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# DETERMINANTAL EXPRESSIONS FOR BERNOULLI POLYNOMIALS

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**Abstract**

In this paper, we study determinantal expressions for Bernoulli polynomials by making use of certain polynomial analogues of the Saalschütz-Gelfand identity for Bernoulli numbers. As a result, we establish quite new determinantal expressions that provide us with various properties of these polynomials.

## 1. Introduction

The Bernoulli numbers $B_n$ and polynomials $B_n(x)$, $n = 0, 1, 2, \ldots$, are defined by the generating functions

$$\begin{aligned} \mathcal{F}(t) &:= \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} \quad (|t| < 2\pi); \\ \mathcal{F}(t,x) &:= \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!} \quad (|t| < 2\pi), \end{aligned} \tag{1.1}$$

respectively. They have numerous important applications in various branches of mathematics, most notably in number theory, combinatorics, numerical and asymptotic analysis, the theory of difference equations, and others.

It is easy to find that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $\ldots$, and these numbers satisfy $B_{2n+1} = 0$ and $(-1)^{n-1} B_{2n} > 0$ for all $n \geq 1$. Further, since $\mathcal{F}(t,x) = \mathcal{F}(t)e^{xt}$, we see that $B_n(0) = B_n$, and $B_n(x)$ is expressed as

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i} \quad (n \geq 0). \tag{1.2}$$

Using this expression, it can be shown that for $n \geq 1$,

$$\text{(i)} \quad \frac{d}{dx}B_n(x) = nB_{n-1}(x); \quad \text{(ii)} \quad \int_0^x B_{n-1}(t)dt = \frac{B_n(x) - B_n}{n}. \tag{1.3}$$

Since $B_0(x) = 1$, the first property (i) in (1.3) tells us that the Bernoulli polynomials form an Appell sequence.

Many kinds of recurrence relations for these numbers and polynomials are known (see, e.g., [19, 14, 12, 8]). Among them, the most basic linear ones are

$$\begin{aligned} &\text{(i)} && \sum_{i=0}^{n-1} \binom{n}{i} \frac{B_{i+1}}{i+1} + \frac{1}{n+1} = 0 \quad (n \geq 1); \\ &\text{(ii)} && \sum_{i=0}^{n-1} \binom{n}{i} \frac{B_{i+1}(x)}{i+1} + \frac{1}{n+1} = x^n \quad (n \geq 1). \end{aligned} \tag{1.4}$$

These identities are easily obtained by expanding both sides of each of the functional relations $\mathcal{F}(t)(e^t - 1) = t$ and $\mathcal{F}(t, x)(e^t - 1) = te^{xt}$ into the Maclaurin power series and then comparing the coefficients of $t^{n+1}$ on both sides, respectively.

The identity (1.4) (i) involves all the preceding Bernoulli numbers up to $B_n$. In contrast to this, the following *shortened* (or *incomplete*) recurrence relation was discovered by Saalschütz [18], and later by M. B. Gelfand [11]:

$$\begin{aligned} \sum_{i=0}^{k} \binom{k}{i} \frac{B_{m+1+i}}{m+1+i} &+ (-1)^{k+m} \sum_{j=0}^{m} \binom{m}{j} \frac{B_{k+1+j}}{k+1+j} \\ &= \frac{(-1)^{m+1}}{k+m+1} \binom{k+m}{k}^{-1}, \end{aligned} \tag{1.5}$$

which is valid for arbitrary integers $k, m \geq 0$. An elementary proof and a brief historical overview can be found in [1, 2, 3]. The most remarkable feature of (1.5) is that the first $\min\{k, m\}$ Bernoulli numbers are completely missing.

Such type recurrences must go back to von Ettingshausen [10] in 1827 and later Stern [21] in 1878. Indeed, they first discovered the following surprising identity that involves only the second half of Bernoulli numbers up to $B_{2m+1}$:

$$\sum_{i=0}^{m+1} \binom{m+1}{i} (m+i+1)B_{m+i} = 0, \quad m \geq 0. \tag{1.6}$$

As will be mentioned details in the next Section 2, both (1.5) and (1.6) can be extended to Bernoulli polynomials and they play important roles in this paper.

There are many interesting approaches to searching for determinantal expressions for Bernoulli polynomials (see, e.g., [6, 5, 7, 16, 17] and the references therein). For instance, by making use of the fundamental difference equation

$$B_k(x+1) - B_k(x) = kx^{k-1} \quad (k \geq 1), \tag{1.7}$$

which is easily shown from the relation $\mathcal{F}(t.x+1) - \mathcal{F}(t,x) = te^{xt}$, we can deduce for $n \geq 1$,

$$\frac{B_n(x)}{n!} = \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{2!} & 1 & 0 & \cdots & 0 & \frac{x}{1!} \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & \frac{x^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & 1 & \frac{x^{n-1}}{(n-1)!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & \frac{x^n}{n!} \end{bmatrix}. \tag{1.8}$$

Indeed, writing (1.7) as

$$\sum_{i=0}^{k-1} \frac{B_i(x)}{i!(k-i)!} = \frac{x^{k-1}}{(k-1)!},$$

which is actually equivalent to (1.4) (ii) with $n = k-1$, consider a system of linear equations by taking $k = 1, 2, \ldots, n+1$. Then, according to Cramer's rule we can obtain (1.8) by using ordinary determinantal operations. See also [6, 7] for a slightly different proof of (1.8).

The main purpose of this paper is to study determinantal expressions for Bernoulli polynomials based on shortened recurrence relations. This paper is organized as follows: In Section 2 we first present a polynomial analogue of (1.5) (Lemma 2.1), and subsequently, applying it we deduce three kinds of recurrence relations (Lemma 2.2). Based on them, we establish, in Section 3, quite new determinantal expressions for Bernoulli polynomials. In Section 4, to confirm the effectiveness of our expressions, by making use of them we recover some of well-known basic properties of Bernoulli polynomials. In Section 5 we discuss special topics related to the golden ratio. We conclude this paper, in Section 6, with some additional remarks on determinantal expressions for Euler polynomials that have a intimate connection with those for Bernoulli polynomials.

We would like to point out beforehand that the methods we will use are quite elementary without use of intricate tools and it does not require any advanced knowledge other than basic linear algebra.

## 2. Recurrence Relations

The following identity is a polynomial analogue of Saalschütz-Gelfand's formula (1.5), in which the first $\min\{m, k\}$ Bernoulli polynomials are completely missing.

**Lemma 2.1.** *For integers $k, m \geq 0$ and $q \geq 1$ we have*

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}\frac{B_{m+1+j}(x)}{m+1+j}q^{k-j}+\sum_{j=0}^{m}(-1)^{m+1+j}\binom{m}{j}\frac{B_{k+1+j}(x)}{k+1+j}q^{m-j}\\ =\sum_{r=0}^{q-1}(x+r)^k(x+r-q)^m+\frac{(-1)^{m+1}}{k+m+1}\binom{k+m}{k}^{-1}q^{k+m+1}.\end{aligned} \tag{2.1}$$

It is clear that (2.1) reduces to (1.5) by setting $x = 0$ and $q = 1$. An elementary proof of (2.1) and more details surrounding can be found, e.g., in [1, 2].

As consequences of (2.1), we can deduce the following three kinds of shortened recurrence relations that (typically) involve only the second half of all the Bernoulli polynomials up to $B_{2k}(x)$ or $B_{2k-1}(x)$.

