# SPRAGUE-GRUNDY VALUES OF MODULAR NIM

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**Abstract**

*Modular Nim*, also known as *Kotzig's Nim*, is an impartial, two-player combinatorial game invented by Anton Kotzig in 1946. The game is played using a token placed on a circular board of $n$ spaces and a set $M$ of possible moves. On his turn, a player selects a move from $M$ and advances the token accordingly around the board to a previously unoccupied space. In normal play, the last player who is able to move wins the game. To date, much research into Modular Nim has focussed on determining the $\mathcal{P}$-positions for various combinations of $n$ and $M$. In this paper, we calculate the Sprague-Grundy values of certain instances of the game.

## 1. Introduction

*Modular Nim*, also known as *Kotzig's Nim*, is a combinatorial game for two players invented by Anton Kotzig in 1946. The game is played with a circular arrangement of $n$ spaces, numbered clockwise from 0 to $n-1$. A token is placed initially on space 0 and the players, moving alternately, advance the token clockwise around the circle. On his move, a player advances the token $m$ spaces where $m$ is selected from a set of possible moves called the *move set* with the proviso that the token may not be moved to a previously visited space. The last player who is able to move is the winner.

Following [2] and [3], we let $\Gamma(M;n)$ denote the game of Modular Nim played on a circle with $n$ spaces and move set $M$. While Modular Nim is easy to describe and play, it is far from being well-understood. The question of determining the outcome class, $\mathcal{N}$ or $\mathcal{P}$, of $\Gamma(M;n)$ for various combinations of $n$ and $M$ has been considered by several researchers in [1], [2], and [3].

Our chief consideration in this paper is the problem of calculating the Sprague-Grundy value of (the starting position of) $\Gamma(M; n)$. In Section 2, we study a related game, called the *Line Game*, as a precursor to determining the values of $\Gamma(\{1, 2\}, n)$ for all values of $n$ in Section 3. Section 4 is devoted to the calculation of the value of $\Gamma(\{1, 3\}, n)$ for all values of $n$. In Section 5, we calculate the value of $\Gamma(\{a, a+1\}, n)$ for all values of $a$ and $n$ with $n \equiv -1 \pmod{2a + 1}$. Finally, in Section 6, we close with some concluding remarks.

We assume that the reader is familiar with the basic facts of combinatorial game theory concerning outcome classes and Sprague-Grundy values as may be found in [1]. Some brief comments concerning notation: we shall call the first and second players, "$A$" and "$B$", respectively. A position in a game of Modular Nim may be denoted by listing, in order, the moves that the players have made. We shall use the notation $\langle a_1\ a_2\ \ldots\ a_r\rangle$ to denote the position obtained after the token is moved $a_1$ spaces initially, then $a_2$ spaces, and so on, up to the $r$th move which is $a_r$ spaces. Let $\langle a^k\rangle$ denote the position obtained after $k$ moves of $a$ spaces each, and let the initial position be denoted $\langle\emptyset\rangle$. We will use mex $S$ to denote the minimum excluded value of a set $S$ of nonnegative integers, and $\mathcal{G}(G)$ for the Sprague-Grundy value (or simply, "value", for short) of the game $G$.

## 2. The Line Game

To aid in our study, we introduce the *Line Game*, an impartial combinatorial game for two players. This game is played using a horizontal line of spaces and a token, initially placed on the leftmost space. Each space is either open or forbidden. A move consists of moving the token $m$ spaces to the right of its current location to an open space where $m$ is selected from a given set, called the *move set*. The game ends when no permissible move may be made, and the last player who made a move is declared the winner.

Let $L_M(n, S)$ denote the Line Game played using a line of $n+1$ spaces numbered $0, 1, 2, \ldots, n$ from left to right. The token is placed initially on space 0. The move set is given by $M$, and $S \subseteq \{1, 2, \ldots, n\}$ denotes the set of forbidden spaces. For convenience, we assume that $n \in S$ and note that this may be done without loss of generality by appending a forbidden space at the right end of the line, if necessary.

In this section, we consider only the move set $M = \{1, 2\}$ and therefore will refer to the game $L_M(n, S)$ simply as $L(n, S)$.

We wish to determine the Sprague-Grundy value of the game $L(n, S)$ for any given $n$ and $S$. This may be done systematically by labelling each open space, starting with the rightmost open space and working towards the left until the initial space is reached. The label assigned to an open space is the Sprague-Grundy value of the line game in which the particular space is considered to be the starting space.

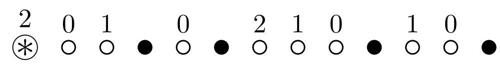


Figure 1: the Line Game $L(12, \{3, 5, 9, 12\})$

As an example, consider the Line Game with $n = 12$ and $S = \{3, 5, 9, 12\}$, depicted in Figure 1. Since the 12th space is forbidden, no move is possible from the 11th space, so we begin by labelling the 11th space with 0. Now the only move from the 10th space is to the 11th space, so the label on the 10th space is $\text{mex}\{0\} = 1$. Continuing in this fashion, we obtain the indicated labelling of the open spaces.

Thus, the value of this game is the label on the 0th space, that is, 2.

