ON THE STRUCTURE OF SETS WHICH HAVE COINCIDING REPRESENTATION FUNCTIONS

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Received: 10/24/17, Revised: 3/20/19, Accepted: 12/11/19, Published: 12/24/19

Abstract

For a set of nonnegative integers A, denote by $R_A(n)$ the number of unordered representations of the integer n as the sum of two different terms from A. In this paper we partially describe the structure of the sets, which have coinciding representation functions.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers. For a given set $A \subseteq \mathbb{N}$, $A = \{a_1, a_2, \dots\}$ $(0 \le a_1 < a_2 < \dots)$, the additive representation functions $R^{(1)}_{h,A}(n)$, $R^{(2)}_{h,A}(n)$ and $R^{(3)}_{h,A}(n)$ are defined in the following way:

$$R^{(1)}_{h,A}(n) = |\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1}, \dots, a_{i_h} \in A\}|,$$

$$R^{(2)}_{h,A}(n) = |\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1} \le a_{i_2} \le \dots \le a_{i_h}, a_{i_1}, \dots, a_{i_h} \in A\}|,$$

$$R^{(3)}_{h,A}(n) = |\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1} < a_{i_2} < \dots < a_{i_h}, a_{i_1}, \dots, a_{i_h} \in A\}|.$$

[1]This author was supported by the National Research, Development and Innovation Office NKFIH Grant No. K115288 and K109789, K129335. This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Supported by the ÚNKP-18-4 New National Excellence Program of the Ministry of Human Capacities. Supported by the ÚNKP-19-4 New National Excellence Program of the Ministry for Innovation and Technology.

[2]This author was supported by the OTKA Grant No. K109789, K129335. This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

For simplicity we write $R_{2,A}^{(3)}(n) = R_A(n)$. If A is finite, let $|A|$ denote the cardinality of A.

The investigation of the partitions of the set of nonnegative integers with identical representation functions was a popular topic in the last few decades [1,3,4,5,7,9,11, 13,14]. It is easy to see that $R_{2,A}^{(1)}(n)$ is odd if and only if $\frac{n}{2} \in A$. It follows that for every positive integer n, $R_{2,C}^{(1)}(n) = R_{2,D}^{(1)}(n)$ holds if and only if $C = D$, where $C = \{c_1, c_2, \dots\}$ $(c_1 < c_2 < \dots)$ and $D = \{d_1, d_2, \dots\}$ $(d_1 < d_2 < \dots)$ are two sets of nonnegative integers. In [8], Nathanson gave a full description of the sets C and D, which have identical representation functions $R_{2,C}^{(1)}(n) = R_{2,D}^{(1)}(n)$ from a certain point on. Namely, he proved the following theorem. Let $C(z) = \sum_{c\in C} z^c$, $D(z) = \sum_{d\in D} z^d$ be the generating functions of the sets C and D, respectively.

Theorem 1. *Let C and D be different infinite sets of nonnegative integers. Then $R_{2,C}^{(1)}(n) = R_{2,D}^{(1)}(n)$ holds from a certain point on if and only if there exist positive integers n_0, M and finite sets of nonnegative integers F_C, F_D, T with $F_C \cup F_D \subset [0, Mn_0 - 1]$, $T \subset [0, M - 1]$ such that*

$$C = F_C \cup \{lM + t : l \ge n_0, t \in T\},$$
$$D = F_D \cup \{lM + t : l \ge n_0, t \in T\},$$
$$1 - z^M | (F_C(z) - F_D(z))T(z).$$

We conjecture in [6] that the above theorem of Nathanson can be generalized in the following way.

Conjecture 1. For $h > 2$ let C and D be different infinite sets of nonnegative integers. Then $R_{h,C}^{(1)}(n) = R_{h,D}^{(1)}(n)$ holds from a certain point on if and only if there exist positive integers n_0, M and finite sets F_C, F_D, T with $F_C \cup F_D \subset [0, Mn_0 - 1]$, $T \subset [0, M - 1]$ such that

$$C = F_C \cup \{lM + t : l \ge n_0, t \in T\},$$
$$D = F_D \cup \{lM + t : l \ge n_0, t \in T\},$$
$$(1 - z^M)^{h-1} | (F_C(z) - F_D(z))T(z)^{h-1}.$$

For $h = 3$, Kiss, Rozgonyi and Sándor proved [6] Conjecture 1. In the general case when $h > 3$ we proved that if the conditions of Conjecture 1 hold then $R_{h,C}^{(1)}(n) = R_{h,D}^{(1)}(n)$ holds from a certain point on. Later, Rozgonyi and Sándor [10] proved that the above conjecture holds when $h = p^\alpha$, where $\alpha \ge 1$ and p is a prime.

It is easy to see that for any two different sets C, $D \subset \mathbb{N}$ we have $R_{2,C}^{(2)}(n) \ne R_{2,D}^{(2)}(n)$ for some $n \in \mathbb{N}$. Let i denote the smallest index for which $c_i \ne d_i$, thus we may assume that $c_i < d_i$. It is clear that $R_{2,C}^{(2)}(c_1 + c_i) > R_{2,D}^{(2)}(c_1 + c_i)$, which implies that there exists a nonnegative integer n such that $R_{2,C}^{(2)}(n) \ne R_{2,D}^{(2)}(n)$. We pose a problem about this representation function.

Problem 1. Determine all the sets of nonnegative integers C and D such that $R^{(2)}_{2,C}(n) = R^{(2)}_{2,D}(n)$ holds from a certain point on.

In this paper, we focus on the representation function $R_A(n)$. We partially describe the structure of the sets, which have identical representation functions. To do this, we define the Hilbert cube which plays a crucial role in our results. Let $\{h_1, h_2, \dots\}$ $(h_1 < h_2 < \dots)$ be finite or infinite set of positive integers. The set

$$H(h_1, h_2, \dots) = \Big\{ \sum_i \varepsilon_i h_i : \varepsilon_i \in \{0, 1\} \Big\}$$

is called the Hilbert cube. The even part of a Hilbert cube is the set

$$H_0(h_1, h_2, \dots) = \Big\{ \sum_i \varepsilon_i h_i : \varepsilon_i \in \{0, 1\}, 2 | \sum_i \varepsilon_i \Big\},$$

and the odd part of a Hilbert cube is

$$H_1(h_1, h_2, \dots) = \Big\{ \sum_i \varepsilon_i h_i : \varepsilon_i \in \{0, 1\}, 2 \nmid \sum_i \varepsilon_i \Big\}.$$

We say a Hilbert cube $H(h_1, h_2, \dots)$ is half non-degenerated if the representation of any integer in $H_0(h_1, h_2, \dots)$ and $H_1(h_1, h_2, \dots)$ is unique, that is $\sum_i \varepsilon_i h_i \neq \sum_i \varepsilon_i^{'} h_i$ whenever $\sum_i \varepsilon_i \equiv \sum_i \varepsilon_i^{'} \bmod 2$, where $\varepsilon_i^{'} \in \{0, 1\}$.

Many years ago, Selfridge and Straus [12] proved the following theorem about the cardinality of sets with identical representation functions. For the sake of completeness, we present the proof in Section 2.

Theorem 2. *Let C and D be different finite sets of nonnegative integers such that for every positive integer n, $R_C(n) = R_D(n)$ holds. Then we have $|C| = |D| = 2^l$ for a nonnegative integer l.*

If $0 \in C$ and for $D = \{d_1, d_2, \dots\}$, $0 \le d_1 < d_2 < \dots$, we have $R_C(m) = R_D(m)$ (sequences C and D are different), then $d_1 > 0$. Otherwise let us suppose that $c_i = d_i$ for $i = 1, 2, \dots, n-1$, but $c_n < d_n$ which implies that $R_C(c_1 + c_n) > R_D(c_1 + c_n)$, a contradiction.

If $|C| = |D| = 1$ and $0 \in C$ with $R_C(n) = R_D(n)$, then we have $C = \{0\}$ and $D = \{d_1\}$. Therefore, $C = H_0(d_1)$ and $D = H_1(d_1)$.

If $|C| = |D| = 2$ and $0 \in C$ with $R_C(n) = R_D(n)$, then $C = \{0, c_2\}$ and $D = \{d_1, d_2\}$. In this case $1 = R_C(0 + c_2)$, and for $n \neq c_2$ we have $R_C(n) = 0$. Moreover, $1 = R_C(d_1 + d_2)$ and for $n \neq d_1 + d_2$ we have $R_D(n) = 0$. This implies that $d_1 + d_2 = c_1 + c_2 = c_2$, that is $C = \{0, d_1 + d_2\} = H_0(d_1, d_2)$ and $D = \{d_1, d_2\} = H_1(d_1, d_2)$.

If $|C| = |D| = 4$ and $0 \in C$ with $R_C(n) = R_D(n)$, then let $C = \{c_1, c_2, c_3, c_4\}$, $c_1 = 0$ and $D = \{d_1, d_2, d_3, d_4\}$, where $d_1 > 0$. Then we have

$$c_1 + c_2 < c_1 + c_3 < c_1 + c_4, c_2 + c_3 < c_2 + c_4 < c_3 + c_4$$

and

$$d_1 + d_2 < d_1 + d_3 < d_1 + d_4, d_2 + d_3 < d_2 + d_4 < d_3 + d_4,$$

which implies that $c_1 + c_2 = d_1 + d_2$. Therefore, $c_2 = d_1 + d_2$ and $c_1 + c_3 = d_1 + d_3$, thus we have $c_3 = d_1 + d_3$. If $c_2 + c_3 = d_2 + d_3$, then $(d_1 + d_2) + (d_1 + d_3) = d_2 + d_3$, that is $d_1 = 0$, a contradiction. Hence $c_2 + c_3 = d_1 + d_4$, that is $(d_1 + d_2) + (d_1 + d_3) = d_1 + d_4$. This implies that $d_4 = d_1 + d_2 + d_3$. Finally $c_1 + c_4 = d_2 + d_3$, that is $c_4 = d_2 + d_3$. Thus we have $C = \{0, d_1 + d_2, d_1 + d_3, d_2 + d_3\} = H_0(d_1, d_2, d_3)$ and $D = \{d_1, d_2, d_3, d_1 + d_2 + d_3\} = H_1(d_1, d_2, d_3)$. In the next step we prove that if the sets are even and odd parts of a Hilbert cube, then the corresponding representation functions are identical.