**Lemma 2.2.** *For an integer $k \geq 1$ we have*

$$\begin{aligned}&\text{(i)} \quad \sum_{i=0}^{\lfloor (k-1)/2\rfloor}\binom{k}{2i+1}\frac{B_{2(k-i)}(x)}{2(k-i)}=X_k(x);\\ &\text{(ii)} \quad \sum_{i=0}^{\lfloor (k-1)/2\rfloor}\binom{k}{2i+1}B_{2(k-i)-1}(x)=Y_k(x);\\ &\text{(iii)} \quad \sum_{i=0}^{\lfloor (k-1)/2\rfloor}\binom{k}{2i+1}(2(k-i)-1)B_{2(k-i-1)}(x)=Z_k(x),\end{aligned} \tag{2.2}$$

*where*

$$\begin{aligned}X_k(x) &:= \frac{1}{2}(x^2-x)^k+\frac{(-1)^{k+1}}{2(2k+1)}\binom{2k}{k}^{-1};\\ Y_k(x) &:= \frac{k}{2}(2x-1)(x^2-x)^{k-1};\\ Z_k(x) &:= \frac{k}{2}\left\{(4k-2)x^2-(4k-2)x+k-1\right\}(x^2-x)^{k-2}\\ &= \binom{2k}{2}(x^2-x)^{k-1}+\binom{k}{2}(x^2-x)^{k-2}.\end{aligned} \tag{2.3}$$

*Proof.* In particular, taking $m = k \geq 0$ and $q = 1$ in (2.1), we have

$$\sum_{j=0}^{k}\binom{k}{j}\left(1+(-1)^{k+1+j}\right)\frac{B_{k+1+j}(x)}{k+1+j}=(x^2-x)^k+\frac{(-1)^{k+1}}{2k+1}\binom{2k}{k}^{-1}. \tag{2.4}$$

Since all the terms on the left-hand side of this identity involving odd-index Bernoulli polynomials vanish, we easily see that (2.4) leads to (2.2) (i) after dividing by 2. In order to derive (2.2) (ii) and (iii), differentiate successively both sides of (2.2) (i) with respect to $x$ based on (1.3) (i). □

As is easily seen, (2.2) (iii) is a polynomial analogue of von Ettingshausen-Stern's formula (1.6). Indeed, when $k > 2$, we can deduce (1.6) by setting $x = 0$.

## 3. Determinantal Expressions

In this section, based on shortened recurrence relations in (2.2) we establish new determinant expressions for Bernoulli polynomials. In addition, we discuss mutual relations between two adjacent even-index or odd-index Bernoulli polynomials.

For an integer $k \geq 1$, let us denote by $\boldsymbol{p}_k$ the row vector of length $\lfloor (k+1)/2 \rfloor$ constituted of alternate binomial coefficients, namely

$$\boldsymbol{p}_k := \begin{cases} (\binom{k}{0}, \binom{k}{2}, \ldots, \binom{k}{k-3}, \binom{k}{k-1}) & \text{if } k \text{ is odd;} \\ (\binom{k}{1}, \binom{k}{3}, \ldots, \binom{k}{k-3}, \binom{k}{k-1}) & \text{if } k \text{ is even.} \end{cases}$$

Using this vector for $k = 1, 2, \ldots, n$, consider the lower triangular matrix of order $n$ such that

$$P_n := \begin{bmatrix} \boldsymbol{p}_1 & & & & O \\ & \boldsymbol{p}_2 & & & \\ & \boldsymbol{p}_3 & & & \\ & & \boldsymbol{p}_4 & & \\ & & \boldsymbol{p}_5 & & \\ O & & & \ddots & \\ & & & & \boldsymbol{p}_n \end{bmatrix} \quad (n \geq 1).$$

Since the diagonal entries are $\binom{1}{0}, \binom{2}{1}, \ldots, \binom{n}{n-1}$, we have $\det P_n = n!$.

Further, let $P_n(x)$ be the polynomial matrix, replacing the last column of $P_n$ by $(X_1(x), \ldots, X_n(x))^{\mathrm{T}}$, where "T" means the transpose of a vector, namely

$$P_n(x) := \begin{bmatrix} \boldsymbol{p}_1 & & & & & X_1(x) \\ & \boldsymbol{p}_2 & & & & X_2(x) \\ & \boldsymbol{p}_3 & & O & & X_3(x) \\ & & \boldsymbol{p}_4 & & & \cdot \\ & & \boldsymbol{p}_5 & & & \cdot \\ O & & & \ddots & & \cdot \\ & & & & \boldsymbol{p}_n^* & X_n(x) \end{bmatrix} \quad (n \geq 1). \tag{3.1}$$

Here, we denote by $\boldsymbol{p}_n^*$ the row vector of length $\lfloor (n-1)/2 \rfloor$ obtained by removing the last entry $\binom{n}{n-1}$ from $\boldsymbol{p}_n$, that is,

$$\boldsymbol{p}_n^* := \begin{cases} \left(\binom{n}{0}, \binom{n}{2}, \ldots, \binom{n}{n-3}\right) & \text{if } n \text{ is odd;} \\ \left(\binom{n}{1}, \binom{n}{3}, \ldots, \binom{n}{n-3}\right) & \text{if } n \text{ is even.} \end{cases}$$

Similarly, let $Q_n(x)$ and $R_n(x)$ be the matrices, replacing the last column of $P_n(x)$ by $(Y_1(x), \ldots, Y_n(x))^{\mathrm{T}}$ and $(Z_1(x), \ldots, Z_n(x))^{\mathrm{T}}$, respectively.

Then, according to Cramer's rule, we obtain from Lemma 2.2 that for $n \geq 1$,

$$\begin{aligned} &\text{(i)} \quad \frac{B_{2n}(x)}{2n} = \frac{\det P_n(x)}{n!}; \quad \text{(ii)} \quad B_{2n-1}(x) = \frac{\det Q_n(x)}{n!}; \\ &\text{(iii)} \quad (2n-1)B_{2(n-1)}(x) = \frac{\det R_n(x)}{n!}. \end{aligned} \tag{3.2}$$

Now expand $\det P_n(x)$ along the first row. Since $\boldsymbol{p}_1 = (\binom{1}{0}) = (1)$ and the first column of $P_n(x)$ is $(1, O_{n-1})^{\mathrm{T}}$, where $O_k$ means the zero vector of length $k$, we see that the $(1, n)$ minor of $P_n(x)$ completely vanishes. So that, denoting newly

$$\widetilde{P}_{n-1}(x) := \begin{bmatrix} \boldsymbol{p}_2 & & & & X_2(x) \\ \boldsymbol{p}_3 & & & & X_3(x) \\ & \boldsymbol{p}_4 & & O & \cdot \\ & \boldsymbol{p}_5 & & & \cdot \\ O & & \ddots & & \cdot \\ & & & \boldsymbol{p}_n^* & X_n(x) \end{bmatrix} \quad (n \geq 2), \tag{3.3}$$

we have $\det P_n(x) = \det \widetilde{P}_{n-1}(x)$. Similarly, denoting by $\widetilde{Q}_{n-1}(x)$ and $\widetilde{R}_{n-1}(x)$ the matrices replaced the last column of $\widetilde{P}_{n-1}(x)$ by

$$(Y_2(x), \ldots, Y_n(x))^{\mathrm{T}} \quad \text{and} \quad (Z_2(x), \ldots, Z_n(x))^{\mathrm{T}},$$

respectively, we have $\det Q_n(x) = \det \widetilde{Q}_{n-1}(x)$ and $\det R_n(x) = \det \widetilde{R}_{n-1}(x)$.