### 2.1. Preliminary Results

We note that since our move set is $\{1, 2\}$, play cannot proceed beyond two consecutive forbidden spaces. Therefore, in the sequel, we assume that the forbidden set $S$ does not contain two consecutive elements.

Let $L = L(n, S)$ be a line game in which $S$ does not contain two consecutive elements. Suppose that $S = \{a_1, a_2, \ldots, a_s\}$ where $1 \le a_1 < a_2 < \cdots < a_s = n$.

We decompose $L$ into $s$ components as follows. Let $P_1$ be the portion of $L$ consisting of spaces 0 up to $a_1$, inclusive. For $i = 2, 3, \ldots, s$, let $P_i$ be the portion of $L$ consisting of spaces $a_{i-1} + 1$ up to $a_i$, inclusive.

Let $d_i$ denote the number of spaces, including the forbidden space at the right end, that are in $P_i$. Now $d_1 = a_1 + 1$ and, for $i = 2, 3, \ldots, s$, we have $d_i = a_i - a_{i-1}$. For example, the game $L = L(12, \{3, 5, 9, 12\})$ has $s = 4$ and $(d_1, d_2, d_3, d_4) = (4, 2, 4, 3)$.

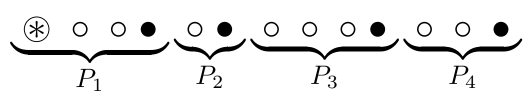


Figure 2: the portions of the Line Game $L(12, \{3, 5, 9, 12\})$

We define the *label* on a space in a line game $L$ to be the value of the game obtained by setting the token initially on that particular space. The value of the game $L$ then is equal to the label on space 0. As described in the previous section, the labels on the spaces may be calculated systematically by working from right to left. We note that, from any space, at most two moves are possible so, in particular, the label on any space is either 0, 1, or 2.

Let $l_i$ denote the label on the leftmost space of $P_i$, that is, the label on the space 0 if $i = 1$ and space $a_{i-1} + 1$ if $2 \le i \le s$. Let $r_i$ denote the label on the rightmost open space of $P_i$, that is, the label on the space $a_i - 1$. For example, for the game $L = L(12, \{3, 5, 9, 12\})$, we see from Figure 1 that $(l_4, r_4) = (1, 0)$, $(l_3, r_3) = (2, 0)$, $(l_2, r_2) = (0, 0)$, and $(l_1, r_1) = (2, 1)$.

No move is possible from the rightmost open space in the game so $r_s = 0$ and, moreover, $r_i = \text{mex}\{l_{i+1}\}$ if $1 \le i \le s - 1$ since the only move from the rightmost open space in $P_i$ is to the leftmost space in $P_{i+1}$. Thus, in particular, the label on the rightmost open space in any portion is never equal to 2. We wish to determine the value of the game, that is, $l_1$, the label on the leftmost space of $P_1$.

Consider a portion $P_i$ and suppose that $r_i = 0$. We may now work from right to left, calculating the labels on the other spaces in $P_i$ as follows:

$$\begin{array}{cccccccc} & 2 & 1 & 0 & 2 & 1 & r_i{=}0 & \\ \cdots & \circ & \circ & \circ & \circ & \circ & \circ & \bullet \end{array}$$

On the other hand, if $r_i = 1$ we obtain the following labels:

$$\begin{array}{cccccccc} & 2 & 0 & 1 & 2 & 0 & r_i{=}1 & \\ \cdots & \circ & \circ & \circ & \circ & \circ & \circ & \bullet \end{array}$$

We see that the labels are periodic with a period length of 3. Our calculations may be summarized then as follows. Recall that $d_i$ is the number of spaces in the portion $P_i$, including the forbidden space at the right end.

$$\text{If } \; r_i = 0 \; \text{ then } \; l_i = \begin{cases} 1 & \text{if } \; d_i \equiv 0 \pmod 3, \\ 2 & \text{if } \; d_i \equiv 1 \pmod 3, \\ 0 & \text{if } \; d_i \equiv 2 \pmod 3. \end{cases} \tag{1}$$

$$\text{If } \; r_i = 1 \; \text{ then } \; l_i = \begin{cases} 0 & \text{if } \; d_i \equiv 0 \pmod 3, \\ 2 & \text{if } \; d_i \equiv 1 \pmod 3, \\ 1 & \text{if } \; d_i \equiv 2 \pmod 3. \end{cases} \tag{2}$$

As an example, consider the line game $L = L(37, \{4, 10, 13, 15, 20, 24, 32, 35, 37\})$. It is possible, but somewhat tedious, to calculate the value of this game using the systematic labelling method described earlier. Instead, we streamline the calculations by using Equations (1) and (2).