Theorem 3. *Let $H(h_1, h_2, \dots)$ be a half non-degenerated Hilbert cube. If $C = H_0(h_1, h_2, \dots)$ and $D = H_1(h_1, h_2, \dots)$, then for every positive integer n, $R_C(n) = R_D(n)$ holds.*

It is easy to see that Theorem 3 is equivalent to Lemma 1 of Chen and Lev in [2]. First, they proved the finite case $H(h_1, \dots, h_n)$ by induction on n, and the infinite case was a corollary of the finite case. For the sake of completeness, we give a different proof by using generating functions. Chen and Lev asked whether Theorem 3 described all different sets C and D of nonnegative integers such that $R_C(n) = R_D(n)$. The following conjecture is a simple generalization of the above question formulated by Chen and Lev but we use a different terminology.

Conjecture 2. Let C and D be different infinite sets of nonnegative integers with $0 \in C$. If for every positive integer n, $R_C(n) = R_D(n)$ holds, then there exist positive integers $d_{i_1}, d_{i_2}, \dots \in D$, where $d_{i_1} < d_{i_2} < \dots$, and a half non-degenerated Hilbert cube $H(d_{i_1}, d_{i_2}, \dots)$ such that

$$C = H_0(d_{i_1}, d_{i_2}, \dots),$$

$$D = H_1(d_{i_1}, d_{i_2}, \dots).$$

We showed above that Conjecture 2 is true for the finite case $l = 0, 1, 2$. Unfortunately we could not settle the cases $l \geq 3$, which seems to be very complicated. In Section 4 we prove the following weaker version of the above conjecture.

Theorem 4. *Let $D = \{d_1, \dots, d_{2^n}\}$, $(0 < d_1 < d_2 < \dots < d_{2^n})$ be a set of nonnegative integers, where $d_{2^k+1} \geq 4d_{2^k}$, for $k = 0, \dots, n-1$ and $d_{2^k} \leq d_1 + d_2 + d_3 + d_5 + \dots + d_{2^i+1} + \dots + d_{2^{k-1}+1}$ for $k = 2, \dots, n$. Let C be a finite set of nonnegative integers such that $0 \in C$. If for every positive integer m, $R_C(m) = R_D(m)$ holds, then*

$$C = H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{n-1}+1}),$$

and

$$D = H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{n-1}+1}).$$

For any sets of nonnegative integers A and B we define the sumset $A+B$ by

$$A + B = \{a + b : a \in A, b \in B\}.$$

The special case $b+A$ denotes the set $\{b+a : a \in A\}$, where b is a fixed nonnegative integer. Let $q\mathbb{N}$ denote the dilate of the set $\mathbb{N}$ by the factor q, that is, $q\mathbb{N}$ is the set of nonnegative integers divisible by q. Let $r_{A+B}(n)$ denote the number of solutions of the equation $a + b = n$, where $a \in A$, $b \in B$. In [2], Chen and Lev proved the following nice result.

Theorem 5. *Let l be a positive integer. Then there exist sets C and D of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$ and for every positive integer n, $R_C(n) = R_D(n)$ holds.*

This theorem is an easy consequence of Theorem 3 by putting

$$H(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l} - 1, 2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots).$$

The details can be found in the first part of the proof of Theorem 6. Chen and Lev [2] formulated the following conjecture.

Conjecture 3. Let C and D be different sets of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$, $m \geq 2$. If for every positive integer n, $R_C(n) = R_D(n)$ holds, then there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$.

We formulate the following conjecture, which is a stronger version of the above conjecture of Chen and Lev.

Conjecture 4. Let C and D be different sets of nonnegative integers such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$, $m \geq 2$. If for every positive integer n, $R_C(n) = R_D(n)$ holds, then there exists an integer $l \geq 1$ such that

$$C = H_0(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l}-1, 2^{2l+1}-1, 2(2^{2l+1}-1), 4(2^{2l+1}-1), 8(2^{2l+1}-1), \dots),$$

and

$$D = H_1(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l}-1, 2^{2l+1}-1, 2(2^{2l+1}-1), 4(2^{2l+1}-1), 8(2^{2l+1}-1), \dots).$$

We prove that Conjecture 2 implies Conjecture 4.

Theorem 6. *Assume that Conjecture 2 holds. Then there exist C and D, different infinite sets of nonnegative integers, such that $C \cup D = \mathbb{N}$, $C \cap D = r + m\mathbb{N}$ with integers $r \geq 0$, $m \geq 2$ and for every positive integer n, $R_C(n) = R_D(n)$ if and only if there exists an integer $l \geq 1$ such that*

$$C = H_0(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l}-1, 2^{2l+1}-1, 2(2^{2l+1}-1), 4(2^{2l+1}-1), 8(2^{2l+1}-1), \dots)$$

and

$$D = H_1(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l}-1, 2^{2l+1}-1, 2(2^{2l+1}-1), 4(2^{2l+1}-1), 8(2^{2l+1}-1), \dots).$$

2. Proof of Theorem 2

Proof. By using the generating functions of the sets C and D, we get that

$$\sum_{n=1}^{\infty} R_C(n)z^n = \frac{C(z)^2 - C(z^2)}{2},$$

$$\sum_{n=1}^{\infty} R_D(n)z^n = \frac{D(z)^2 - D(z^2)}{2}.$$

It follows that

$$R_C(n) = R_D(n) \text{ if and only if } C(z)^2 - D(z)^2 = C(z^2) - D(z^2). \tag{1}$$

Let $l+1$ be the largest exponent of the factor $(z-1)$ in $C(z) - D(z)$, i.e.,

$$C(z) - D(z) = (z-1)^{l+1}p(z), \tag{2}$$

where $p(z)$ is a polynomial and $p(1) \neq 0$. Substituting (2) back to (1) gives

$$(C(z) + D(z))(z-1)^{l+1}p(z) = (z^2-1)^{l+1}p(z^2).$$

Then we have $(C(z) + D(z))p(z) = (z+1)^{l+1}p(z^2)$. Substituting $z = 1$, we have $C(1) + D(1) = 2^{l+1}$, which implies that $|C| + |D| = 2^{l+1}$. On the other hand,

$$\binom{|C|}{2} = \sum_m R_C(m) = \sum_m R_D(m) = \binom{|D|}{2},$$

which gives $|C| = |D|$. □

3. Proof of Theorem 3

Proof. By (1) we have to prove that $C(z)^2 - D(z)^2 = C(z^2) - D(z^2)$. It is easy to see from the definition of C and D that

$$\prod_i (1 - z^{h_i}) = \sum_{i_1 < \ldots < i_t} (-1)^t z^{h_{i_1} + \ldots + h_{i_t}} = C(z) - D(z).$$

On the other hand, clearly we have $C(z) + D(z) = \prod_i (1 + z^{h_i})$. Then we have

$$C(z)^2 - D(z)^2 = \Big(C(z) - D(z)\Big)\Big(C(z) + D(z)\Big) = \prod_i (1 - z^{h_i}) \cdot \prod_i (1 + z^{h_i})$$

$$= \prod_i (1 - z^{2h_i}) = C(z^2) - D(z^2).$$

The proof is completed. □

4. Proof of Theorem 4

We apply induction on n. If $n = 0$, then $C = \{0\}$ and $D = \{d_1\}$. Therefore, for every positive integer m, we have $R_C(m) = R_D(m) = 0$. If $n = 1$, then $C = \{0, c_2\}$ and $D = \{d_1, d_2\}$. Since $R_C(m) = R_D(m)$ for every positive integer m, it follows that $R_D(d_1 + d_2) = 1 = R_C(d_1 + d_2)$. Then we have $C = \{0, d_1 + d_2\} = H_0(d_1, d_2)$ and $D = \{d_1, d_2\} = H_1(d_1, d_2)$. Assume that the statement of Theorem 4 holds for $n = N - 1$. We will prove it for $n = N$. Let $D = \{d_1, \dots, d_{2^N}\}$ be a set of nonnegative integers, where $d_{2^k+1} \ge 4d_{2^k}$, for $k = 0, \dots, N-1$ and $d_{2^k} \le d_1 + d_2 + d_3 + d_5 + \dots + d_{2^i+1} + \dots + d_{2^{k-1}+1}$ for $k = 2, \dots, N$. If C is a set of nonnegative integers such that $0 \in C$ and for every positive integer m, $R_C(m) = R_D(m)$ holds, then we have to prove that

$$C = H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{N-1}+1}),$$

and

$$D = H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{N-1}+1}).$$

Define the sets

$$C_1 = \{c_1, \dots, c_{2^{N-1}}\}, \qquad C_2 = C \setminus C_1,$$

and

$$D_1 = \{d_1, \dots, d_{2^{N-1}}\}, \qquad D_2 = D \setminus D_1.$$

We prove that for every positive integer m, we have

$$R_{C_1}(m) = R_{D_1}(m). \tag{3}$$

Since $d_{2^{N-1}} \le \frac{1}{4} d_{2^{N-1}+1}$, it follows that for any d_i, $d_j \in D_1$ we have

$$d_i + d_j \le \frac{1}{4} d_{2^{N-1}+1} + \frac{1}{4} d_{2^{N-1}+1} = \frac{1}{2} d_{2^{N-1}+1}.$$

This implies that for every $\frac{1}{2} d_{2^{N-1}+1} \le m \le d_{2^{N-1}+1}$, we have $R_D(m) = 0$, which yields $R_C(m) = 0$. As $0 \in C$, we have a representation $m = 0 + m$. It follows that $m \notin C$ for $\frac{1}{2} d_{2^{N-1}+1} \le m \le d_{2^{N-1}+1}$. We will show that

$$C_1 = \Big[0, \frac{1}{3} d_{2^{N-1}+1}\Big[\cap C, \qquad D_1 = \Big[0, \frac{1}{3} d_{2^{N-1}+1}\Big[\cap D.$$