Therefore, from (3.2) we can deduce the following determinantal expressions:

**Theorem 3.1.** *We have $B_0(x) = Z_1(x)$, $B_1(x) = Y_1(x)$, and for $n \geq 2$,*

$$\begin{aligned} &\text{(i)} \quad B_{2n}(x) = \frac{2 \det \widetilde{P}_{n-1}(x)}{(n-1)!}; \qquad \text{(ii)} \quad B_{2n-1}(x) = \frac{\det \widetilde{Q}_{n-1}(x)}{n!}; \\ &\text{(iii)} \quad B_{2(n-1)}(x) = \frac{\det \widetilde{R}_{n-1}(x)}{(2n-1) \cdot n!}. \end{aligned} \tag{3.4}$$

Since both (i) and (iii) in (3.4) are the expressions for Bernoulli polynomials with even-indices, it follows that

$$B_{2n}(x) = \frac{2 \det \widetilde{P}_{n-1}(x)}{(n-1)!} = \frac{\det \widetilde{R}_n(x)}{(2n+1) \cdot (n+1)!} \quad (n \geq 2).$$

In addition, since $\frac{d}{dx}X_k(x) = Y_k(x)$ and $\frac{d}{dx}Y_k(x) = Z_k(x)$, we have

$$\frac{d}{dx}\det \widetilde{P}_{n-1}(x) = \det \widetilde{Q}_{n-1}(x) \;\text{ and }\; \frac{d}{dx}\det \widetilde{Q}_{n-1}(x) = \det \widetilde{R}_{n-1}(x).$$

We next rephrase all the determinantal expressions in (3.4) in terms of $X_k(x)$, $Y_k(x)$ and $Z_k(x)$ as follows:

**Theorem 3.2.** *For $n \geq 2$, there exist uniquely $c_k \in \mathbb{Z}$, $k = 2, 3, \ldots, n$, such that*

$$\begin{aligned} &\text{(i)} \quad B_{2n}(x) = \frac{2}{(n-1)!}\sum_{k=2}^{n} c_k X_k(x); \quad \text{(ii)} \quad B_{2n-1}(x) = \frac{1}{n!}\sum_{k=2}^{n} c_k Y_k(x); \\ &\text{(iii)} \quad B_{2(n-1)}(x) = \frac{1}{(2n-1)\cdot n!}\sum_{k=2}^{n} c_k Z_k(x). \end{aligned} \tag{3.5}$$

*Proof.* For an integer $k$ with $2 \leq k \leq n$, let $c_k$ be the $(k-1, n-1)$ cofactor of $\widetilde{P}_{n-1}(x)$. Then, it is clear that $c_k$ is also the common $(k-1, n-1)$ cofactor of $\widetilde{Q}_{n-1}(x)$ and $\widetilde{R}_{n-1}(x)$. By expanding all the determinants appeared in (3.4) along their last columns, we can derive the expressions in (3.5). The uniqueness can be shown by the following inductive method. Suppose that there exists another integer $(n-1)$-tuple $(d_2, d_3, \ldots, d_n)$ satisfying (3.5) (i), that is,

$$\sum_{k=2}^{n} c_k X_k(x) = \sum_{k=2}^{n} d_k X_k(x).$$

Comparing the coefficients of the highest degree term $x^{2n}$ on the both sides, we see that $c_n = d_n$; thus, the above relation reduces to the one replaced $n$ by $n-1$. By the same argument, we can show that $c_{n-1} = d_{n-1}$. Continue such procedures until to arrive at $c_2 = d_2$. □

Since the sum parts on the right-hand sides of (i), (ii) and (iii) in (3.5) keep the same forms of linear combinations, it is possible to obtain $B_{2n-1}(x)$ and $B_{2n}(x)$ automatically by utilizing the linear form of $B_{2(n-1)}(x)$; and hence of $\det \widetilde{R}_{n-1}(x)$. We present below an easy example in the case when $n = 4$. At first, find the form of $\det \widetilde{R}_3(x)$. By a direct calculation one has

$$\det \widetilde{R}_3(x) = \det \begin{bmatrix} \binom{2}{1} & 0 & Z_2(x) \\ \binom{3}{0} & \binom{3}{2} & Z_3(x) \\ 0 & \binom{4}{1} & Z_4(x) \end{bmatrix} = 4Z_2(x) - 8Z_3(x) + 6Z_4(x), \tag{3.6}$$

and thus, $(c_2, c_3, c_4) = (4, -8, 6)$. Therefore, (3.6) yields from (3.5) (iii) that

$$\begin{aligned} B_6(x) &= \frac{1}{7 \cdot 4!}(4Z_2(x) - 8Z_3(x) + 6Z_4(x)) \\ &= \frac{1}{42}\left\{42(x^2-x)^3 - 21(x^2-x)^2 + 1\right\} \\ &= \frac{1}{42}\left(42x^6 - 126x^5 + 105x^4 - 21x^2 + 1\right). \end{aligned}$$

By utilizing this form we automatically obtain from (3.5) (ii),

$$\begin{aligned} B_7(x) &= \frac{1}{4!}(4Y_2(x) - 8Y_3(x) + 6Y_4(x)) \\ &= \frac{1}{6}(2x-1)(x^2-x)\left\{3(x^2-x)^2 - 6(x^2-x) + 4\right\} \\ &= \frac{1}{6}\left(6x^7 - 21x^6 + 21x^5 - 7x^3 + x\right) \end{aligned}$$

and also from (3.5) (i),

$$\begin{aligned} B_8(x) &= \frac{2}{3!}(4X_2(x) - 8X_3(x) + 6X_4(x)) \\ &= \frac{1}{3}\left\{3(x^2-x)^4 - 4(x^2-x)^3 + 2(x^2-x)^2 + (6s_4 - 8s_3 + 4s_2)\right\} \\ &= \frac{1}{3}\left(3x^8 - 12x^7 + 14x^6 - 7x^4 + 2x^2 - 1/10\right), \end{aligned}$$

where $s_k$ is the constant term of $X_k(x)$, i.e.,

$$s_k := X_k(0) = \frac{(-1)^{k+1}}{2(2k+1)}\binom{2k}{k}^{-1} \quad (k \geq 1). \tag{3.7}$$

We next search for mutual relations between two adjacent even-index or odd-index Bernoulli polynomials based on the expressions in (3.4). For this purpose, consider the matrix $\widetilde{P}_n(x)$ of order $n$ and expand its determinant along the $(n-1)$th column (i.e., the second column from the last). Since this column vector is

$$\left(O_{n-2}, \binom{n}{n-1}, \binom{n+1}{n-2}\right)^{\mathrm{T}} = (O_{n-2}, n, (n+1)n(n-1)/6)^{\mathrm{T}},$$

we have

$$\det \widetilde{P}_n(x) = n \det L_{n-1}(x) - \frac{(n+1)n(n-1)}{6} \det \widetilde{P}_{n-1}(x), \tag{3.8}$$

where $L_{n-1}(x)$ is the matrix of order $n-1$ obtained by deleting the $(n-1)$th row and column from $\widetilde{P}_n(x)$, namely

$$L_{n-1}(x) := \begin{bmatrix} \boldsymbol{p}_2 & & & & X_2(x) \\ \boldsymbol{p}_3 & & O & & X_3(x) \\ & \boldsymbol{p}_4 & & & \cdot \\ & \boldsymbol{p}_5 & & & \cdot \\ & & \ddots & & \cdot \\ & O & & \boldsymbol{p}_{n-1} & X_{n-1}(x) \\ & & & \boldsymbol{p}_{n+1}^{**} & X_{n+1}(x) \end{bmatrix} \tag{3.9}$$

with

$$\boldsymbol{p}_{n+1}^{**} := \begin{cases} (\binom{n+1}{0}, \binom{n+1}{2}, \ldots, \binom{n+1}{n-4}) & \text{if } n \text{ is even;} \\ (\binom{n+1}{1}, \binom{n+1}{3}, \ldots, \binom{n+1}{n-4}) & \text{if } n \text{ is odd.} \end{cases} \tag{3.10}$$

Note that $\boldsymbol{p}_{n+1}^{**}$ is the row vector removed the last two entries $\binom{n+1}{n-2}$ and $\binom{n+1}{n}$ from $\boldsymbol{p}_{n+1}$. In particular, $\boldsymbol{p}_2^{**}$, $\boldsymbol{p}_3^{**}$ and $\boldsymbol{p}_4^{**}$ become the empty vectors of length 0.