The game $L$ decomposes into 9 portions with

$$(d_1, d_2, \ldots, d_9) = (5, 6, 3, 2, 5, 4, 8, 3, 2) \equiv (2, 0, 0, 2, 2, 1, 2, 0, 2) \pmod 3,$$

and now, using Equations (1) and (2), we obtain the following:

$$\begin{array}{rlll} r_9 = 0, & d_9 \equiv 2 \pmod 3 & \text{implies} & l_9 = 0 \\ r_8 = \text{mex}\{0\} = 1, & d_8 \equiv 0 \pmod 3 & \text{implies} & l_8 = 0 \\ r_7 = \text{mex}\{0\} = 1, & d_7 \equiv 2 \pmod 3 & \text{implies} & l_7 = 1 \\ r_6 = \text{mex}\{1\} = 0, & d_6 \equiv 1 \pmod 3 & \text{implies} & l_6 = 2 \\ r_5 = \text{mex}\{2\} = 0, & d_5 \equiv 2 \pmod 3 & \text{implies} & l_5 = 0 \\ r_4 = \text{mex}\{0\} = 1, & d_4 \equiv 2 \pmod 3 & \text{implies} & l_4 = 1 \\ r_3 = \text{mex}\{1\} = 0, & d_3 \equiv 0 \pmod 3 & \text{implies} & l_3 = 1 \\ r_2 = \text{mex}\{1\} = 0, & d_2 \equiv 0 \pmod 3 & \text{implies} & l_2 = 1 \\ r_1 = \text{mex}\{1\} = 0, & d_1 \equiv 2 \pmod 3 & \text{implies} & l_1 = 0. \end{array}$$

Therefore, the value of the game $L$ is $l_1 = 0$.

### 2.2. An Algorithm

We now make some observations that will allow us to further streamline these calculations.

Consider a portion $P_i$ for which $d_i \equiv 1 \pmod 3$. From Equations (1) and (2), we have $l_i = 2$ for such a portion, regardless of the value of $r_i$. This means that when determining the value of a line game $L$, *we may ignore all portions that are to the right of the leftmost portion that has length congruent to 1 modulo 3.*

Therefore, if the first portion $P_1$ of the line game $L$ has $d_1 \equiv 1 \pmod 3$, then the value of $L$ is determined immediately to be 2. Otherwise, we analyze the portions to the left of the leftmost occurrence of a portion having length congruent to 1 modulo 3; we call these *initial* portions. An initial portion then will have length congruent to 0 or 2 modulo 3.

Let $P_j$ denote the rightmost initial portion. Then either $P_j$ is the rightmost portion in the game or else the portion to the right of $P_j$ has length congruent to 1 modulo 3. In the first case, clearly $r_j = 0$, and, in the second case, $l_{j+1} = 2$ so $r_j = \text{mex}\{2\} = 0$. Thus, in either case, we have $r_j = 0$ and now, by Equation (1), $l_j = 1$ or 0 depending on whether $d_j$ is congruent to 0 or 2 modulo 3.

We now consider how successive values of the sequence $\{l_i\}$ for the initial portions relate to each other. Since the length of any initial portion $P_i$ is congruent to 0 or 2 modulo 3, we see from Equations (1) and (2) that $l_i$ is either 0 or 1.

If $l_{i+1} = 0$ then $r_i = \text{mex}\{0\} = 1$ so, by Equation (2),

$$l_i = \begin{cases} 0 & \text{if } d_i \equiv 0 \pmod 3, \\ 1 & \text{if } d_i \equiv 2 \pmod 3. \end{cases} \tag{3}$$

On the other hand, if $l_{i+1} = 1$ then $r_i = \text{mex}\{l_{i+1}\} = 0$ so, by Equation (1),

$$l_i = \begin{cases} 1 & \text{if } d_i \equiv 0 \pmod 3, \\ 0 & \text{if } d_i \equiv 2 \pmod 3. \end{cases} \tag{4}$$

Equations (3) and (4) imply that $l_i$ and $l_{i+1}$ have the same value if $d_i \equiv 0 \pmod 3$ and different values if $d_i \equiv 2 \pmod 3$. Thus, the value of $l_0$ depends upon the parity of the number of initial portions that have length 2 (mod 3). As noted above, if $P_j$ denotes the rightmost initial portion then $l_j = 1$ or 0 as $d_j \equiv 0$ or 2 (mod 3) respectively. Therefore, if the number of initial portions of length 2 (mod 3) is even then $l_0$ will be 1; if this number is odd then $l_0 = 0$. We summarize this procedure in the following algorithm.

Algorithm to calculate the value of $L = L(n, S)$:

1. Decompose $L$ into portions $P_1, P_2, \ldots, P_s$ and calculate the lengths $d_1, d_2, \ldots, d_s$ of these portions. For each $1 \le i \le s$, let $a_i \in \{0, 1, 2\}$ be such that $a_i \equiv d_i \pmod 3$.
2. Let $t = \min\{i \mid a_i = 1\}$. If none of $a_1, a_2, \ldots, a_s$ is equal to 1, then set $t = s + 1$.
3. If $t = 1$ then the value of $L$ is 2 and the algorithm is terminated. Otherwise, go on to Step 4.
4. Count the number of $a_1, a_2, \ldots, a_{t-1}$ that are equal to 2. If this number is even then the value of $L$ is 1; otherwise, the value of $L$ is 0.

For an example, we take $L = L(66, \{4, 10, 13, 18, 27, 35, 37, 41, 47, 54, 59, 62, 66\})$. Now $(d_1, d_2, \ldots, d_{13}) = (5, 6, 3, 5, 9, 8, 2, 4, 6, 7, 5, 3, 4)$ and so $(a_1, a_2, \ldots, a_{13}) = (2, 0, 0, 2, 0, 2, 2, 1, 0, 1, 2, 0, 1)$. We see that $t = 8$ and that there are 4 occurrences of 2 to the left of $a_8$. Since 4 is even, the value of the game is 1.