We distinguish two cases. In the first case we assume that $c_{2^{N-1}+1} \le \frac{d_{2^{N-1}+1}}{2}$. Then we have

$$\binom{2^{N-1}+1}{2} \le \sum_{m < d_{2^{N-1}+1}} R_C(m) = \sum_{m < d_{2^{N-1}+1}} R_D(m) = \binom{2^{N-1}}{2}$$

which is a contradiction. In the second case we assume that $c_{2^{N-1}} > \frac{d_{2^{N-1}+1}}{2}$, which implies that $c_{2^{N-1}} \geq d_{2^{N-1}+1}$. Then we have

$$\binom{2^{N-1}-1}{2} \geq \sum_{m<d_{2^{N-1}+1}} R_C(m) = \sum_{m<d_{2^{N-1}+1}} R_D(m) = \binom{2^{N-1}}{2},$$

which is impossible. Then we have $c_{2^{N-1}} \leq \frac{1}{2} d_{2^{N-1}+1} < c_{2^{N-1}+1}$ and $d_{2^{N-1}+1} < c_{2^{N-1}+1}$, which implies that

$$R_{C_1}(m) = \begin{cases} 0, \text{ if } m \geq d_{2^{N-1}+1} \\ R_C(m), \text{ if } m < d_{2^{N-1}+1} \end{cases},$$

and

$$R_{D_1}(m) = \begin{cases} 0, \text{ if } m \geq d_{2^{N-1}+1} \\ R_D(m), \text{ if } m < d_{2^{N-1}+1} \end{cases}.$$

It follows that for every positive integer m, $R_{C_1}(m) = R_{D_1}(m)$, which proves (3). By the induction hypothesis we get that

$$C_1 = H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-2}+1})$$

and

$$D_1 = H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-2}+1}).$$

By Theorem 4, $d_{2^{k-1}+1} \leq d_{2^k} \leq \frac{1}{4} d_{2^k+1}$ for $1 \leq k \leq N-1$. This implies that $d_{2^{N-i}+1} \leq \frac{1}{4^{i-1}} d_{2^{N-1}+1}$ for $i = 2, \dots, N$ and $d_1 \leq \frac{1}{4^N} d_{2^{N-1}+1}$. It follows that the largest element of the set $H(d_1, d_2, d_3, d_5, d_9, \dots, d_{2^{N-2}+1})$ is

$$\begin{aligned} & d_1 + d_2 + d_3 + d_5 + d_9 + \dots + d_{2^{N-2}+1} \\ \leq \; & \frac{1}{4^N} d_{2^{N-1}+1} + \frac{1}{4^{N-1}} d_{2^{N-1}+1} + \dots + \frac{1}{4} d_{2^{N-1}+1} < \frac{1}{3} d_{2^{N-1}+1}, \end{aligned}$$

which implies that

$$C_1 = \left[0, \frac{1}{3} d_{2^{N-1}+1}\right[\cap C, \qquad D_1 = \left]0, \frac{1}{3} d_{2^{N-1}+1}\right[\cap D. \tag{4}$$

Therefore,

$$C_1 + C_1 \subset \left[0, \frac{2}{3} d_{2^{N-1}+1}\right[, \qquad D_1 + D_1 \subset \left[0, \frac{2}{3} d_{2^{N-1}+1}\right[. \tag{5}$$

Furthermore, for every $d \in D_2$, we have

$$\begin{aligned} & d_{2^{N-1}+1} \leq d \leq d_{2^N} \leq d_{2^{N-1}+1} + d_{2^{N-2}+1} + \dots + d_{2^i+1} + \dots + d_2 + d_1 \\ \leq \; & d_{2^{N-1}+1} + \frac{1}{4} d_{2^{N-1}+1} + \dots + \frac{1}{4^{N-i-1}} d_{2^{N-1}+1} + \dots \\ + \; & \frac{1}{4^{N-1}} d_{2^{N-1}+1} + \frac{1}{4^N} d_{2^{N-1}+1} \\ < \; & \frac{4}{3} d_{2^{N-1}+1}. \end{aligned}$$

Thus, we conclude

$$D_1 + D_2 \subset \Big[d_{2^{N-1}+1}, \frac{5}{3}d_{2^{N-1}+1}\Big[, \tag{6}$$

and

$$D_2 + D_2 \subset \Big[2d_{2^{N-1}+1}, \frac{8}{3}d_{2^{N-1}+1}\Big[. \tag{7}$$

It follows that

$$R_C(m) = 0 \text{ for } m \geq \frac{8}{3}d_{2^{N-1}+1}. \tag{8}$$

We now show that $c_{2^{N-1}+1} = d_{2^{N-1}+1} + d_1$. Assume that

$$c_{2^{N-1}+1} < d_{2^{N-1}+1} + d_1. \tag{9}$$

Obviously, $c_{2^{N-1}+1} > d_{2^{N-1}+1}$. Since $c_{2^{N-1}+1} = c_{2^{N-1}+1} + 0$, then we have $1 \leq R_C(c_{2^{N-1}+1}) = R_D(c_{2^{N-1}+1})$, which implies that $c_{2^{N-1}+1} = d_i + d_j$, $i < j$, $d_i, d_j \in D$. If $j \leq 2^{N-1}$, then by using the first condition in Theorem 4 we have

$$c_{2^{N-1}+1} = d_i + d_j \leq 2d_{2^{N-1}} \leq \frac{1}{2}d_{2^{N-1}+1},$$

which contradicts the inequality $c_{2^{N-1}+1} \geq d_{2^{N-1}+1}$. Moreover, when $j \geq 2^{N-1}+1$, we have

$$c_{2^{N-1}+1} = d_i + d_j \geq d_1 + d_{2^{N-1}+1},$$

which contradicts (9).

Assume that $c_{2^{N-1}+1} > d_{2^{N-1}+1} + d_1$. Obviously, $1 \leq R_D(d_{2^{N-1}+1} + d_1) = R_C(d_{2^{N-1}+1} + d_1)$, which implies that $d_1 + d_{2^{N-1}+1} = c_i + c_j$, $i < j$, $c_i, c_j \in C$. If $j \leq 2^{N-1}$, then we have

$$d_1 + d_{2^{N-1}+1} = c_i + c_j \leq 2c_{2^{N-1}} \leq d_{2^{N-1}+1},$$

which is impossible. Moreover, when $j \geq 2^{N-1}+1$, we have

$$d_1 + d_{2^{N-1}+1} = c_i + c_j \geq c_{2^{N-1}+1} > d_{2^{N-1}+1} + d_1$$

which is a contradiction.

It follows that for every $c \in C$ with $c > c_{2^{N-1}+1}$, we have $c \leq \frac{5}{3}d_{2^{N-1}+1}$. Otherwise, $c + c_{2^{N-1}+1} \geq \frac{8}{3}d_{2^{N-1}+1}$ and then $R_C(c + c_{2^{N-1}+1}) \geq 1$, which contradicts (8). By (4) and (8) we have

$$C_1 + C_2 \subset \Big[d_{2^{N-1}+1}, 2d_{2^{N-1}+1}\Big[, \tag{10}$$

and

$$(C_2 + C_2) \setminus \{2c_{2^N}\} \subset \Big[2d_{2^{N-1}+1}, \frac{8}{3}d_{2^{N-1}+1}\Big[. \tag{11}$$

It sufficies to prove that

$$C_2 = d_{2^{N-1}+1} + H_1(d_1, d_2, d_3, d_5, \dots, d_{2^{N-2}+1}) = d_{2^{N-1}+1} + D_1,$$

and

$$D_2 = d_{2^{N-1}+1} + H_0(d_1, d_2, d_3, d_5, \dots, d_{2^{N-2}+1}) = d_{2^{N-1}+1} + C_1.$$

Define the sets

$$C_{2,n} = \{c_{2^{N-1}+1}, c_{2^{N-1}+2}, \dots, c_{2^{N-1}+n}\},$$

and

$$D_{2,n} = \{d_{2^{N-1}+1}, d_{2^{N-1}+2}, \dots, d_{2^{N-1}+n}\}.$$

Furthermore, define the sets

$$C_1 + C_{2,n} = \{p_1, p_2, \dots\},\ (p_1 < p_2 < \dots),$$

$$C_{2,n} + C_{2,n} = \{t_1, t_2, \dots\},\ (t_1 < t_2 < \dots),$$

and

$$D_1 + D_{2,n} = \{q_1, q_2, \dots\},\ (q_1 < q_2 < \dots),$$

$$D_{2,n} + D_{2,n} = \{s_1, s_2, \dots\},\ (s_1 < s_2 < \dots).$$

Denote the first $2^{N-1} + n$ elements of the set

$$H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{N-1}+1})$$

by $H_0^{(n)}$, and let $H_1^{(n)}$ denote the first $2^{N-1} + n$ elements of the set

$$H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, d_{2^{N-1}+1}).$$

Now we prove by induction on n that

$$H_0^{(n)} = C_1 \cup C_{2,n} \text{ and } H_1^{(n)} = D_1 \cup D_{2,n}$$

for $1 \le n \le 2^{N-1}$. For $n = 1$ we have already proved that $D_{2,1} = \{d_{2^{N-1}+1}\}$ and $C_{2,1} = \{d_{2^{N-1}+1} + d_1\}$. It follows that $H_0^{(1)} = C_1 \cup C_{2,1}$ and $H_1^{(1)} = D_1 \cup D_{2,1}$. Let us suppose that $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$ and we are going to prove that $H_0^{(n+1)} = C_1 \cup C_{2,n+1}$ and $H_1^{(n+1)} = D_1 \cup D_{2,n+1}$. To prove $H_0^{(n+1)} = C_1 \cup C_{2,n+1}$ and $H_1^{(n+1)} = D_1 \cup D_{2,n+1}$, we need three lemmas. Let i be the smallest index u such that $r_{C_1+C_{2,n}}(p_u) > r_{D_1+D_{2,n}}(p_u)$. If such an i does not exist, then $p_i = +\infty$. Let j be the smallest index v such that $r_{C_1+C_{2,n}}(q_v) < r_{D_1+D_{2,n}}(q_v)$. If such a j does not exist, then $q_j = +\infty$. Let k be the smallest index w such that $R_{C_{2,n}}(t_w) > R_{D_{2,n}}(t_w)$. If such a k does not exist, then $t_k = +\infty$. Let l be the smallest index x such that $R_{C_{2,n}}(s_x) < R_{D_{2,n}}(s_x)$. If such an l does not exist, then $s_l = +\infty$. The following observations play a crucial role in the proof.

Lemma 1. *Let us suppose that* $H_0^{(n)} = C_1 \cup C_{2,n}$ *and* $H_1^{(n)} = D_1 \cup D_{2,n}$*. Then we have*

(i) $min\{p_i, c_{2^{N-1}+n+1}\} = min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$,

(ii) $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$.