Similar to the above, denote by $M_{n-1}(x)$ and $N_{n-1}(x)$ the submatrices of $\widetilde{Q}_n(x)$ and $\widetilde{R}_n(x)$ obtained by replacing the last column of $L_{n-1}(x)$ by

$$(Y_2(x), \ldots, Y_{n-1}(x), Y_{n+1}(x))^{\mathrm{T}}; \quad (Z_2(x), \ldots, Z_{n-1}(x), Z_{n+1}(x))^{\mathrm{T}},$$

respectively. Expanding $\det \widetilde{Q}_n(x)$ and $\det \widetilde{R}_n(x)$ along their $(n-1)$th columns, one can get the relations

$$\det \widetilde{Q}_n(x) = n \det M_{n-1}(x) - \frac{(n+1)n(n-1)}{6} \det \widetilde{Q}_{n-1}(x); \tag{3.11}$$

$$\det \widetilde{R}_n(x) = n \det N_{n-1}(x) - \frac{(n+1)n(n-1)}{6} \det \widetilde{R}_{n-1}(x). \tag{3.12}$$

Applying (3.8), (3.11) and (3.12) to the corresponding expressions in (3.4), we are able to obtain the following mutual relations between two adjacent even-index or odd-index Bernoulli polynomials.

**Theorem 3.3.** *For $n \geq 2$, we have*

$$\begin{aligned} &\text{(i)} \quad B_{2(n+1)}(x) + \frac{(n+1)(n-1)}{6} B_{2n}(x) = \frac{2 \det L_{n-1}(x)}{(n-1)!}; \\ &\text{(ii)} \quad B_{2n+1}(x) + \frac{n(n-1)}{6} B_{2n-1}(x) = \frac{n \det M_{n-1}(x)}{(n+1)!}; \\ &\text{(iii)} \quad B_{2n}(x) + \frac{n(n-1)(2n-1)}{6(2n+1)} B_{2(n-1)}(x) = \frac{n \det N_{n-1}(x)}{(2n+1) \cdot (n+1)!}. \end{aligned} \tag{3.13}$$

As is easily seen, using repeatedly these relations, it is possible to derive three types of recurrences for 'any' number of consecutive even-index or odd-index Bernoulli polynomials. For example, taking $n = k, k+1, \ldots, k+m-1$ in (3.13) (i) for any integers $k, m \geq 2$ and then adding up all of them, we get

$$\begin{aligned} B_{2(m+k)}(x) + \sum_{i=1}^{m-1} \frac{(k+m-i)^2+5}{6} B_{2(m+k-i)}(x) + \frac{k^2-1}{6} B_{2k}(x) \\ = 2 \sum_{j=1}^{m} \frac{\det L_{k+m-1-j}(x)}{(k+m-1-j)!}. \end{aligned} \tag{3.14}$$

In the same way as the above, using (3.13) (ii) and (iii), slightly different types of recurrences of arbitrary length can be obtained.

If there exists a common zero $\theta$ of $B_{2(n+1)}(x)$ and $B_{2n}(x)$, then $\theta$ must satisfy $\det L_{n-1}(\theta) = \det N_n(\theta) = 0$ by (3.13) (i) and (iii). Similarly, we can say from (3.13) (ii) that if there exists a common zero $\xi$ of $B_{2n+1}(x)$ and $B_{2n-1}(x)$, then $\det M_{n-1}(\xi) = 0$. Note that even if there exist such common zeros, they are never multiple zeros. Indeed, it has been proved by Dilcher [9] that Bernoulli polynomials have no multiple zeros. For the trivial common zeros $0, 1/2$ and $1$ of $B_{2n+1}(x)$, see (h) in the next Section 4.

Using the above results on Bernoulli polynomials, we next discuss determinantal expressions for Bernoulli numbers.

Since $Y_1(0) = 1/2$ and $Y_k(0) = 0$ for $k \geq 2$, we see from (3.5) (ii) that $B_1 = -1/2$ and $B_{2n-1} = 0$ for $n \geq 2$. Next, by setting $x = 0$ in (3.4) (i) and (iii) we have for $n \geq 2$,

$$\text{(i)} \quad B_{2n} = \frac{2 \det \widetilde{P}_{n-1}(0)}{(n-1)!}; \quad \text{(ii)} \quad B_{2(n-1)} = \frac{\det \widetilde{R}_{n-1}(0)}{(2n-1) \cdot n!}. \tag{3.15}$$

Here, it is obvious that $\det \widetilde{R}_n(0) = 2n(n+1)(2n+1) \det \widetilde{P}_{n-1}(0)$.

To obtain more terse expressions, we now attempt to expand $\det \widetilde{P}_{n-1}(0)$ and $\det \widetilde{R}_{n-1}(0)$ along the first row and the last column, respectively. Setting $x = 0$ in (3.3) and using $s_k$ defined in (3.7), we have

$$\widetilde{P}_{n-1}(0) = \begin{bmatrix} \boldsymbol{p}_2 & & & & s_2 \\ \boldsymbol{p}_3 & & & & s_3 \\ & \boldsymbol{p}_4 & & O & \cdot \\ & \boldsymbol{p}_5 & & & \cdot \\ O & & \ddots & & \cdot \\ & & & \boldsymbol{p}_n^* & s_n \end{bmatrix} \quad (n \geq 2). \tag{3.16}$$

First, expand $\det \widetilde{P}_{n-1}(0)$ along the first row in consideration of $\boldsymbol{p}_2 = (\binom{2}{1}) = (2)$

and $s_2 = -1/60$. Then we obtain the determinantal identity

$$\det \widetilde{P}_{n-1}(0) = 2\det C_{n-2} + \frac{(-1)^{n-1}}{60}\det D_{n-2} \quad (n \geq 2), \tag{3.17}$$

where $C_{n-2}$ and $D_{n-2}$ are the submatrices of $\widetilde{P}_{n-1}(0)$ defined by $C_0 := [0]$ and $D_0 := [1]$ by convention, and for $n \geq 3$, using the last entry $\binom{3}{2} = 3$ of $\boldsymbol{p}_3$,

$$C_{n-2} := \begin{bmatrix} 3 & & & & s_3 \\ \boldsymbol{p}_4 & & O & & s_4 \\ \boldsymbol{p}_5 & & & & s_5 \\ & \boldsymbol{p}_6 & & & \cdot \\ & \boldsymbol{p}_7 & & & \cdot \\ O & & \ddots & & \cdot \\ & & & \boldsymbol{p}_n^* & s_n \end{bmatrix}; \quad D_{n-2} := \begin{bmatrix} \boldsymbol{p}_3 & & & & O \\ & \boldsymbol{p}_4 & & & \\ & \boldsymbol{p}_5 & & & \\ & & \boldsymbol{p}_6 & & \\ & & \boldsymbol{p}_7 & & \\ O & & & \ddots & \\ & & & & \boldsymbol{p}_n^* \end{bmatrix},$$

respectively. In particular, $\det C_1 = \det[s_3] = 1/280$ and $\det D_1 = \det[\binom{3}{0}] = 1$. Next, expand $\det \widetilde{R}_{n-1}(0)$ along the last column $(1, O_{n-2})^{\mathrm{T}}$. Then, since $Z_2(0) = 1$ and $Z_k(0) = 0$ for $k \geq 3$, we get immediately

$$\det \widetilde{R}_{n-1}(0) = (-1)^n \det D_{n-2}. \tag{3.18}$$

Consequently, by applying (3.17) and (3.18) to (3.15), we can finally obtain the following two kinds of determinantal expressions for nonzero Bernoulli numbers.