## 3. Values of Modular Nim With Move Set $\{1, 2\}$

In this section, let $\Gamma(n)$ denote the game of Modular Nim with the move set $\{1, 2\}$. It is known (see [1] or [2]) that $\Gamma(n) \in \mathcal{P}$ if and only if $n \in \{1, 3, 7\}$. Our main result in this section is the calculation of the value of $\Gamma(n)$ for all $n$.

### 3.1. Reduction to the Line Game

We begin by noting some connections between Modular Nim and the Line Game.

1. A move of one space, following any number of moves of two spaces from the initial position in Modular Nim, creates a barrier consisting of two neighbouring spaces that cannot be jumped over on the next pass around the circle because the move set is $\{1, 2\}$. We note that the position obtained by such a sequence of moves then, is equivalent to a line game. For example, in $\Gamma(8)$, the position $\langle 2^2\ 1 \rangle$ is equivalent to the line game $L(7, \{3, 5, 7\})$.

2. If $n$ is even then, in the game $\Gamma(n)$, players can alternate moving the token two spaces a total of $n/2-1$ times before a move of 1 space is forced. On the other hand, if $n$ is odd, then $(n-1)/2$ moves of 2 spaces can be made before the token is adjacent to the initial space. Therefore, the game $\Gamma(n)$ will reduce to a line game after at most $\lfloor n/2 \rfloor$ moves and this will happen on the first pass around the circle.

Consider the position in $\Gamma(n)$ that is reached after a move of one space following $k \le \lfloor n/2 \rfloor - 1$ moves of two spaces. As noted above, this position is equivalent to the line game $L(n-1, S)$ where $S = \{n-2k-1, n-2k+1, \dots, n-3, n-1\}$.

We now decompose this line game into portions, as in Section 2.1. The lengths of these portions are $n-2k, \underbrace{2, 2, \dots, 2}_{k}$, and now the value of this line game may be calculated using the algorithm of Section 2.2. The results are given in the following lemma.

**Lemma 1.** *The value of the position in the game $\Gamma(n)$ that is reached after a move of one space following $k \le \lfloor n/2 \rfloor - 1$ moves of two spaces is*

$$\begin{cases} k+1 \pmod 2 & \text{if } n-2k \equiv 0 \pmod 3 \\ 2 & \text{if } n-2k \equiv 1 \pmod 3 \\ k \pmod 2 & \text{if } n-2k \equiv 2 \pmod 3. \end{cases}$$

### 3.2. Game Tree

In order to determine the value of a game of Modular Nim, we consider the game tree. The method of solution is to determine the values of the positions at leaf nodes and then to work up the tree, eventually determining the value of the root node. Since we are able to compute the value of any position that is equivalent to a line game, such positions are leaves in the game tree.

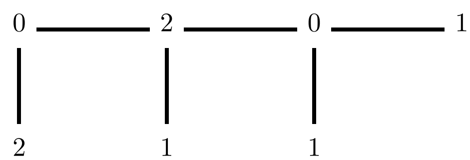
Pictorially, a move of one space is represented by a downward branch, and a move of two spaces is a rightward branch. The root node is in the upper left. For example, the game tree for $\Gamma(7)$ is shown below.

$$\begin{tikzcd} \langle \emptyset \rangle \arrow[r, dash] \arrow[d, dash] & \langle 2 \rangle \arrow[r, dash] \arrow[d, dash] & \langle 2^2 \rangle \arrow[r, dash] \arrow[d, dash] & \langle 2^3 \rangle \\ \langle 1 \rangle & \langle 2\ 1 \rangle & \langle 2^2\ 1 \rangle & \end{tikzcd}$$

Using Lemma 1, we may calculate the values of the positions along the bottom. We have $n = 7$ and, for the position $\langle 1 \rangle$, we have $k = 0$ moves of two spaces before a move of one space. Thus $n - 2k = 7 \equiv 1 \pmod 3$ so, by the lemma, the value of this position is 2. Similarly, for the positions $\langle 2\ 1 \rangle$ and $\langle 2^2\ 1 \rangle$, we have $k = 1$

and $k = 2$, respectively, and, using the lemma, we determine that both of these positions have value 1. Lemma 1 may not be used to calculate the value of the position $\langle 2^3 \rangle$. However, from this position there is only one move, namely to $\langle 2^4 \rangle$ which is a $\mathcal{P}$-position, so the value of $\langle 2^3 \rangle$ is 1.

Having determined the values of the positions on the leaves, it is straightforward to calculate the values of the other positions, working up towards the root, as shown below.

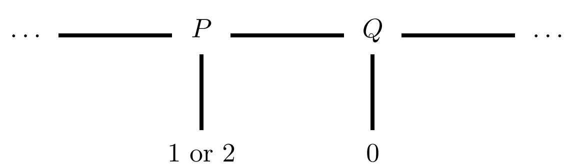



Therefore, the value of the game $\Gamma(7)$ is 0.

### 3.3. A Shortcut

The method used to calculate the value of $\Gamma(7)$ in the previous section could be used for any Modular Nim game. But this could be quite tedious if the game tree is large.