Proof. In the first step we prove (i). We will prove that $p_i = +\infty$ is equivalent to $q_j = +\infty$ and for $p_i = q_j = +\infty$, we have $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$. Assume that $p_i = +\infty$. Then by the definition of p_i, we have $r_{C_1+C_{2,n}}(p_f) \le r_{D_1+D_{2,n}}(p_f)$ for every positive integer f. It follows that

$$r_{C_1+C_{2,n}}(m) \le r_{D_1+D_{2,n}}(m)$$

for every positive integer m. On the other hand,

$$2^{N-1} \cdot n = \sum_m r_{C_1+C_{2,n}}(m) \le \sum_m r_{D_1+D_{2,n}}(m) = 2^{N-1} \cdot n.$$

Then $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ for every positive integer m, which implies that $q_j = +\infty$. Suppose that $q_j = +\infty$. Then by the definition of q_j, we have $r_{D_1+D_{2,n}}(q_g) \le r_{C_1+C_{2,n}}(q_g)$ for every positive integer g. It follows that

$$r_{C_1+C_{2,n}}(m) \ge r_{D_1+D_{2,n}}(m)$$

for every positive integer m. Moreover,

$$2^{N-1} \cdot n = \sum_m r_{C_1+C_{2,n}}(m) \ge \sum_m r_{D_1+D_{2,n}}(m) = 2^{N-1} \cdot n.$$

Then $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ for every positive integer m, which implies that $p_i = +\infty$.

We distinguish two cases.

`Case 1.` $p_i = +\infty$, $q_j = +\infty$, that is, $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ for every positive integer m. Now we prove that $c_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+n+1}$. Assume that $c_{2^{N-1}+n+1} < d_1 + d_{2^{N-1}+n+1}$. Since $c_{2^{N-1}+n+1} = 0 + c_{2^{N-1}+n+1}$, where $0 \in C_1$ but $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$ it follows from (5), (6), (7) and (10) that $R_D(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1})$ and $R_C(c_{2^{N-1}+n+1}) > r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1})$. Then we have

$$R_D(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+n+1}),$$

which is impossible. Similarly, if $c_{2^{N-1}+n+1} > d_1 + d_{2^{N-1}+n+1}$, then $R_D(d_1 + d_{2^{N-1}+n+1}) > r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$ because $d_1 \in D_1$, $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$. It follows from (5), (6), (10) and (11) that $R_C(d_1 + d_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1})$. Then we have

$$R_C(d_1 + d_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$$

$$< R_D(d_1 + d_{2^{N-1}+n+1}),$$

which is a contradiction.

`Case 2.` $p_i < +\infty$ and $q_j < +\infty$. We have two subcases.

`Case 2a.` $min\{p_i, c_{2^{N-1}+n+1}\} < min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$.
If $p_i \le c_{2^{N-1}+n+1}$, then obviously $p_i < d_1 + d_{2^{N-1}+n+1}$, which implies by (5), (6), (7) and (10) that $R_D(p_i) = r_{D_1+D_{2,n}}(p_i)$. By using the above facts and the definition of p_i, we get that

$$R_C(p_i) \ge r_{C_1+C_{2,n}}(p_i) > r_{D_1+D_{2,n}}(p_i) = R_D(p_i),$$

which contradicts the fact that $R_C(m) = R_D(m)$ for every positive integer m. On the other hand, if $p_i > c_{2^{N-1}+n+1}$, then by the definition of p_i, $r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) \le r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1})$. Since $c_{2^{N-1}+n+1} = 0 + c_{2^{N-1}+n+1}$, $0 \in C_1$ and $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$, we have

$$R_C(c_{2^{N-1}+n+1}) > r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}). \tag{12}$$

The condition $min\{p_i, c_{2^{N-1}+n+1}\} < min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$ implies that $q_j > c_{2^{N-1}+n+1}$. It follows from the definition of q_j that

$$r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) \le r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}).$$

We conclude that

$$r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}). \tag{13}$$

It follows from $0 + c_{2^{N-1}+n+1} < d_1 + d_{2^{N-1}+n+1}$, $0 \in C_1$, (5), (6), (7) and (10) that $r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) = R_D(c_{2^{N-1}+n+1})$. Furthermore, we obtain from (12) and (13) that

$$R_D(c_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(c_{2^{N-1}+n+1}) = r_{C_1+C_{2,n}}(c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+n+1}),$$

which contradicts the fact that $R_C(m) = R_D(m)$ for every positive integer m.

`Case 2b.` $min\{p_i, c_{2^{N-1}+n+1}\} > min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$.
If $q_j \le d_1 + d_{2^{N-1}+n+1}$, then obviously $q_j < c_{2^{N-1}+n+1}$, which implies from (5), (6), (10) and (11) that $R_C(q_j) = r_{C_1+C_{2,n}}(q_j)$. By using the above facts and the definition of q_j, we get that

$$R_C(q_j) = r_{C_1+C_{2,n}}(q_j) < r_{D_1+D_{2,n}}(q_j) \le R_D(q_j),$$

which contradicts the fact that $R_C(m) = R_D(m)$ for every positive integer m.
Moreover, if $q_j > d_1 + d_{2^{N-1}+n+1}$, then by the definition of q_j,

$$r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) \ge r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}).$$

Since $d_1 \in D_1$, $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$, we have

$$R_D(d_1 + d_{2^{N-1}+n+1}) > r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}). \tag{14}$$

The assumption $min\{p_i, c_{2^{N-1}+n+1}\} > min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$ implies that $p_i > d_1 + d_{2^{N-1}+n+1}$. It follows from the definition of p_i that

$$r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) \le r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}).$$

We conclude that

$$r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1}). \tag{15}$$

It follows from $min\{p_i, c_{2^{N-1}+n+1}\} > min\{q_j, d_1 + d_{2^{N-1}+n+1}\}$ that $c_{2^{N-1}+n+1} > d_1 + d_{2^{N-1}+n+1}$. Therefore, it follows from (5), (6), (10) and (11) that

$$r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = R_C(d_1 + d_{2^{N-1}+n+1}). \tag{16}$$

Furthermore, from (14),(15) and (16) we have

$$R_D(d_1 + d_{2^{N-1}+n+1}) > r_{D_1+D_{2,n}}(d_1 + d_{2^{N-1}+n+1})$$
$$= r_{C_1+C_{2,n}}(d_1 + d_{2^{N-1}+n+1}) = R_C(d_1 + d_{2^{N-1}+n+1}),$$

which contradicts the fact that $R_C(m) = R_D(m)$ for every positive integer m. The proof of (i) in Lemma 1 is completed.

The proof of (ii) in Lemma 1 is similar to the proof of (i). For the sake of completeness, we present it. We prove that $s_l = +\infty$ is equivalent to $t_k = +\infty$ and in this case $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every positive integer m. If $t_k = +\infty$, then by the definition of t_k, we have $R_{C_{2,n}}(t_f) \le R_{D_{2,n}}(t_f)$ for every positive integer f. Then

$$R_{C_{2,n}}(m) \le R_{D_{2,n}}(m)$$

for every positive integer m. On the other hand, we have

$$\binom{n}{2} = \sum_m R_{C_{2,n}}(m) \le \sum_m R_{D_{2,n}}(m) = \binom{n}{2}.$$

It follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every positive integer m, which implies that $s_l = +\infty$. If $s_l = +\infty$, then by the definition of s_l, we have $R_{C_{2,n}}(s_g) \ge R_{D_{2,n}}(s_g)$ for every positive integer g. Then

$$R_{C_{2,n}}(m) \ge R_{D_{2,n}}(m)$$

for every positive integer m. Furthermore,

$$\binom{n}{2} = \sum_m R_{C_{2,n}}(m) \ge \sum_m R_{D_{2,n}}(m) = \binom{n}{2}.$$

Then, we have $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every positive integer m, which implies that $t_k = +\infty$. We distinguish two cases.

Case 1. $t_k = +\infty$, $s_l = +\infty$, that is, $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ for every positive integer m. Now, we prove that $d_{2^{N-1}+n+1} = d_1 + c_{2^{N-1}+n+1}$. Assume that $d_{2^{N-1}+n+1} > d_1 + c_{2^{N-1}+n+1}$. As $d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1} = c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$, where $c_{2^{N-1}+1} \in C_{2,n}$ and $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$, it follows that

$$R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) > R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}).$$

On the other hand, we will show that

$$d_{2^{N-1}+1} + d_{2^{N-1}+n+1} > d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1} = c_{2^{N-1}+1} + c_{2^{N-1}+1}$$

and (5), (6), (7), (11) imply

$$R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}).$$

It is clear from (11) that

$$c_{2^{N-1}+1} + c_{2^{N-1}+n+1} \in (C_2 + C_2) \setminus \{2c_{2^N}\} \subset \Big[2d_{2^{N-1}+1}, \frac{8}{3}d_{2^{N-1}+1}\Big[.$$

By (5) and (6), we have $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} \notin (D_1 + D_1) \cup (D_1 + D_2)$. This implies that $R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_2}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$. Moreover, $D_2 = D_{2,n} \cup (D_2 \setminus D_{2,n})$, thus we have

$$R_{D_2}(m) = R_{D_{2,n}}(m) + 2r_{D_2+(D_2\setminus D_{2,n})}(m) + R_{D_2\setminus D_{2,n}}(m)$$

for any positive integer m. We conclude that for any positive integer m with $2d_{2^{N-1}+1} \le m < d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$, we have $r_{D_2+(D_2\setminus D_{2,n})}(m) = 0$ and

$$R_{D_2\setminus D_{2,n}}(m) = 0.$$

This implies that $R_{D_2}(m) = R_{D_{2,n}}(m)$, and then

$$R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}).$$

Therefore,

$$R_D(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1})$$
$$= R_{C_{2,n}}(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+1} + c_{2^{N-1}+n+1}),$$

which is impossible. Similarly, if $d_1 + c_{2^{N-1}+n+1} > d_{2^{N-1}+n+1}$, then it follows that $R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) > R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$ because $d_{2^{N-1}+1} \in D_{2,n}$ and $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$. It follows from (5), (7), (10), (11) and

$$d_{2^{N-1}+1} + d_{2^{N-1}+n+1} < d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1} = c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$$

that $R_{C_{2,n}}(d_{2^{N-1}+1}+d_{2^{N-1}+n+1}) = R_C(d_{2^{N-1}+1}+d_{2^{N-1}+n+1})$. Then, we have

$$R_C(d_{2^{N-1}+1}+d_{2^{N-1}+n+1}) = R_{C_{2,n}}(d_{2^{N-1}+1}+d_{2^{N-1}+n+1})$$

$$= R_{D_{2,n}}(d_{2^{N-1}+1}+d_{2^{N-1}+n+1}) < R_D(d_{2^{N-1}+1}+d_{2^{N-1}+n+1}),$$

which is a contradiction.