**Corollary 3.4.** *For an integer $n \geq 2$ we have*

$$\begin{aligned} &\text{(i)} \quad B_{2n} = \frac{1}{(n-1)!}\left(4\det C_{n-2} + (-1)^{n-1}\det D_{n-2}/30\right); \\ &\text{(ii)} \quad B_{2(n-1)} = \frac{(-1)^n \det D_{n-2}}{(2n-1)\cdot n!}. \end{aligned} \tag{3.19}$$

Subsequently, we discuss mutual relations between adjacent nonzero Bernoulli numbers by utilizing the expressions in (3.19). Consider the submatrix $T_{n-2}$ of order $n-2$ of $D_{n-1}$ obtained by deleting the $(n-2)$th row and the $(n-1)$th column from $D_{n-1}$, namely

$$T_{n-2} := \begin{bmatrix} \boldsymbol{p}_3 & & & & \\ & \boldsymbol{p}_4 & & O & \\ & \boldsymbol{p}_5 & & & \\ & & \ddots & & \\ & O & & \boldsymbol{p}_{n-1} & \\ & & & & \boldsymbol{p}_{n+1}^{**} \end{bmatrix} \quad (n \geq 2),$$

where $\boldsymbol{p}_{n+1}^{**}$ is the row vector defined in (3.10). Since both $\boldsymbol{p}_3^{**}$ and $\boldsymbol{p}_4^{**}$ are the empty vectors of length 0, assume that $T_0 = T_1 = [0]$ as convention.

Using the above matrices, we can state the following two kinds of mutual relations between adjacent nonzero Bernoulli numbers:

**Corollary 3.5.** *For $n \geq 2$ we have*

$$\begin{aligned} &\text{(i)} \quad B_{2n} + \frac{n(2n-1)}{30} B_{2(n-1)} = \frac{4 \det C_{n-2}}{(n-1)!}; \\ &\text{(ii)} \quad B_{2n} + \frac{n(n-1)(2n-1)}{6(2n+1)} B_{2(n-1)} = \frac{(-1)^n n \det T_{n-2}}{(2n+1)\cdot(n+1)!}. \end{aligned} \tag{3.20}$$

*Proof.* In order to obtain (3.20) (i), we have only to eliminate $\det D_{n-2}$ from two expressions in (3.19). For the proof of (3.20) (ii), consider the matrix $D_{n-1}$ of order $n-1$ and expand its determinant along the last column

$$\left(O_{n-3}, \binom{n}{n-1}, \binom{n+1}{n-2}\right)^{\mathrm{T}} = (O_{n-3}, n, (n+1)n(n-1)/6)^{\mathrm{T}}.$$

Then we have

$$\det D_{n-1} = \frac{(n+1)n(n-1)\det D_{n-2}}{6} - n \det T_{n-2}, \tag{3.21}$$

and this leads to (3.20) (ii) from (3.19) (ii). □

By repeatedly making use of the expressions in (3.20), we can also derive two kinds of recurrence relations for ‘any’ number of consecutive nonzero Bernoulli numbers. On the occasion of practical use of them, it requires in advance the knowledge of some corresponding values of $\det C_n$ or $\det T_n$. Since $\det C_n$ can be obtained by using $\det D_n$, it is desirable to explore a method of how to efficiently calculate $\det D_n$ and $\det T_n$. As is observed empirically, they are growing so rapidly.

The following is a list of the first few values of $\det D_n$ and $\det T_n$ with $\det D_0 = 1$ and $\det T_0 = 0$ by convention:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\det D_n$ | 1 | 1 | 4 | 36 | 600 | 16584 | 705600 | 43751232 | 3790108800 |
| $\det T_n$ | 0 | 0 | 1 | 24 | 736 | 31872 | 1939896 | 162226560 | 18182807424 |

Here, we note that each $\det T_n$ can be calculated by using $\det D_n$ and $\det D_{n+1}$. Indeed, the relation (3.21) tells us that

$$\det T_n = \frac{(n+3)(n+1)}{6} \det D_n - \frac{1}{n+2} \det D_{n+1} \quad (n \geq 0).$$

A square matrix $H_n = (h_{i,j})_{1 \leq i,j \leq n}$ is called a lower *Hessenberg matrix* if $h_{i,j} = 0$ for all $i, j$ with $j - i > 1$ and its determinant can be calculated based on a very effective recursive algorithm. As a matter of fact, letting $\det H_0 = 1$ by convention, it is easy to show by induction that $\det H_1 = h_{1,1}$ and for $n \geq 2$,

$$\det H_n = h_{n,n} \det H_{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k} h_{n,k} \prod_{j=k}^{n-1} h_{j,j+1} \det H_{k-1}. \tag{3.22}$$

Since both $D_n$ and $T_n$ are just the lower Hessenberg-type matrices, (3.22) is available for computing their determinants recursively. However, it is needed to calculate successively the determinants of submatrices of $D_n$ and $T_n$ corresponding to $H_1, H_2, \ldots, H_{n-1}$. Note that $D_k$ (resp. $T_k$) is not always the submatrix of $D_n$ (resp. $T_n$) corresponding to $H_k$ for each $k = 1, 2, \ldots, n-1$.

By way of experiment, we now attempt to utilize (3.22) for computing $\det T_n$. Denoting $T_n = (t_{i,j})_{1 \leq i,j \leq n}$ with the $(i, j)$ entry $t_{i,j}$, we see from the definition of $T_n$ that

$$\begin{aligned} &t_{k,1} = t_{k,2} = \cdots = t_{k,\lfloor k/2 \rfloor} = 0, \quad k = 2, 3, \ldots, n-1; \\ &t_{n,1} = t_{n,2} = \cdots = t_{n,n-\lfloor n/2 \rfloor} = 0. \end{aligned} \tag{3.23}$$

For each $k = 1, 2, \ldots, n-1$, let $T'_k := (t_{i,j})_{1 \leq i,j \leq k}$ be the submatrix of order $k$ of $T_n$ corresponding to $H_k$. Taking account of the conditions in (3.23) and using (3.22), we can derive the following recursive formulas for $\det T'_k$ and $\det T_n$.

**Lemma 3.6.** *For simplicity, denote $\alpha_k := \det T'_k$. Then, for an integer $n \geq 2$ we have, with $\alpha_0 = 1$ by convention,*

$$\begin{aligned} &\text{(i)} \quad \alpha_k = t_{k.k} \alpha_{k-1} + \sum_{l=\lfloor k/2 \rfloor + 1}^{k-1} (-1)^{k-l} t_{k,l} \prod_{j=l}^{k-1} t_{j,j+1} \alpha_{l-1} \ \ (1 \leq k \leq n-1); \\ &\text{(ii)} \quad \det T_n = t_{n,n} \alpha_{n-1} + \sum_{k=n+1-\lfloor n/2 \rfloor}^{n-1} (-1)^{n-k} t_{n,k} \prod_{j=k}^{n-1} t_{j,j+1} \alpha_{k-1}. \end{aligned} \tag{3.24}$$

Here, in particular, $t_{k,k} = \binom{k+2}{k-1}$ unless $k = n$ and $t_{n,n} = \binom{n+3}{n-2}$.

To illustrate the procedure of how to calculate $\det T_n$ based on Lemma 3.6, we present below a concrete example for $n = 2, 3, \ldots, 6$. By applying the recursive algorithm (3.24) (i), the following values of $\alpha_k$ $(1 \leq k \leq 5)$ can be easily obtained:

$$\begin{aligned} &\alpha_1 = \binom{3}{0} \alpha_0 = 1; \quad \alpha_2 = \binom{4}{1} \alpha_1 = 4; \quad \alpha_3 = \binom{5}{2} \alpha_2 - \binom{5}{0} \binom{4}{3} \alpha_1 = 36; \\ &\alpha_4 = \binom{6}{3} \alpha_3 - \binom{6}{1} \binom{5}{4} \alpha_2 = 600; \\ &\alpha_5 = \binom{7}{4} \alpha_4 - \binom{7}{2} \binom{6}{5} \alpha_3 + \binom{7}{0} \binom{5}{4} \binom{6}{5} \alpha_2 = 16584. \end{aligned}$$

By using these values, it is shown from (3.24) (ii) that

$$\det T_2 = \binom{5}{0}\alpha_1 = 1; \quad \det T_3 = \binom{6}{1}\alpha_2 = 24;$$
$$\det T_4 = \binom{7}{2}\alpha_3 - \binom{7}{0}\binom{5}{4}\alpha_2 = 736;$$
$$\det T_5 = \binom{8}{3}\alpha_4 - \binom{8}{1}\binom{6}{5}\alpha_3 = 31872;$$
$$\det T_6 = \binom{9}{4}\alpha_5 - \binom{9}{2}\binom{7}{6}\alpha_4 + \binom{9}{0}\binom{6}{5}\binom{7}{6}\alpha_3 = 1939896,$$

as indicated in the above table.