However, we note that a particular pattern in the bottom values of the game tree may be exploited to streamline the calculations. Suppose that we have two consecutive positions in the bottom row of the game tree that have values 1 or 2, and 0 as shown below.




We claim that the value of $P$ is 0. To see this, note that $\mathcal{G}(Q)$ is nonzero since $Q$ has an option of value 0. Now both options of $P$ have nonzero values so $\mathcal{G}(P) = 0$. Therefore, should this pattern occur, we may ignore everything to the right of it in the game tree.

### 3.4. Main Results

In this section, we determine the value of $\Gamma(n)$ for all $n$, excluding a small number of exceptional cases which will be dealt with in Section 3.5.

**Theorem 1.** *If* $n \equiv 0 \pmod 3$ *and* $n \geq 6$ *then* $\mathcal{G}(\Gamma(n)) = 2$.

*Proof.* Using Lemma 1, we calculate the first three values along the bottom of the game tree:

$$\begin{array}{lllll}
k=0 & \Longrightarrow & n-2k=n-0\equiv 0 \pmod 3 & \Longrightarrow & \text{value} = 0+1 \pmod 2 = 1 \\
k=1 & \Longrightarrow & n-2k=n-2\equiv 1 \pmod 3 & \Longrightarrow & \text{value} = 2 \\
k=2 & \Longrightarrow & n-2k=n-4\equiv 2 \pmod 3 & \Longrightarrow & \text{value} = 2 \pmod 2 = 0.
\end{array}$$

The game tree then is as follows.

$$\begin{tikzcd}
\langle\emptyset\rangle \arrow[r, dash] \arrow[d, dash] & \langle 2\rangle \arrow[r, dash] \arrow[d, dash] & \langle 2^2\rangle \arrow[r, dash] \arrow[d, dash] & \cdots \\
1 & 2 & 0 &
\end{tikzcd}$$

Using the shortcut, we have $\mathcal{G}(\langle 2\rangle)=0$ and then $\mathcal{G}(\langle\emptyset\rangle)=\text{mex}\{0,1\}=2$, which is the value of the game. $\square$

**Theorem 2.** *If* $n\equiv 1 \pmod 3$ *and* $n\geq 10$, *then* $\mathcal{G}(\Gamma(n))=1$.

*Proof.* Using Lemma 1, we calculate the first 5 values along the bottom of the game tree to be $2,1,1,2,0$. Now using the shortcut, we determine that the fourth position along the top row, namely $\langle 2^3\rangle$, has value 0. Finally, we compute the remaining values along the top row.

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] \arrow[d, dash] & \neq 0 \arrow[r, dash] \arrow[d, dash] & \cdots \\
2 & 1 & 1 & 2 & 0 &
\end{tikzcd}$$

Therefore, the value of the game is 1. $\square$

**Theorem 3.** *If* $n\equiv 2 \pmod 3$ *and* $n\geq 14$ *then* $\mathcal{G}(\Gamma(n))=1$.

*Proof.* Using Lemma 1, we determine that $0,0,2,1,1,2,0$ are the initial values along the bottom of the game tree. Using the shortcut, the sixth value along the top, namely $\mathcal{G}(\langle 2^5\rangle)$, is then 0 and the other values in the top row may then be calculated from right to left.

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 1 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] \arrow[d, dash] & \neq 0 \arrow[r, dash] \arrow[d, dash] & \cdots \\
0 & 0 & 2 & 1 & 1 & 2 & 0 &
\end{tikzcd}$$

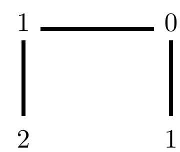
Thus, the value of the game is 1. $\square$

### 3.5. Exceptional Cases

The results in the previous section give the value of $\Gamma(n)$ for all positive integers $n$, except $n \in \{1, 2, 3, 4, 5, 7, 8, 11\}$. In this section, we examine these exceptional cases.

The cases $n = 1, 3, 7$ are easily dispensed with; as noted earlier, these are precisely the $\mathcal{P}$-positions, so each one has value 0. Moreover, we have $\mathcal{G}(\Gamma(2)) = 1$ since the first player has only the terminal position as an option. We now construct the game trees for the remaining exceptions.

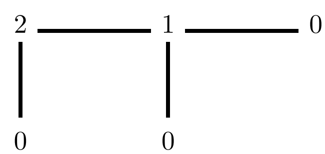
For $n = 4$, we make use of the values along the bottom as computed in Theorem 2.




Thus $\mathcal{G}(\Gamma(4)) = 1$.

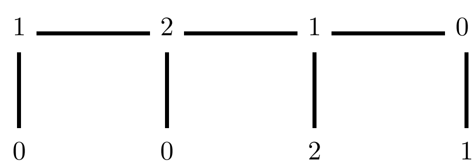
The remaining exceptions, namely 5, 8, and 11, are all congruent to 2 modulo 3, so we use the bottom values calculated in Theorem 3.

For $n = 5$, the position $\langle 2^2 \rangle$ is a $\mathcal{P}$-position so the game tree is as follows.



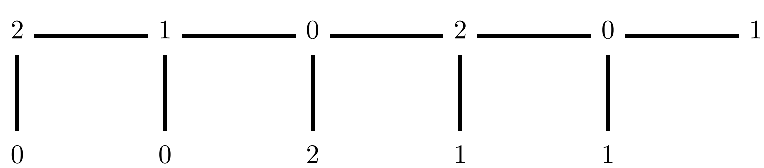

Thus $\mathcal{G}(\Gamma(5)) = 2$.