`Case 2.` $t_k < +\infty$ and $s_l < +\infty$. We have two subcases.

`Case 2a.` $min\{t_k, c_{2^{N-1}+1}+c_{2^{N-1}+n+1}\} < min\{s_l, d_{2^{N-1}+1}+d_{2^{N-1}+n+1}\}$. If $t_k \le c_{2^{N-1}+1}+c_{2^{N-1}+n+1}$, then obviously $t_k < d_{2^{N-1}+1}+d_{2^{N-1}+n+1}$. By using (5), (6), (7), (11) we have $R_D(t_k) = R_{D_{2,n}}(t_k)$. By the definition of t_k, we get that

$$R_C(t_k) \ge R_{C_{2,n}}(t_k) > R_{D_{2,n}}(t_k) = R_D(t_k),$$

which contradicts the fact that $R_C(t_k) = R_D(t_k)$.
On the other hand, if $t_k > c_{2^{N-1}+1}+c_{2^{N-1}+n+1}$, then by the definition of t_k, we have $R_{C_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) \le R_{D_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1})$. Moreover, it follows from $c_{2^{N-1}+1} \in C_{2,n}$ and $c_{2^{N-1}+n+1} \in C_2 \setminus C_{2,n}$ that

$$R_C(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) > R_{C_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}).$$

Since $min\{t_k, c_{2^{N-1}+1}+c_{2^{N-1}+n+1}\} < min\{s_l, d_{2^{N-1}+1}+d_{2^{N-1}+n+1}\}$, we have $c_{2^{N-1}+1}+c_{2^{N-1}+n+1} < s_l$. By the definition of s_l, $R_{D_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) \le R_{C_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1})$. Then, we have

$$R_{C_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}).$$

By

$$c_{2^{N-1}+1}+c_{2^{N-1}+n+1} = min\{t_k, c_{2^{N-1}+1}+c_{2^{N-1}+n+1}\}$$

$$< min\{s_l, d_{2^{N-1}+1}+d_{2^{N-1}+n+1}\} \le d_{2^{N-1}+1}+d_{2^{N-1}+n+1}$$

and (5), (6), (7), (11), we have

$$R_{D_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) = R_D(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}),$$

and then

$$R_D(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) = R_{D_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1})$$

$$= R_{C_{2,n}}(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) < R_C(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}),$$

which contradicts the fact that

$$R_C(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}) = R_D(c_{2^{N-1}+1}+c_{2^{N-1}+n+1}).$$

`Case 2b.` $min\{t_k, c_{2^{N-1}+1}+c_{2^{N-1}+n+1}\} > min\{s_l, d_{2^{N-1}+1}+d_{2^{N-1}+n+1}\}$.

If $s_l \le d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$, then obviously $s_l < c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$. By the definition of s_l and (5), (7), (10), (11), it follows that $R_{C_{2,n}}(s_l) = R_C(s_l)$, and then

$$R_C(s_l) = R_{C_{2,n}}(s_l) < R_{D_{2,n}}(s_l) \le R_D(s_l),$$

which contradicts the fact that $R_C(s_l) = R_D(s_l)$. On the other hand, if $s_l > d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$, then by the definition of s_l, $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) \ge R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$. Since $d_{2^{N-1}+1} \in D_2$, $d_{2^{N-1}+n+1} \in D_2 \setminus D_{2,n}$, we have

$$R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) > R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}).$$

It follows from $min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} > min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$ that $t_k > d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$. By the definition of t_k,

$$R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) \le R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}),$$

and then $R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$. By

$$d_{2^{N-1}+1} + d_{2^{N-1}+n+1} < c_{2^{N-1}+1} + c_{2^{N-1}+n+1}$$

and (5), (7), (10), (11), we have

$$R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}),$$

and then

$$R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_{C_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1})$$

$$= R_{D_{2,n}}(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) < R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}),$$

which contradicts the fact that

$$R_C(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}) = R_D(d_{2^{N-1}+1} + d_{2^{N-1}+n+1}).$$

The proof of (ii) in Lemma 1 is completed. □

Let

$$H = H(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-1}+1})$$

and

$$H_0 = H_0(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-1}+1})$$

$$H_1 = H_1(d_1, d_2, d_3, d_5, \dots, d_{2^k+1}, \dots, , d_{2^{N-1}+1}).$$

If $R_{H_0}(m) > 0$ or $R_{H_1}(m) > 0$, then

$$m = \delta_0 d_1 + \sum_{i=1}^{N} \delta_i d_{2^{i-1}+1},$$

where δ_0, $\delta_i \in \{0,1,2\}$. It follows from $d_2 \geq 4d_1$, $d_{2^k+1} \geq 4d_{2^{k-1}+1}$, $(k = 1, \ldots, N-1)$ that when

$$m^{'} = \delta_0^{'} d_1 + \sum_{i=1}^{N} \delta_i^{'} d_{2^{i-1}+1},$$

where $\delta_0^{'}$, $\delta_i^{'} \in \{0,1,2\}$ and $(\delta_0, \ldots, \delta_N) \neq (\delta_0^{'}, \ldots, \delta_N^{'})$, then $m \neq m^{'}$. On the other hand, if

$$m = \delta_0 d_1 + \sum_{i=1}^{N} \delta_i d_{2^{i-1}+1},$$

where δ_0, $\delta_i \in \{0,1,2\}$, then $m = k + k^{'}$ with

$$k = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1},$$

where ε_0, $\varepsilon_i \in \{0,1\}$ and

$$k^{'} = \varepsilon_0^{'} d_1 + \sum_{i=1}^{N} \varepsilon_i^{'} d_{2^{i-1}+1},$$

where $\varepsilon_0^{'}$, $\varepsilon_i^{'} \in \{0,1\}$ if and only if $\delta_0 = \varepsilon_0 + \varepsilon_0^{'}$ and $\delta_i = \varepsilon_i + \varepsilon_i^{'}$, $1 \leq i \leq N$.

Let $H_{0,n}$ and $H_{1,n}$ denote the $2^{N-1} + n$th elements of H_0 and H_1, respectively. If

$$H_{0,n} = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1},$$

where ε_0, $\varepsilon_i \in \{0,1\}$, then it follows from $d_2 \geq 4d_1$ and $d_{2^k+1} \geq 4d_{2^{k-1}+1}$, $(k = 1, \ldots, N-1)$ that

$$H_{1,n} = (1-\varepsilon_0)d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1}.$$

In the next step, we prove the following lemma.

Lemma 2. *Let us suppose that* $H_0^{(n)} = C_1 \cup C_{2,n}$ *and* $H_1^{(n)} = D_1 \cup D_{2,n}$ *holds for some* $1 \leq n < 2^{N-1}$ *. Let* $H_{0,n+1} = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1}$*. If* $\varepsilon_0 = 0$ *and* $H_{0,n+1} = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$*, where* $1 \leq i_1 < i_2 < \ldots < i_t < N$*, then we have*

(i) $q_j = H_{0,n+1}$,

(ii) $p_i > q_j$.

If $t > 1$*, then*

(iii) $s_l = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$,

(iv) $t_k > s_l$.

If $t = 1$, *then*

(v) $t_k = s_l = +\infty$.

Proof. We prove (i) and (ii) simultaneously. It is enough to show that if $m < H_{0,n+1}$, then $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ and $r_{D_1+D_{2,n}}(H_{0,n+1}) > r_{C_1+C_{2,n}}(H_{0,n+1})$. If $m < d_{2^{N-1}+1}$, then it follows from (6) and (10) that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m) = 0$. If $d_{2^{N-1}+1} \le m < H_{0,n+1}$, then by using (5), (7), (11) and $H_{1,n+1} = H_{0,n+1} + d_1$, it follows that $R_{H_0}(m) = r_{C_1+C_{2,n}}(m)$ and $R_{H_1}(m) = r_{D_1+D_{2,n}}(m)$. It follows from $R_{H_0}(m) = R_{H_1}(m)$ that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$. By using (5), (7), (11) and $H_{0,n+1} < H_{1,n+1}$, we get that $R_{H_1}(H_{0,n+1}) = r_{D_1+D_{2,n}}(H_{0,n+1})$. Since $H_{0,n+1} = 0 + H_{0,n+1}$, where $0, H_{0,n+1} \in H_0$ and $H_{0,n+1} \notin C_{2,n}$, we have $R_{H_0}(H_{0,n+1}) > r_{C_1+C_{2,n}}(H_{0,n+1})$. It follows from $R_{H_0}(H_{0,n+1}) = R_{H_1}(H_{0,n+1})$ that $r_{D_1+D_{2,n}}(H_{0,n+1}) > R_{C_1+C_{2,n}}(H_{0,n+1})$, which proves (i) and (ii).

We prove (iii) and (iv) simultaneously. Let

$$M = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}.$$

It is enough to show that if $m < M$, then $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$ and $R_{C_{2,n}}(M) < R_{D_{2,n}}(M)$. If $m < 2d_{2^{N-1}+1}$, then by using (7) and (11), we have $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. Let

$$2d_{2^{N-1}+1} \le m < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1},$$

and write $m = h + h^{'}$ with $h, h^{'} \in H_0$. By using (5) and (10), we get that $h, h^{'} \in H_0 \setminus C_1$. Since $h \ge d_{2^{N-1}+1}$, we have

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1},$$

thus $h, h^{'} \in C_{2,n}$, which yields $R_{H_0}(m) = R_{C_{2,n}}(m)$. On the other hand, write $m = h + h^{'}$ with $h, h^{'} \in H_1$. By using (5) and (6), we get that $h, h^{'} \in H_1 \setminus D_1$. Since $h \ge d_{2^{N-1}+1}$, we have

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1} < H_{1,n+1},$$

thus $h, h^{'} \in D_{2,n}$, which yields $R_{H_1}(m) = R_{D_{2,n}}(m)$. It follows from $R_{H_0}(m) = R_{H_1}(m)$ that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$. Suppose that

$$d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$$

$$\le m < 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}.$$

We can assume that

$$m = \delta_0 d_1 + \sum_{i=1}^{u} d_{2^{x_j-1}+1} + \sum_{i=1}^{v} 2d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1},$$

where $\delta_0 \in \{0,1,2\}$ and $1 \le x_1 < x_2 < \ldots < x_u < i_1$ and $1 \le y_1 < y_2 < \ldots < y_v < i_1$ and $x_\alpha \ne y_\beta$ are integers; otherwise, $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. Since $H_{0,n+1} = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$, then t is odd, thus we can assume that δ_0+u+t is even; otherwise, $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. We distinguish three cases.