## 4. Basic Properties

In this section, in order to confirm the effectiveness of our determinantal expressions mentioned in Section 3, we recover some of basic properties of Bernoulli polynomials based on these expressions, not relying on the generating function $\mathcal{F}(t,x)$.

We begin by proving the following symmetric property:

**(a)** $B_n(1-x) = (-1)^n B_n(x) \quad (n \geq 0)$.

*Proof.* The case $n=0$ is trivial, so assume $n \geq 1$. Since $(1-x)^2-(1-x) = x^2-x$ and $2(1-x)-1 = -(2x-1)$, we have $X_k(1-x) = X_k(x)$ and $Y_k(1-x) = -Y_k(x)$ for $k \geq 1$, which imply $\det \widetilde{P}_{n-1}(1-x) = \det \widetilde{P}_{n-1}(x)$ and $\det \widetilde{Q}_{n-1}(1-x) = -\det \widetilde{Q}_{n-1}(x)$, respectively. So that (a) follows immediately from (3.4) (i) and (ii). □

**(b)** $B_n(x+1) = (-1)^n B_n(-x) \quad (n \geq 0)$.

*Proof.* Since $(x+1)^2 - (x+1) = x^2 + x$ and $2(x+1) - 1 = 2x + 1$, we have for $k \geq 1$,

$$X_k(x+1) = \frac{1}{2}(x^2+x)^k + s_k = X_k(-x);$$
$$Y_k(x+1) = \frac{k}{2}(2x+1)(x^2+x)^{k-1} = -Y_k(-x),$$

which lead to

$$\det \widetilde{P}_{n-1}(x+1) = \det \widetilde{P}_{n-1}(-x); \quad \det \widetilde{Q}_{n-1}(x+1) = -\det \widetilde{Q}_{n-1}(-x),$$

respectively. Thus, make use of (3.4) (i) and (ii) in order to deduce (b). □

The following is the fundamental difference equation as mentioned in (1.7).

**(c)** $B_n(x+1) - B_n(x) = nx^{n-1} \quad (n \geq 0)$.

*Proof.* In view of (b), we see that (c) is equivalent to $(-1)^n B_n(-x) - B_n(x) = nx^{n-1}$. By a direct calculation we have

$$X_k(-x) - X_k(x) = \frac{1}{2}\left((x^2+x)^k - (x^2-x)^k\right) = \frac{1}{2}\sum_{i=0}^{k}\binom{k}{i}\left(1-(-1)^{k-i}\right)x^{k+i}$$
$$= \sum_{j=0}^{\lfloor (k-1)/2\rfloor}\binom{k}{2j+1}x^{2k-1-2j} \quad (k \geq 2).$$

Here, we notice the fact that the ordered array of binomial coefficients is exactly the same as that of coefficients of $B_{2(k-i)}(x)/2(k-i)$, $i = 0, 1, \ldots, \lfloor (k-1)/2\rfloor$, on the left-hand side of (2.2) (i). Assuming that $n \geq 2$ is an even integer, consider the matrix $A_{n-1}(x)$ of order $n-1$, replacing the last column of $\widetilde{P}_{n-1}(x)$ by

$$(X_2(-x) - X_2(x), \ldots, X_n(-x) - X_n(x))^{\mathrm{T}}.$$

Multiply the $j$th column of $\det A_{n-1}(x)$ by $x^{2j+1}$ and subtract it from the last column for each $j = 1, 2, \ldots, n-2$. Then, from the above fact we see that the last column of $\det A_{n-1}(x)$ becomes $(O_{n-2}, nx^{2n-1})^{\mathrm{T}}$. So that, expanding $\det A_{n-1}(x)$ along this column and using the last entry $\binom{k}{k-1} = k$ of $\boldsymbol{p}_k$ $(2 \leq k \leq n-1)$, we get

$$\det A_{n-1}(x) = nx^{2n-1}\det\begin{bmatrix} \boldsymbol{p}_2 & & & O \\ \boldsymbol{p}_3 & & & \\ & \boldsymbol{p}_4 & & \\ & \boldsymbol{p}_5 & & \\ O & & \ddots & \\ & & & \boldsymbol{p}_{n-1} \end{bmatrix} = nx^{2n-1}\prod_{k=2}^{n-1} k = n!x^{2n-1},$$

which yields from (3.4) (i) that

$$B_{2n}(-x) - B_{2n}(x) = \frac{2\det A_{n-1}(x)}{(n-1)!} = 2nx^{2n-1},$$

and this proves (c) in the case when $n$ is even. The proof for an odd $n \geq 1$ can be given in almost the same way as done above by using $Y_k(x)$ instead of $X_k(x)$. □

Incidentally, by setting $x = 0$ in (c), we have

$$B_n(1) - B_n(0) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n = 0 \text{ or } n \geq 2, \end{cases}$$

which tells us that $B_1(1) = 1/2$ and $B_n(1) = B_n$ otherwise.

**(d)** $B_n(x+1/2) = (-1)^n B_n(-x+1/2)$ $(n \geq 0)$.

*Proof.* Take $x - 1/2$ in place of $x$ in (b). $\square$

This property shows that $B_n(x+1/2)$ is an even or an odd function of $x$ depending on the parity of $n$. We may prove (d) independently of (b) by using the facts that

$$X_k(x+1/2) = \frac{1}{2}(x^2 - 1/4)^k + s_k \quad \text{and} \quad Y_k(x+1/2) = kx(x^2-1/4)^{k-1}.$$

The following two properties are the repetition of (1.3) (i) and (ii).

**(e)** $\dfrac{d}{dx}B_n(x) = nB_{n-1}(x) \quad (n \geq 1).$

*Proof.* Since $\frac{d}{dx}X_k(x) = Y_k(x)$ and $\frac{d}{dx}Y_k(x) = Z_k(x)$, from Theorem 3.2 we have

$$\frac{d}{dx}B_{2n}(x) = \frac{2}{(n-1)!}\sum_{k=2}^{n} c_k Y_k(x) = 2nB_{2n-1}(x);$$
$$\frac{d}{dx}B_{2n-1}(x) = \frac{1}{n!}\sum_{k=2}^{n} c_k Z_k(x) = (2n-1)B_{2(n-1)}(x),$$

and thus, (e) follows. $\square$

**(f)** $\displaystyle\int_0^x B_n(t)dt = \frac{B_{n+1}(x) - B_{n+1}}{n+1} \quad (n \geq 0).$

*Proof.* Since $\int_0^x Y_k(t)dt = X_k(x) - X_k(0)$ and $\int_0^x Z_k(t)dt = Y_k(x) - Y_k(0)$ for $k \geq 2$, it follows from Theorem 3.2 that

$$\begin{aligned}\int_0^x B_{2n-1}(t)dt &= \frac{1}{n!}\sum_{k=2}^{n} c_k \int_0^x Y_k(t)dt = \frac{1}{n!}\sum_{k=2}^{n} c_k\{X_k(x) - X_k(0)\} \\ &= \frac{B_{2n}(x) - B_{2n}(0)}{2n};\end{aligned}$$
$$\begin{aligned}\int_0^x B_{2(n-1)}(t)dt &= \frac{1}{(2n-1)\cdot n!}\sum_{k=2}^{n} c_k \int_0^x Z_k(t)dt \\ &= \frac{1}{(2n-1)\cdot n!}\sum_{k=2}^{n} c_k\{Y_k(x) - Y_k(0)\} = \frac{B_{2n-1}(x) - B_{2n-1}(0)}{2n-1},\end{aligned}$$

which imply (f) by summarizing the both cases. $\square$

**(g)** (1) $\displaystyle\int_0^1 B_n(x)dx = 0 \quad (n \geq 1)$; (2) $\displaystyle\int_0^{1/2} B_n(x)dx = 0 \quad (n \geq 2 \text{ an even}).$