For $n = 8$, the position $\langle 2^3 \rangle$ is a $\mathcal{P}$-position so the game tree is as follows.




Thus $\mathcal{G}(\Gamma(8)) = 1$.

For $n = 11$, the position $\langle 2^5 \rangle$ has value 1 since, once this position is reached, the remaining 5 moves in the game are all forced.




Thus $\mathcal{G}(\Gamma(11)) = 2$.

## 4. Values of Modular Nim With Move Set $\{1, 3\}$

In this section, let $\Gamma(n)$ denote the game of Modular Nim played on a circle with $n$ spaces and move set $\{1, 3\}$. The game $\Gamma(n)$ is in $\mathcal{P}$ precisely when $n \equiv 1, 3 \pmod 6$; the explanation, due to Richard Nowakowski, is found in [1].

This section is devoted to determining the Sprague-Grundy value of $\Gamma(n)$ for all $n$. Our main result is the following theorem.

**Theorem 4.** *Let $\Gamma(n)$ denote the game of Modular Nim played on a circle of $n$ spaces with move set $\{1, 3\}$. The Sprague-Grundy values of $\Gamma(n)$ are periodic with period length 6, specifically*

$$\mathcal{G}(\Gamma(n)) = \begin{cases} 0 & \textit{if } \; n \equiv 1, 3 \pmod 6 \\ 1 & \textit{if } \; n \equiv 2, 4, 6 \pmod 6 \\ 2 & \textit{if } \; n \equiv 5 \pmod 6. \end{cases}$$

The result is immediate in the case $n \equiv 1, 3 \pmod 6$. The other two cases are considered in the following subsections.

### 4.1. Proof of Theorem 4 in the Case $n \equiv 2, 4, 6 \pmod 6$

We wish to show that $\mathcal{G}(\Gamma(n)) = 1$ in the case that $n \equiv 2, 4, 6 \pmod 6$, that is, when $n$ is even. This result will follow immediately once we establish that both options from $\Gamma(n)$, namely $\langle 1 \rangle$ and $\langle 3 \rangle$, are $\mathcal{P}$-positions.

We claim that player $B$ has a winning strategy from the position $\langle 1 \rangle$ by always moving 1 space forward. Since $n$ is even, $A$ will always play on even numbered spaces and $B$ on odd numbered spaces. Therefore, as long as $A$ is able to move, $B$ always has a response and so $B$ will eventually win the game. In other words, $\langle 1 \rangle$ is a $\mathcal{P}$-position.

Similarly, $\langle 3 \rangle$ is also a $\mathcal{P}$-position; in this case, $B$'s winning strategy is always to move 3 spaces.

### 4.2. Proof of Theorem 4 in the case $n \equiv 5 \pmod 6$

In this case, we wish to show that $\mathcal{G}(\Gamma(n)) = 2$. The result will follow immediately after we show that $\mathcal{G}(\langle 1 \rangle) = 1$ and $\mathcal{G}(\langle 3 \rangle) = 0$. We will require the following lemma.

**Lemma 2.** *Suppose that $n$ is odd. If, on the first pass of the circle, two consecutive moves of 1 are made, the resulting position is in $\mathcal{P}$.*

*Proof.* The position that is obtained after two consecutive moves of 1 are made for the first time, on the first pass around the circle, has the form $\langle 3^m \; 1^2 \rangle$ for some $m \leq (n-3)/3$. The players cannot move past the block just created and so the game has become a line game, namely $L_{\{1,3\}}(n-2, \{n-3m-2, n-3m+1, \ldots, n-2\})$.

Neither player will be blocked during this line game since there are two open spaces between every pair of consecutive forbidden spaces. Moreover, the first player always plays on odd numbered spaces and the second player on even numbered spaces. Since $n$ is odd, the rightmost open space, numbered $n-3$, is even, so it is the second player who will eventually win the game. $\square$

We may now show that $\mathcal{G}(\langle 1\rangle) = 1$. First, it follows immediately from Lemma 2 that $\langle 1^2\rangle$ is a $\mathcal{P}$-position.

Second, consider the position $\langle 1\ 3\rangle$. On the first pass around the circle, neither player will move 1 space lest his opponent reply with the same move and win by Lemma 2. Therefore, the players will alternate moving 3 spaces each until $B$ moves to $n-1$. Player $A$ is now forced to move to 2 and $B$ responds by moving to 3. From this point, the players have no choice and must alternate moves of 3 spaces until $B$ moves to $n-2$. But now $A$ is stuck as neither $n-1$ nor 1 is available and so $B$ wins.

Thus $\mathcal{G}(\langle 1\rangle) = 1$ since both options, namely $\langle 1^2\rangle$ and $\langle 1\ 3\rangle$, are $\mathcal{P}$-positions.

Finally, we show that $\mathcal{G}(\langle 3\rangle) = 0$. As above, the players will alternate moving 3 spaces on the first pass around the circle until eventually $B$ reaches $n-2$. Now $A$ may move to either $n-1$ or 1 and, in either case, $B$ responds by moving to 2. On the second pass of the circle, the players are forced to alternate moves of 3 spaces until $B$ reaches $n-3$. But now $A$ has no move and $B$ wins.