`Case 1.` $\delta_0 = 0$. Then, u is odd. If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \ne \emptyset$ and $w + v + t + 1$ is even. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$.
Furthermore, if $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_1$, then it follows from (5) and (6) that $h, h^{'} \in H_0 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \ne \emptyset$ and $w + v + t + 1$ is odd. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $R_{C_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

`Case 2.` $\delta_0 = 1$. Then, u is even. If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0,1\}$ and $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\}$ and $\varepsilon_0 + w + v + t + 1$ is even. If $u = 0$, then for a suitable ε_0, there is only one possibility for $h^{'}$, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 1$. If $u > 0$, to choose the pairs $(\varepsilon_0, \{z_1, \ldots, z_w\})$, we have $2 \cdot 2^{u-1} = 2^u$ possibilities, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 2^u$. Moreover, if $m = h + h^{'}$, with $h < h^{'}$ and $h, h^{'} \in H_1$, then it follows from (5) and (6) that $h, h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0,1\}$ and $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\}$ and $\varepsilon_0 + w + v + t + 1$ is odd. If $u = 0$, then for a suitable ε_0, there is only one possibility for $h^{'}$ thus we have $R_{D_{2,n}}(m) = R_{H_1}(m) - 1$. When $u > 0$, to choose the pairs $(\varepsilon_0, \{z_1, \ldots, z_w\})$, we have $2 \cdot 2^{u-1} = 2^u$ possibilities, thus we have $R_{D_{2,n}}(m) = R_{H_1}(m) - 2^u$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$.

Case 3. $\delta_0 = 2$. Then, u is odd. If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \neq \emptyset$ and $1 + w + v + t + 1$ is even. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $R_{C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$.
On the other hand, if $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_1$, then it follows from (5) and (6) that $h, h^{'} \in H_0 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \neq \emptyset$ and $1 + w + v + t + 1$ is odd. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $R_{C_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$. Let

$$M = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}.$$

Now, we prove $R_{D_{2,n}}(M) = R_{H_1}(M)$. Assume that

$$M = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1} = h + h^{'},$$

where $h, h^{'} \in H_1$ with $h < h^{'}$. Then, it follows from (5) and (6) that $h, h^{'} \in H_1 \setminus D_1$. It follows that

$$h^{'} = d_{2^{i_1-1}+1} + d_{2^{z_1-1}+1} + \ldots + d_{2^{z_w-1}+1} + d_{2^{N-1}},$$

where $\{z_1, \ldots, z_w\} \subset \{i_2, \ldots, i_t\}$. Thus, we have

$$h^{'} \leq d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1} < H_{1,n+1},$$

which implies that $R_{D_{2,n}}(M) = R_{H_1}(M)$. On the other hand,

$$M = (d_{2^{i_1-1}+1} + d_{2^{N-1}+1}) + (d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}),$$

where $d_{2^{i_1-1}+1} + d_{2^{N-1}+1} \in C_{2,n}$ and

$$d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \notin C_{2,n},$$

and we have

$$d_{2^{i_1-1}+1} + d_{2^{N-1}+1} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$$

because $t \geq 2$. This gives $R_{H_0}(M) > R_{C_{2,n}}(M)$. It follows from $R_{H_0}(M) = R_{H_1}(M)$ that $R_{D_{2,n}}(M) > R_{C_{2,n}}(M)$. Assume that $t = 1$, that is, $H_{0,n+1} = d_{2^{i_1-1}+1} + d_{2^{N-1}+1}$. The previous argument shows that for $m < 2d_{2^{i_1-1}+1} + 2d_{2^{N-1}+1}$, we have $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$. Moreover, if $m \geq 2d_{2^{i_1-1}} + 2d_{2^{N-1}+1} = 2H_{0,n+1}$ and $R_{C_{2,n}}(m) \neq 0$ or $R_{D_{2,n}}(m) \neq 0$, then

$$m = \delta_0 d_1 + \sum_{u=1}^{s} \delta_u d_{2^{j_u-1}+1} + 2d_{2^{N-1}+1},$$

where $\delta_0 \in \{0, 1, 2\}$, $\delta_u \in \{1, 2\}$, $1 \leq j_1 < j_2 < \ldots < j_s < N$ and $j_s \geq i_1$. If $m = h + h^{'}$ with $h, h^{'} \in H_0$ or $h, h^{'} \in H_1$, $h < h^{'}$, then

$$h^{'} = \varepsilon_0 d_1 + \sum_{l=1}^{r} d_{2^{h_l-1}+1} + d_{2^{N-1}+1},$$

where $1 \leq h_1 < j_2 < \ldots < h_r$ and $h_r = j_s \geq i_1$. Hence, we have $h^{'} \geq H_{0,n+1}$. It follows that $h^{'} \notin C_{2,n}$ and $h^{'} \notin D_{2,n}$, which implies that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m) = 0$. This proves $s_l = t_k = +\infty$. □

Lemma 3. *For $1 \leq n < 2^{N-1}$, let $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$. Let $H_{0,n+1} = \varepsilon_0 d_1 + \sum_{i=1}^{N} \varepsilon_i d_{2^{i-1}+1}$. If $\varepsilon_0 = 1$ and $H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$, where $1 \leq i_1 < i_2 < \ldots < i_t < N$, then we have*

(i) $p_i = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$,

(ii) $q_j > p_i$,

(iii) $t_k = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}$,

(iv) $s_l > t_k$.

Proof. We prove (i) and (ii) simultaneously. It is enough to show that for $\varepsilon_0 = 1$ and

$$H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

if $m < 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$, then $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$ and $r_{D_1+D_{2,n}}(K) < r_{C_1+C_{2,n}}(K)$, where

$$K = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}.$$

If $m < d_{2^{N-1}+1}$, then it follows from (6) and (10) that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m) = 0$. Assume that

$$d_{2^{N-1}+1} \leq m < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}.$$

If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (11) that $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. Since

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} < H_{0,n+1},$$

we have $h^{'} \in C_{2,n}$, which yields $R_{H_0}(m) = r_{C_1+C_{2,n}}(m)$. Upon writing $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_1$, it follows from (5) and (7) that $h \in D_1$ and $h^{'} \in H_1 \setminus D_1$. Since

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{1,n+1},$$

we have $h^{'} \in D_{2,n}$. As $R_{H_0}(m) = R_{H_0}(m)$, we have $r_{D_1+D_{2,n}}(m) = r_{C_1+C_{2,n}}(m)$. Suppose that

$$\begin{aligned} &d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \\ \leq\ & m < 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}. \end{aligned}$$

Then, we may assume that m can be written in the form

$$m = \delta_0 d_1 + \sum_{i=1}^{u} d_{2^{x_j-1}+1} + \sum_{i=1}^{v} 2d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\delta_0 \in \{0, 1, 2\}$ and $1 \leq x_1 < x_2 < \ldots < x_u < i_1$ and $1 \leq y_1 < y_2 < \ldots < y_v < i_1$, and $x_\alpha \neq y_\beta$ are integers; otherwise, $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m) = 0$.

Since $H_{0,n+1} = d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$, we have t is even, thus $\delta_0 + u + 2v + t + 1$ is even, which implies that $\delta_0 + u$ is odd. We distinguish three cases.

`Case 1.` $\delta_0 = 0$. Then, u is odd. If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \neq \emptyset$ and $w + v + t + 1$ is even. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $r_{C_1+C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$. Furthermore, if $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_1$, then it follows from (5) and (7) that $h \in D_1$ and $h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \neq \emptyset$ and $w + v + t + 1$ is odd. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $r_{D_1+D_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$.

Case 2. $\delta_0 = 1$. Then, $1 + u + 2v + t + 1$ is even, which implies that u is even.

If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0, 1\}$ and $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\}$ and $\varepsilon_0 + w + v + t + 1$ is even. If $u = 0$, then $\{z_1, \ldots, z_w\} = \emptyset$ and for a suitable ε_0, there is only one way to choose $h^{'}$, and so $r_{C_1+C_{2,n}}(m) = R_{H_0}(m) - 1$. When $u > 0$ is even, to choose the pairs $(\varepsilon_0, \{z_1, \ldots, z_w\})$ we have $2 \cdot 2^{u-1} = 2^u$ possibilities, thus we have $r_{C_{2,n}}(m) = R_{H_0}(m) - 2^u$.

On the other hand, if $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then $h \in D_1$ and $h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = \varepsilon_0 d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0, 1\}$ and $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\}$ and $\varepsilon_0 + w + v + t + 1$ is odd. If $u = 0$, then $\{z_1, \ldots, z_w\} = \emptyset$, and for a suitable ε_0, there is only one way to choose $h^{'}$, and so $r_{D_1+D_{2,n}}(m) = R_{H_1}(m) - 1$. For $u > 0$ even, to choose the pairs $(\varepsilon_0, \{z_1, \ldots, z_w\})$ we have $2 \cdot 2^{u-1} = 2^u$ possibilities, thus we have $r_{D_1+D_{2,n}}(m) = R_{H_1}(m) - 2^u$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$.