*Proof.* Since $\int_0^1 Y_k(x)dx = [X_k(x)]_0^1 = 0$ for $k \geq 1$ and $\int_0^1 Z_k(x)dx = [Y_k(x)]_0^1 = 0$ for $k \geq 2$, we obtain (1) immediately from (3.5) (ii) and (iii). Further, since $\int_0^{1/2} Z_k(x)dx = [Y_k(x)]_0^{1/2} = 0$ for $k \geq 2$, we can deduce (2) from (3.5) (iii). $\square$

**(h)** (Trivial zeros) $B_1(1/2)=0$, $B_{2n+1}(0)=B_{2n+1}(1/2)=B_{2n+1}(1)=0$ $(n\geq 1)$.

*Proof.* Since $(2x-1)\mid 2Y_1(x)$ and $(2x-1)(x^2-x)\mid 2Y_k(x)$ for $k\geq 2$, the results obviously hold true from (3.5) (ii). □

It should be noted here that Nörlund [15, p. 22] has shown that these trivial zeros are all the zeros of $B_{2n+1}(x)$, $n\geq 1$, in the interval $[0,1]$. Furthermore, Inkeri [13] has proved that these are the only rational zeros of any Bernoulli polynomial.

Although all the above trivial zeros of $B_{2n+1}(x)$ are single ones, from (3.5) (ii) one can see that the improved polynomials

$$\mathcal{B}_{2n+1}(x) := (n+1)!B_{2n+1}(x) + (-1)^n(2x-1)(x^2-x)\det D_{n-1},\ \ n=1,2,3,\dots,$$

have always the zeros $0, 1/2$ and 1 whose multiplicities are $2, 1$ and 2, respectively, because of $c_2=(-1)^{n+1}\det D_{n-1}$ and $(2x-1)x^2(x-1)^2\mid Y_k(x)$ for all $k\geq 3$. For example, if we take $n=2$ and 3, then

$$\begin{aligned}
\mathcal{B}_5(x) &= 3!B_5(x) + (2x-1)(x^2-x)\det\left[\tbinom{3}{0}\right] = 3(2x-1)x^2(x-1)^2;\\
\mathcal{B}_7(x) &= 4!B_7(x) - (2x-1)(x^2-x)\det\begin{bmatrix}\tbinom{3}{0} & \tbinom{3}{2}\\ 0 & \tbinom{4}{1}\end{bmatrix}\\
&= 12(2x-1)x^2(x-1)^2(x+1)(x-2).
\end{aligned}$$

It is possible to prove even more properties of Bernoulli polynomials by applying our determinantal expressions, but let us move to the next subject.

## 5. An Encounter With the Golden Ratio

In this section we will talk about some special topics related to the golden ratio referring to the discussion in [4].

Although the zeros of Bernoulli polynomials have been studied quite extensively from various points of view, it is very curious that the concretely known odd-index Bernoulli polynomial that has non-trivial zeros is only $B_{11}(x)$. As a matter of fact, denoting $\phi := (1+\sqrt{5})/2$ (the golden ratio), it can be shown that

**Theorem 5.1.** *We have $B_{11}(\phi)=B_{11}(\widehat{\phi})=0$, where $\widehat{\phi}$ is the conjugate of $\phi$.*

*Proof.* Since $B_n(x)\in\mathbb{Q}[x]$, it suffices to prove only $B_{11}(\phi)=0$. From (3.4) (ii) we have $B_{11}(\phi)=\det\widetilde{Q}_5(\phi)/5!$. In addition, since $\phi^2-\phi=1$, it is clear that

$$Y_k(\phi)=\frac{k}{2}(2\phi-1)(\phi^2-\phi)^{k-1}=\frac{2\phi-1}{2}k\ \ (k\geq 1).$$

Thus, by a direct calculation we can confirm that

$$\det \widetilde{Q}_5(\phi) = \det \begin{bmatrix} \binom{2}{1} & 0 & 0 & 0 & Y_2(\phi) \\ \binom{3}{0} & \binom{3}{2} & 0 & 0 & Y_3(\phi) \\ 0 & \binom{4}{1} & \binom{4}{3} & 0 & Y_4(\phi) \\ 0 & \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & Y_5(\phi) \\ 0 & 0 & \binom{6}{1} & \binom{6}{3} & Y_6(\phi) \end{bmatrix} = \frac{2\phi-1}{2} \det \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 & 3 \\ 0 & 4 & 4 & 0 & 4 \\ 0 & 1 & 10 & 5 & 5 \\ 0 & 0 & 6 & 20 & 6 \end{bmatrix} = 0,$$

which completes the proof. □

Of course we may prove the above fact by directly substituting $x = \phi$ into (1.2) with $n = 11$, but it requires in advance the knowledge of $B_n$ for $n = 0, 1, \ldots, 11$.

Since $X_k(\phi) = 1/2 + s_k$, we see from (3.5) (i) that all the values of even-index Bernoulli polynomials at $x = \phi$ are rational numbers, i.e., $B_{2n}(\phi) \in \mathbb{Q}$ for all $n \geq 0$. More generally, if $\theta_b$ is a real or complex root of the quadratic equation $x^2 - x = b$ for any given $b \in \mathbb{Q}$, then the same is true for the values at $x = \theta_b$, i.e., $B_{2n}(\theta_b) \in \mathbb{Q}$, because $B_0(\theta_b) = 1$ and $X_k(\theta_b) \in \mathbb{Q}$ for $k \geq 1$. Needless to say, there is no such guarantee for odd-index Bernoulli polynomials. However, we can state the following fact by applying (3.4) (ii).

($*$) *There exist infinitely many $b \in \mathbb{Q}$ such that $B_{2n-1}(\theta_b) \in \mathbb{Q}$ for all $n \geq 1$.*

Indeed, if we put $b = r(r+1)$ for an integer $r \geq 1$, then $\sqrt{4b+1} = 2r+1$; thus, we have for $k \geq 1$,

$$\frac{2}{k} Y_k(\theta_b) = (2\theta_b - 1)(\theta_b^2 - \theta_b)^{k-1} = \sqrt{4b+1} \cdot b^{k-1} = (2r+1)(r^2+r)^{k-1} \in \mathbb{Z},$$

which implies that $\det \widetilde{Q}_{n-1}(\theta_b) = n! B_{2n-1}(\theta_b) \in \mathbb{Z}$ for all $n \geq 1$ from (3.4) (ii). Since one can give infinitely many such the integers $b$, the above ($*$) holds true.

We cannot say for certain at this time, but the possibility of that the above $\theta_b$ with $b \neq 0, 1$ satisfies either one of the equations $\det \widetilde{P}_{n-1}(x) = 0, \det \widetilde{Q}_{n-1}(x) = 0$ and $\det \widetilde{R}_{n-1}(x) = 0$ for some $n \geq 2$ cannot be denied, as well as in the case of $\det \widetilde{Q}_5(\phi) = 0$ for $b = 1$ and $n = 6$. If it happens once, then we are able to find a new non-trivial zero of the corresponding Bernoulli polynomial. Just to be sure, it may be worth mentioning that for $k \geq 2$,

$$X_k(\theta_b) = \frac{1}{2} b^k + s_k; \; Y_k(\theta_b) = \frac{k}{2} \sqrt{1+4b} \cdot b^{k-1}; \; Z_k(\theta_b) = \binom{2k}{2} b^{k-1} + \binom{k}{2} b^{k-2}.$$

Next, we discuss a certain infinite series constructed by Bernoulli numbers that has a closed-form expression in terms of the golden ratio. The series we will deal with is not directly related to our determinantal expressions, but it concerns with the special case of (1.5) for $k = m$; that is, the identity (2.2) (i) with $x = 0$.

We prepare in advance the following lemma.