## 5. Values of Modular Nim With Move Set $\{a, a+1\}$

In this section, we consider the game of Modular Nim played with the move set $\{a, a+1\}$ on a circle with $n$ spaces, where $n \equiv -1 \pmod{2a+1}$. Let us denote this game as $\Gamma(n)$.

In the sequel, we assume that $a \geq 2$. The case of $a = 1$ corresponds to the move set $\{1, 2\}$, which was dealt with in Section 3.

We set $d = 2a+1$ and note that if a player moves the token to a space $s$, then he may play to $s + d \pmod{n}$ on his next turn (assuming this move is available) by using a *diamond strategy*, that is, by replying with $a$ if his opponent moves $a+1$, or vice-versa. The diamond strategy is a popular one in Modular Nim, and several examples may be found in [2] and [3].

In [2], it is shown that $\Gamma(n)$ is always a first player win. In this section, we determine the Sprague-Grundy value of this game. Our main result is the following.

**Theorem 5.** *Let $\Gamma(n)$ denote the game of Modular Nim played with move set $\{a, a+1\}$ on a circle with $n = kd - 1$ spaces where $d = 2a+1$, $k$ is a positive* *integer, and $a \geq 2$. Then*

$$\mathcal{G}(\Gamma(n)) = \begin{cases} 1 & \text{if } k = 1 \\ 2 & \text{if } k \geq 2. \end{cases}$$

The following lemma will be used below in the proof of Theorem 5.

**Lemma 3.** *The position $\langle a \rangle$ in the game of Modular Nim described in Theorem 5 is a $\mathcal{P}$-position.*

*Proof.* It is shown in Theorem 4 of [2] that the first player can always win the game $\Gamma(n)$ by moving $a$ spaces on his first move. $\square$

### 5.1. The case $n = d - 1$

In this section, we consider the particular case $n = d - 1 = 2a$. We begin with a simple number theoretic lemma.

**Lemma 4.** *Let $m$ be a positive integer. The equation $ax + (a+1)y = 2am$ has a unique solution in integers in which $1 \leq y \leq 2a - 1$, namely $(2m - a - 1, a)$.*

*Proof.* If $(x, y)$ is a solution to the equation then $a \mid (a+1)y$, which implies that $a \mid y$ since $a$ and $a+1$ are coprime. But now $a \mid y$ and $1 \leq y \leq 2a - 1$ gives $y = a$ as the only possibility. Substituting $y = a$ gives $x = 2m - a - 1$ so $(2m - a - 1, a)$ is indeed an integral solution and it is unique. $\square$

The following lemma characterizes the conditions under which a sequence of moves would cause the token to land on a previously visited space.

**Lemma 5.** *Let $m_1, m_2, \ldots, m_r$ be a sequence $S$ of moves in $\Gamma(2a)$ with no two consecutive moves of $a$ spaces. Then $\sum_{i=1}^{r} m_i \equiv 0 \pmod{2a}$ if and only if (i) there are exactly $a$ moves of $a+1$ spaces in $S$, and (ii) the number of moves in $S$ of $a$ spaces and $a$ are of opposite parity.*

*Proof.* If $\sum_{i=1}^{r} m_i = 2am$ for some positive integer $m$ then, by Lemma 4, $S$ must consist of precisely $a$ moves of $a+1$ spaces and $2m - a - 1$ moves of $a$ spaces.

Conversely, suppose that $S$ satisfies (i) and (ii) in the statement of the lemma. Then $S$ consists of $a$ moves of $a+1$ spaces and $r-a$ moves of $a$ spaces, and $r-a$ and $a$ are of opposite parity. Now the moves in $S$ advance the token $((r-a)+(a+1))a$ spaces, which is an even multiple of $a$ since $r-a$ and $a+1$ have the same parity. $\square$

The following lemma will be used in the proof of Theorem 5.

**Lemma 6.** *In $\Gamma(2a)$, the position $\langle (a+1)^m \; a \rangle$, where $1 \leq m \leq a-1$, is in $\mathcal{P}$.*

*Proof.* Since $n = 2a$, we will never encounter two consecutive moves of $a$ spaces during the course of play. Therefore, $A$'s first move from the position $\langle (a+1)^m\ a\rangle$ must be $a+1$ spaces. We now consider two cases, depending on the parity of $m$.

Case 1: $m$ is odd

We claim that $B$'s winning response is to move $a$ spaces after which $A$ is again forced to move $a+1$ spaces. Player $B$'s winning response is to move $a$ spaces and play continues in this fashion until each player has played $a-m-1$ times. We claim that $A$ is now stuck and so $B$ wins.

From the beginning of game then, the sequence $S$ of moves is as follows:

$$\underbrace{a+1, a+1, \ldots, a+1}_{m}, a\,\underbrace{a+1, a, a+1, a, \ldots, a+1, a}_{2(a-m-1)}\ .$$

There have been $a-1$ moves of $a+1$ spaces and $a-m$ moves of $a$ spaces. In total, $2a-m-1$ moves have been made so $2a-m$ spaces have been covered, leaving $m$ spaces uncovered at the end of the game.