Case 3. $\delta_0 = 2$. Then, $2 + u + 2v + t + 1$ is even, thus u is odd. If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then $h \in C_1$ and $h^{'} \in H_0 \setminus C_1$. It is clear that $h^{'} \notin C_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \neq \emptyset$ and $1 + w + v + t + 1$ is even. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $r_{C_1+C_{2,n}}(m) = R_{H_0}(m) - 2^{u-1}$. Moreover, if $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_1$, then $h \in D_1$, $h^{'} \in H_1 \setminus D_1$. It is clear that $h^{'} \notin D_{2,n}$ if and only if $h^{'}$ can be written in the form

$$h^{'} = d_1 + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + \sum_{j=1}^{v} d_{2^{y_j-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \ldots, z_w\} \subset \{x_1, \ldots, x_u\} \neq \emptyset$ and $1 + w + v + t + 1$ is odd. There are 2^{u-1} ways to choose the set $\{z_1, \ldots, z_w\}$, thus we have $R_{D_1+D_{2,n}}(m) = R_{H_1}(m) - 2^{u-1}$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $r_{C_1+C_{2,n}}(m) = r_{D_1+D_{2,n}}(m)$. If

$$K = 2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = h + h^{'}$$

with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h \in C_1$, $h^{'} \in H_0 \setminus C_1$ and $h^{'}$ can be written in the form

$$h^{'} = d_{2^{i_1-1}+1} + \sum_{j=1}^{w} d_{2^{z_j-1}+1} + d_{2^{N-1}+1},$$

where $\{z_1, \dots, z_w\} \subset \{i_2, \dots, i_t\} \neq \emptyset$. Thus, we have

$$h^{'} \le d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}$$

$$< d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{0,n+1},$$

which implies $h^{'} \in C_{2,n}$ and $R_{H_0}(K) = r_{C_1+C_{2,n}}(K)$. In the last step, we prove $r_{C_1+C_{2,n}}(K) > r_{D_1+D_{2,n}}(K)$. It is clear that

$$K = d_{2^{i_1-1}+1} + (d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}) = d_{2^{i_1-1}+1} + H_{1,n+1},$$

where $d_{2^{i_1-1}+1}, H_{1,n+1} \in H_1$. Since $H_{1,n+1} \notin D_{2,n}$, we have $R_{H_1}(K) > r_{D_1+D_{2,n}}(K)$. It follows from $R_{H_0}(K) = R_{H_1}(K)$ that $r_{D_1+D_{2,n}}(K) < r_{C_1+C_{2,n}}(K)$.

We will prove (iii) and (iv) simultaneously. Let

$$L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}.$$

We have to prove that if $m < L$, then $R_{D_{2,n}}(m) = R_{C_{2,n}}(m)$ and $R_{D_{2,n}}(L) < R_{C_{2,n}}(L)$. If $m < 2d_{2^{N-1}+1}$, then by using (7) and (11), we get that $R_{D_{2,n}}(m) = R_{C_{2,n}}(m) = 0$. Assume that

$$2d_{2^{N-1}+1} \le m < L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}.$$

If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. This implies that $h \ge d_{2^{N-1}+1}$ and

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} < H_{0,n+1}.$$

It follows that $h, h^{'} \in C_{2,n}$, which yields $R_{H_0}(m) = R_{C_{2,n}}(m)$. If $m = h + h^{'}$ with $h < h^{'}$ and $h, h^{'} \in H_1$, then it follows from (5) and (6) that $h, h^{'} \in H_1 \setminus D_1$. Since $h \ge d_{2^{N-1}+1}$ and

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{1,n+1},$$

it follows that $h, h^{'} \in D_{2,n}$, which yields $R_{H_1}(m) = R_{D_{2,n}}(m)$. As $R_{H_0}(m) = R_{H_1}(m)$, it follows that $R_{C_{2,n}}(m) = R_{D_{2,n}}(m)$. If

$$L = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \dots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1} = h + h^{'}$$

with $h < h^{'}$ and $h, h^{'} \in H_0$, then it follows from (5) and (10) that $h, h^{'} \in H_0 \setminus C_1$. It follows that $h > d_{2^{N-1}+1}$ and

$$h^{'} < d_{2^{i_1-1}+1} + d_{2^{z_1-1}+1} + \ldots + d_{2^{z_w-1}+1} + d_{2^{N-1}} < H_{0,n+1},$$

thus we have $h, h^{'} \in C_{2,n}$, which implies that $R_{H_0}(L) = R_{C_{2,n}}(L)$. On the other hand,

$$\begin{aligned} L &= d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1} \\ &= d_{2^{N-1}+1} + (d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}) \\ &= d_{2^{N-1}+1} + H_{1,n+1}. \end{aligned}$$

Note that $H_{1,n+1}, d_{2^{N-1}+1} \in H_1$ and $H_{1,n+1} \notin D_{2,n}$, which gives $R_{H_1}(L) > R_{C_{2,n}}(L)$. It follows from $R_{H_0}(L) = R_{H_1}(L)$ that $R_{D_{2,n}}(L) > R_{C_{2,n}}(L)$. □

Now we are ready to prove that $H_0^{(n)} = C_1 \cup C_{2,n}$ and $H_1^{(n)} = D_1 \cup D_{2,n}$ hold for every $1 \le n \le 2^{N-1}$. We prove by induction on n that $C_1 \cup C_{2,n} = H_0^{(n)}$ and $D_1 \cup D_{2,n} = H_1^{(n)}$. We have already proved $C_1 \cup C_{2,1} = H_0^{(1)}$ and $D_1 \cup D_{2,1} = H_1^{(1)}$. Assume that $C_1 \cup C_{2,n} = H_0^{(n)}$ and $D_1 \cup D_{2,n} = H_1^{(n)}$ hold for some $1 \le n < 2^{N-1}$. We will prove that $C_1 \cup C_{2,n+1} = H_0^{(n+1)}$ and $D_1 \cup D_{2,n+1} = H_1^{(n+1)}$ hold, i.e., $c_{2^{N-1}+n+1} = H_{0,n+1}$ and $d_{2^{N-1}+n+1} = H_{1,n+1}$. Let

$$H_{0,n+1} = \varepsilon_0 d_1 + \sum_{j=1}^{t} d_{2^{i_j-1}+1} + d_{2^{N-1}+1},$$

where $\varepsilon_0 \in \{0, 1\}$, $(1 \le i_1 < \ldots < i_t < N)$.

`Case 1.` $\varepsilon_0 = 0$, $t = 1$. We know from Lemma 1 that

$$min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}$$

and from Lemma 2 that $t_k = s_l = +\infty$. These facts imply that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$. Furthermore, we know that $c_{2^{N-1}+1} = d_{2^{N-1}+1} + d_1$, thus we have $c_{2^{N-1}+n+1} + d_1 = d_{2^{N-1}+n+1}$, and then $d_{2^{N-1}+n+1} > c_{2^{N-1}+n+1}$. It follows from Lemma 1 that $min\{p_i, c_{2^{N-1}+n+1}\} = min\{q_j, d_{2^{N-1}+n+1}\}$ and from Lemma 2 that $p_i > q_j = H_{0,n+1}$. Then, we have $c_{2^{N-1}+n+1} = q_j = H_{0,n+1}$ and

$$d_{2^{N-1}+n+1} = c_{2^{N-1}+n+1} + d_1 = H_{0,n+1} + d_1 = H_{1,n+1}.$$

`Case 2.` $\varepsilon_0 = 0$, $t > 1$. Applying Lemma 2, we get that $p_i > q_j$, thus from Lemma 1 we have $min\{q_j, d_1 + d_{2^{N-1}+n+1}\} = min\{p_i, c_{2^{N-1}+n+1}\} = c_{2^{N-1}+n+1}$. On the other hand, it follows from Lemma 2 that $s_l < t_k$, thus by Lemma 1, we have

$$min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} = min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\}$$

$$= c_{2^{N-1}+1} + c_{2^{N-1}+n+1}.$$

Assume that $c_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+n+1}$. Then, we have

$$\begin{aligned} & c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+1} + d_1 + d_{2^{N-1}+n+1} \\ = \quad & 2d_1 + d_{2^{N-1}+1} + d_{2^{N-1}+n+1} \\ > \quad & d_{2^{N-1}+1} + d_{2^{N-1}+n+1} \geq min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}, \end{aligned}$$

which is a contradiction. It follows from Lemma 2 that $c_{2^{N-1}+n+1} = q_j = H_{0,n+1}$ and

$$\begin{aligned} c_{2^{N-1}+1} + c_{2^{N-1}+n+1} &= d_1 + d_{2^{N-1}+1} + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \\ &= H_{1,n+1} + d_{2^{N-1}+1} = min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} \\ &= min\{2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\}. \end{aligned}$$

Since

$$\begin{aligned} & d_1 + d_{2^{N-1}+1} + d_{2^{i_1-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \\ & \quad < 2d_{2^{i_1-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}, \end{aligned}$$

it follows that $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} = H_{1,n+1} + d_{2^{N-1}+1}$, thus we have $d_{2^{N-1}+n+1} = H_{1,n+1}$.

`Case 3.` $\varepsilon_0 = 1$. Applying Lemma 3, we get that $q_j > p_i$, thus from Lemma 1, we have $min\{p_i, c_{2^{N-1}+n+1}\} = min\{q_j, d_1 + d_{2^{N-1}+n+1}\} = d_1 + d_{2^{N-1}+n+1}$. On the other hand, it follows from Lemma 3 that $s_l > t_k$, thus by Lemma 1, we have

$$\begin{aligned} min\{t_k, c_{2^{N-1}+1} + c_{2^{N-1}+n+1}\} &= min\{s_l, d_{2^{N-1}+1} + d_{2^{N-1}+n+1}\} \\ &= d_{2^{N-1}+1} + d_{2^{N-1}+n+1}. \end{aligned}$$

Assume that $c_{2^{N-1}+1} + c_{2^{N-1}+n+1} = d_{2^{N-1}+1} + d_{2^{N-1}+n+1}$. Then, we have $d_{2^{N-1}+1} + d_{2^{N-1}+n+1} = d_1 + d_{2^{N-1}+1} + c_{2^{N-1}+n+1}$, thus we have $d_{2^{N-1}+n+1} = d_1 + c_{2^{N-1}+n+1}$. It follows that $d_1 + d_{2^{N-1}+1} = 2d_1 + c_{2^{N-1}+n+1} = min\{p_i, c_{2^{N-1}+n+1}\}$, which is a contradiction because $d_1 > 0$. Then, we have

$$d_{2^{N-1}+1} + d_{2^{N-1}+n+1} = t_k = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + 2d_{2^{N-1}+1}.$$

It follows that

$$d_{2^{N-1}+n+1} = d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} = H_{1,n+1}.$$

Applying Lemma 1 and Lemma 3, we get that

$$\begin{aligned} & d_1 + d_{2^{N-1}+n+1} = H_{0,n+1} \\ = \quad & d_1 + d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1} \\ = \quad & min\{p_i, c_{2^{N-1}+n+1}\} \\ = \quad & min\{2d_{2^{i_1-1}+1} + d_{2^{i_2-1}+1} + \ldots + d_{2^{i_t-1}+1} + d_{2^{N-1}+1}, c_{2^{N-1}+n+1}\} \\ = \quad & c_{2^{N-1}+n+1}, \end{aligned}$$

thus we have $c_{2^{N-1}+n+1} = H_{0,n+1}$. The proof of Theorem 4 has been completed.