**Lemma 5.2.** *It follows that*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{2k}{k}^{-1} = \frac{4\log\phi}{\sqrt{5}}. \tag{5.1}$$

*Proof.* For any numbers $x, y \in \mathbb{C}$ satisfying $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, denote by $B(x, y)$ the beta function defined by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma$ is the Gamma function. Setting here $x = y = k+1$ for an integer $k \geq 0$, we have

$$B(k+1, k+1) = \frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+2)} = \frac{(k!)^2}{(2k+1)!} = \frac{1}{2k+1}\binom{2k}{k}^{-1}.$$

Therefore, the alternating sum taken over $k = 0, 1, 2, \ldots$ can be given by

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k B(k+1,k+1) &= \int_0^1 \sum_{k=0}^{\infty}(-1)^k t^k(1-t)^k dt \\ &= \int_0^1 \frac{1}{1+t-t^2}dt = \frac{4\log\phi}{\sqrt{5}},\end{aligned}$$

which proves (5.1). ☐

Note that (5.1) is not new. Indeed, Sprugnoli [20] has derived a number of closed formulas for infinite sums involving the reciprocals of central binomial coefficients; we can find (5.1) in [20, Theorem 3.5]). See also [4].

For given integers $m, a$ with $m \geq a \geq 1$ let us denote

$$\gamma_{m,a} := \sum_{i=0}^{a-1} \binom{m+i}{2i+1}.$$

In particular, when $a = m$, we simply write as $\gamma_m := \gamma_{m,m}$, i.e., in plain terms,

$$\gamma_m = \binom{m}{1} + \binom{m+1}{3} + \cdots + \binom{2m-1}{2m-1} \quad (m \geq 1).$$

Using such binomial sums, we will prove the following formula:

**Theorem 5.3.** *We have*

$$\sum_{m=1}^{\infty} \gamma_m \frac{B_{2m}}{2m} = \frac{1}{2} - \frac{2\log\phi}{\sqrt{5}} \approx 0.069591059\cdots. \tag{5.2}$$

*Proof.* Setting $x = 0$ in (2.2) (i), we get

$$\sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} \frac{B_{2(k-i)}}{2(k-i)} = s_k \quad (k \geq 1), \tag{5.3}$$

where $s_k$ is the constant term of $X_k(x)$ defined in (3.7). For an integer $a \geq 1$, take $k = 1, 2, \ldots, a$ in (5.3) and add up all of them. Then, gathering the terms involving the same Bernoulli numbers in one place, the sum of the left-hand side of (5.3) can be written accurately as

$$\begin{aligned} \sum_{k=1}^{a} \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} \frac{B_{2(k-i)}}{2(k-i)} = & \sum_{j=1}^{\lfloor (a+1)/2 \rfloor} \gamma_j \, \frac{B_{2j}}{2j} \\ & + \sum_{j=\lfloor (a+1)/2 \rfloor + 1}^{a} \gamma_{j,a+1-j} \, \frac{B_{2j}}{2j}. \end{aligned} \tag{5.4}$$

Here we notice that as $a$ increases the terms in the second sum on the right-hand side of (5.4) are embedded successively into the first sum, because the coefficient of $B_{2m}/2m$ grows taking a step according to $0, \gamma_{m,1}, \gamma_{m.2}, \ldots, \gamma_{m,m-1}$, and eventually, it arrives at

$$\gamma_{m,m-1} + \binom{2m-1}{2m-1} = \gamma_m.$$

From the above, we see that the limiting sum of the left-hand side of (5.4) as $a \to \infty$ can be expressed as follows:

$$\lim_{a \to \infty} \sum_{k=1}^{a} \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2i+1} \frac{B_{2(k-i)}}{2(k-i)} = \sum_{j=1}^{\infty} \gamma_j \, \frac{B_{2j}}{2j}. \tag{5.5}$$

On the other hand, from (5.1) the limiting sum of the right-hand side of (5.3) as $k \to \infty$ is given by

$$\lim_{a \to \infty} \sum_{k=1}^{a} s_k = \sum_{k=1}^{\infty} s_k = -s_0 + \sum_{k=0}^{\infty} s_k = \frac{1}{2} - \frac{2 \log \phi}{\sqrt{5}}. \tag{5.6}$$

So that, by equating (5.4) and (5.6) we can derive (5.2). □

## 6. Additional Remarks

In this final section, as additional remarks, we wish to make a brief summary on determinantal expressions for Euler polynomials that have an intimate connection with those for Bernoulli polynomials discussed in Section 3.

The Euler polynomials $E_n(x)$, $n = 0, 1, 2, \ldots$, are defined by means of the generating function

$$\mathcal{E}(t, x) := \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)t^n}{n!} \quad (|t| < \pi).$$

A large number of recurrence relations for Euler polynomials have been developed over the years in parallel with those for Bernoulli polynomials (see, e.g., [19, 14, 12]). Among them, the most basic linear one is

$$\sum_{i=0}^{n} \binom{n}{i} E_i(x) + E_n(x) = E_n(x+1) + E_n(x) = 2x^n \quad (n \geq 0), \tag{6.1}$$

which can be easily shown from the relation $\mathcal{E}(t, x+1) + \mathcal{E}(t, x) = 2e^{xt}$.

Many kinds of determinantal expressions for Euler polynomials can be derived by making use of various recurrence relations. As an easy example, dividing (6.1) by $2 \cdot n!$, we have

$$\frac{1}{n!} E_n(x) + \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{i!(n-i)!} E_i(x) = \frac{x^n}{n!}.$$

Similar to (1.8), according to Cramer's rule this identity yields that, by using ordinary determinantal operations,

$$\frac{2^n}{n!} E_n(x) = \det \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 & 1 \\ \frac{1}{1!} & 2 & 0 & \cdots & 0 & \frac{x}{1!} \\ \frac{1}{2!} & \frac{1}{1!} & 2 & \cdots & 0 & \frac{x^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & 2 & \frac{x^{n-1}}{(n-1)!} \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{1!} & \frac{x^n}{n!} \end{bmatrix} \quad (n \geq 0).$$

We now pick out the following shortened recurrence relation instead of (6.1), in which some of the preceding Euler polynomials are completely excluded:

$$\sum_{i=0}^{k} \binom{k}{i} E_{m+i}(x) + \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} E_{k+i}(x) = 2x^k (x-1)^m, \tag{6.2}$$

which is valid for arbitrary integers $k, m \geq 0$ (see [1, 2] for an elementary proof). In particular, setting $m = k$ in (6.2) and dividing by 2, we get immediately

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} E_{2(k-i)}(x) = (x^2 - x)^k = 2(X_k(x) - s_k), \tag{6.3}$$

which is the recurrence relation for even-index Euler polynomials.

Since $E_0(x) = 1$ and $\frac{d}{dx}E_n(x) = nE_{n-1}(x)$ for $n \geq 1$, one sees that the Euler polynomials form an Appell sequence. Based on this fact, taking a derivative of both sides of (6.3) with respect to $x$, we can obtain the following recurrence relation for odd-index Euler polynomials:

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{2i} 2(k-i)E_{2(k-i)-1}(x) = k(2x-1)(x^2-x)^{k-1} = 2Y_k(x). \tag{6.4}$$

It can be seen that (6.3) and (6.4) are just corresponding to (2.2) (i) and (ii), respectively, in the Bernoulli polynomial case. So that, by using these shortened recurrence relations we are able to establish new determinantal expressions for Euler polynomials in almost the same way as done in Section 3.

Furthermore, from the functional identity $t\mathcal{E}(t,x) = 2\{\mathcal{F}(t,x) - \mathcal{F}(2t, x/2)\}$ we obtain a mutual relation between Euler and Bernoulli polynomials such that

$$(n+1)E_n(x) = 2\left(B_{n+1}(x) - 2^{n+1}B_{n+1}(x/2)\right) \quad (n \geq 0).$$

By way of this identities, one can observe some interesting relationships between determinantal expressions for Euler and Bernoulli polynomials.

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