acta Arith.* **23** (1973), 207–215.

[7] A. Petho, Simple continued fractions for the Fredholm numbers, *J. Number Theory* **14** (1982), 232–236.

[8] J. Rattanamoong, V. Laohakosol and T. Chaichana, Generalized folding lammas in the field of formal series and their applications, *Integers.* **15** (2014), 30 pages.

[9] P. Riyapan, V. Laohakosol and T. Chaichana, Two types of explicit continued fractions, *Period. Math. Hungar* **52** (2006), 51–72.

[10] W. M. Schmidt, On continued fractions and diophantine approximation in power series fields, *Acta Arith.* **95** (2000), 139–166.

[11] J. O. Shallit, Simple continued fractions for some irrational numbers, *J. Number Theory* **11** (1979), 209–217.

[12] J. O. Shallit, Simple continued fractions for some irrational numbers II, *J. Number Theory* **14** (1982), 228–264.

[13] J. Tamura, Symmetric continued fractions related to certain series, *J. Number Theory* **38** (1991), 251–264.

[14] D. Thakur, Exponential and continued fractions, *J. Number Theory* **59** (1996), 248–261.

[15] A. J. Van der Poorten, Symmetry and folding of continued fractions, *Jour. Théorie des Nombres de Bordeaux* **14** (2002), 603–611.

[16] A. J. Van der Poorten and J. O. Shallit, Folded continued fractions, *J. Number Theory* **40** (1992), 237–250.

[17] A. J. Van der Poorten and J. O. Shallit, A specialised continued fraction, *Canad. J. Math.* **45** (1993), 1067–1079.