5. Proof of Theorem 6

First we prove that for

$$H = H(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l}-1, 2^{2l+1}-1, 2(2^{2l+1}-1), 4(2^{2l+1}-1), 8(2^{2l+1}-1), \dots),$$

$C = H_0$ and $D = H_1$, we have $C \cup D = \mathbb{N}$, $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$ and $R_C(m) = R_D(m)$. It is easy to see that for $H' = H(h_1, h_2, \dots, h_{2l+1}) = H(1, 2, 4, 8, \dots, 2^{2l-1}, 2^{2l}-1)$, $C' = H_0'$ and $D' = H_1'$, we have $C' \cup D' = [0, 2^{2l+1}-2]$ and $C' \cap D' = \{2^{2l} - 1\}$ because

$$2^{2l} - 1 = h_{2l+1} = h_1 + h_2 + \dots + h_{2l} = 1 + 2 + 4 + \dots + 2^{2l-1}.$$

Furthermore, for $H'' = H(2^{2l+1} - 1, 2(2^{2l+1} - 1), 4(2^{2l+1} - 1), 8(2^{2l+1} - 1), \dots)$, $C'' = H_0''$ and $D'' = H_1''$, we have $C'' \cup D'' = (2^{2l+1} - 1)\mathbb{N}$ and $C'' \cap D'' = \emptyset$, which implies $C \cup D = \mathbb{N}$ and $C \cap D = 2^{2l} - 1 + (2^{2l+1} - 1)\mathbb{N}$. Moreover, by Theorem 3, $R_C(m) = R_D(m)$ for every positive integer m. On the other hand, let us suppose that for some sets C and D, we have $C \cup D = \mathbb{N}$ and $C \cap D = r + m\mathbb{N}$. By Conjecture 2, we may assume that for some Hilbert cube $H(h_1, h_2, \dots)$, we have $C = H_0$ and $D = H_1$. We have to prove the existence of integer l such that $h_i = 2^{i-1}$ for $1 \le i \le 2l$, $h_{2l+1} = 2^{2l} - 1$ and $h_{2l+2+j} = 2^j(2^{2l+1} - 1)$ for every nonnegative integer j. We may suppose that $h_1 = 1$ and $h_2 = 2$. Consider the Hilbert cube $H(1, 2, 4, \dots, 2^u, h_{u+2}, \dots)$, where $h_{u+2} \ne 2^{u+1}$. Let us write $v = h_{u+2}$. We will prove that $v = 2^{u+1} - 1$. Assume that $v > 2^{u+1}$. Then, it is clear that $2^{u+1} \notin H$ because $1 + 2 + \dots + 2^u = 2^{u+1} - 1 < 2^{u+1}$. Thus, we have $v < 2^{u+1}$ i.e., $v \le 2^{u+1} - 1$. Assume that $v \le 2^{u+1} - 2$. Considering v as a one term sum, it follows that $v \in D$. Moreover, if $v = \sum_{i=0}^{u} \lambda_i 2^i$, $\lambda_i \in \{0, 1\}$, then $\sum_{i=0}^{u} \lambda_i$ must be even; otherwise, v would have two different representations from D. It follows that $v \in C$ and $v + 1 = h_1 + h_{u+2} \in C$. Furthermore, if we have a representation $v + 1 = \sum_{i=0}^{u} \delta_i 2^i$, $\delta_i \in \{0, 1\}$, then $\sum_{i=0}^{u} \delta_i$ must be odd; otherwise, v would have two different representations from C. This implies that $v + 1 \in D$, thus we have v, $v+1 \in C \cap D$. It follows that $C \cap D = \{v, v+1, \dots\}$ is an arithmetic progression with common difference 1. This implies that the generating functions of the sets C and D are of the form

$$C(z) = p(z) + \frac{z^v}{1 - z},$$

where $p(z)$ is a polynomial, and

$$D(z) = q(z) + \frac{z^v}{1 - z},$$

where $q(z)$ is a polynomial, and

$$p(z)+q(z)=1+z+z^2+\ldots+z^{v-1}=\frac{1-z^v}{1-z}.$$

Since $R_C(n)=R_D(n)$, we have $C^2(z)-D^2(z)=C(z^2)-D(z^2)$. It follows that

$$\Big(p(z)+\frac{z^v}{1-z}\Big)^2-\Big(q(z)+\frac{z^v}{1-z}\Big)^2=p(z^2)+\frac{z^{2v}}{1-z^2}-q(z^2)-\frac{z^{2v}}{1-z^2},$$

which implies

$$p^2(z)-q^2(z)+\frac{2z^v}{1-z}(p(z)-q(z))=p(z^2)-q(z^2).$$

Thus we have

$$(p(z)-q(z))\cdot\frac{1+z^v}{1-z}=p(z^2)-q(z^2).$$

We get

$$(p(z)-q(z))\cdot(1+z^v)=(p(z^2)-q(z^2))\cdot(1-z).$$

The leading coefficient in one side is -1 and the other side is 1, which is a contradiction. Thus we get that $v=2^{u+1}-1$. It follows that the Hilbert cube is of the form $H(1,2,4,8,\ldots,2^u,2^{u+1}-1,\ldots)$. As $h_{u+2}=2^{u+1}-1=1+2+\ldots+2^u=h_1+h_2+\cdots+h_{u+1}$, and $2^{u+1}-1$ is regarded as a one-term sum contained in D, we have $u+1$ must be even, i.e., $u+1=2l$. It follows that there exists an integer l such that $h_i=2^{i-1}$ for $1\le i\le 2l$ and $h_{2l+1}=2^{2l}-1$. It follows that $2^{2l}-1\in C\cap D$ and $r=2^{2l}-1$.

We apply induction on j to show that $h_{2l+2+j}=2^j(2^{2l+1}-1)$ for every nonnegative integer j. For $j=0$, take the Hilbert cube of the form $H(1,2,4,8,\ldots,2^{2l-1},2^{2l}-1,h_{2l+2},\ldots)$. Write $w=h_{2l+2}$. We prove that $w=2^{2l+1}-1$. Suppose that $w>2^{2l+1}-1$. Since $1+2+\ldots+2^{2l-1}+2^{2l}-1<2^{2l+1}-1$, it follows that $2^{2l+1}-1\notin H=C\cup D$, which is impossible. Therefore, $w\le 2^{2l+1}-1$. Suppose that $w\le 2^{2l+1}-3$. We will show that $w\in C\cap D$. Obviously, w is a one-term sum contained in D. Since w has a representation from $H(h_1,\ldots,h_{2l+1})$, w must be an element of C; otherwise, w would have two different representations from D, which is absurd. In the next step, we prove $w+1\in C\cap D$. Obviously, $w+1=h_1+h_{2l+2}$ as a two-term sum contained in C. Since $w+1$ can be written in terms of the Hilbert cube $H(h_1,\ldots,h_{2l+1})$ and $w+1\le 2^{2l+1}-2$, we have $w+1\in D$. It follows that w, $w+1\in C\cap D$, which is a contradiction. It follows that the only possible values of w are $w=2^{2l+1}-2$, and $w=2^{2l+1}-1$. Suppose that $w=2^{2l+1}-2$. Then, it is clear that $w\in D$. On the other hand, $2^{2l}-2=1+2+\ldots+2^{2l-1}+2^{2l}-1=h_1+h_2+\cdots+h_{2l+1}$, where in the right hand side, there are $2l+1$ terms, which is impossible. Thus we have $w=2^{2l+1}-1$. In this case,

$2^{2l}-1$, $(2^{2l}-1)+(2^{2l+1}-1) \in C \cap D$, $(C\cap D)\cap\{1,2,\dots,2^{2l+1}-1\} = \{2^{2l}-1\}$. It follows that $m \mid 2^{2l+1}-1$. If $m \le \frac{2^{2l+1}-1}{2}$, then $(C\cap D)\cap\{1,2,\dots,2^{2l+1}-1\} \ne \{2^{2l}-1\}$, a contradiction. Then, we have $r = 2^{2l}-1$ and $m = 2^{2l+1}-1$.

In the induction step, we assume that for some k, we know that $h_{2l+2+j} = 2^j(2^{2l+1}-1)$ holds for $j=0,1,\dots,k$, and we prove $h_{2l+2+k+1} = 2^{k+1}(2^{2l+1}-1)$. Let $H^{(k)} = H(1,2,4,8,\dots,2^{2l-1},2^{2l}-1,2^{2l+1}-1,2(2^{2l+1}-1),4(2^{2l+1}-1),8(2^{2l+1}-1),\dots,2^k(2^{2l+1}-1))$, $C^{(k)} = H_0^{(k)}$ and $D^{(k)} = H_0^{(k)}$. Then

$$C^{(k)} \cap D^{(k)} = \{2^{2l}-1+i(2^{2l+1}-1) : i = 0,1,\dots,2^k-1\}.$$

If $C = H_0(1,2,4,8,\dots,2^{2l-1},2^{2l}-1,2^{2l+1}-1,2(2^{2l+1}-1),4(2^{2l+1}-1),8(2^{2l+1}-1),\dots,2^k(2^{2l+1}-1),h_{2l+2+k+1},\dots)$, then $C\cap D = \{e_1,e_2,\dots\}$, where $e_i = 2^{2l}-1+(i-1)(2^{2l+1}-1)$ for $i \ge 1$, and $e_{2^{k+1}+1} = 2^{2l}-1+2^{k+1}(2^{2l+1}-1)$. Furthermore, $e_{2^{k+1}+1} = 2^{2l}-1+h_{2l+1+k+1}$, and then $h_{2l+1+k+1} = 2^{k+1}(2^{2l+1}-1)$, which completes the proof.

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