We have noted that all of $A$'s moves in the game were forced. It remains to show that each of $B$'s moves was possible and that, when all moves in $S$ have been made, $A$ has no available move.

First, there were $a-1 < a$ moves of $a+1$ spaces so by Lemma 5, all moves in the game were possible.

Second, consider the position reached after the sequence $S$ of moves has been made. It is now $A$'s turn. He cannot move $a$ since $B$'s last move was to move $a$. If we append $a+1$ to $S$ then $S$ has $a$ moves of size $a+1$ and $a-m$ moves of $a$. Since $m$ is odd, $a$ and $a-m$ have opposite parity, so this sequence is not possible by Lemma 5. An example of this case is given in Figure **??**.

Case 2: $m$ is even

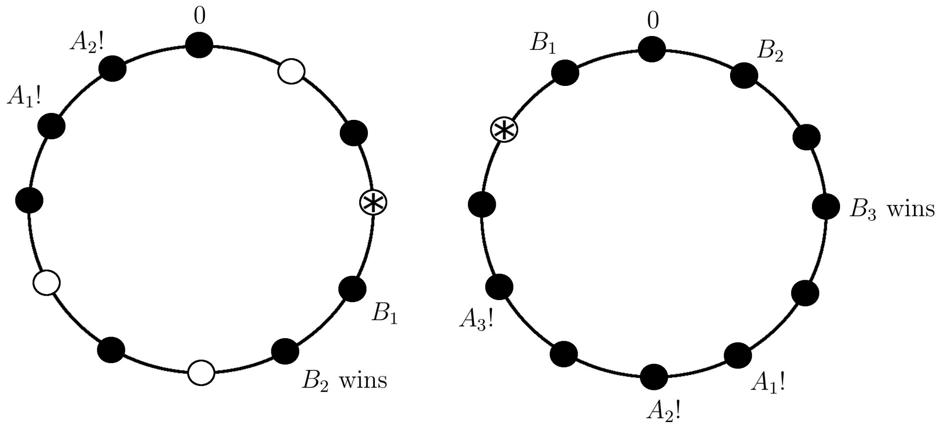
As in the odd $m$ case, from the position $\langle (a+1)^m\ a\rangle$, play begins with each player making $a-m-1$ moves. Player $B$'s strategy is to always move $a$ spaces and $A$ is forced to move $a+1$ spaces each time.

We claim that, from this point on, the players have no choice. The only move available each turn is $a+1$ and each player does this $m/2$ times until all spaces are covered and $B$, who was the last to play, is the winner.

From the beginning of the game, the sequence $S$ of moves is as follows:

$$\underbrace{a+1, a+1, \ldots, a+1}_{m}, a\,\underbrace{a+1, a, a+1, a, \ldots, a+1, a}_{2(a-m-1)}\underbrace{a+1, a+1, \ldots, a+1}_{m}\ .$$

At the end of the game, there have been $a+m-1$ moves of $a+1$ spaces and $a-m$ moves of $a$ spaces. In all, $2a-1$ moves have been made in all so all spaces have been covered.




(a) $n = 12, a = 6, m = 3$ (b) $n = 12, a = 6, m = 4$

Figure 3: Examples of Lemma 6

We need to show that all of $A$'s moves are forced and that all of $B$'s moves are possible. First, to show that all moves in $S$ are possible, we show that no subsequence of consecutive moves in $S$ satisfies the hypotheses of Lemma 5. To this end, suppose that there is such a subsequence. It would have to contain $a$ occurrences of the move $a + 1$ so it would have to run across all three bracketed portions of $S$. Therefore, this subsequence would have to also include all $a - m$ occurrences of the move $a$. But since $m$ is even, $a$ and $a - m$ have the same parity so this subsequence would not satisfy the hypotheses of the lemma.

We have already noted that $A$'s moves in the second bracketed portion of $S$ are forced. It remains to show that $A$'s moves in the third bracketed portion are also forced. To this end, we will show, in fact, that both player's moves are forced in this final section. Suppose that there is a move of $a$ spaces in the final section and consider the subsequence of $S$ consisting of $a$ occurrences of the move $a + 1$ and ending in the first such move of $a$ spaces. This subsequence must span all three bracketed sections so it contains $a - m + 1$ occurrences of the move $a$. Since $m$ is even, $a$ and $a - m + 1$ are of opposite parity so this subsequence of moves satisfies the hypotheses of Lemma 5 and is, therefore, not possible. An example of this case is given in Figure **??**. □

#### 5.1.1. Proof of Theorem 5 in the Case $k = 1$

We consider two cases according to whether $a$ is odd or even.

#### 5.1.2. $a$ is Odd

If the game begins with $a-1$ moves of $a+1$ spaces then, since $a$ is odd, it is not possible to move $a+1$ spaces once more by Lemma 5. Therefore, the game tree has the following form. Moves of $a$ and $a+1$ spaces are down and to the right, respectively.

$$\begin{tikzcd}
\langle \emptyset \rangle \arrow[r, dash] \arrow[d, dash] & \langle a+1 \rangle \arrow[r, dash] \arrow[d, dash] & \langle (a+1)^2 \rangle \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \langle (a+1)^{a-1} \rangle \arrow[d, dash] \\
\langle a \rangle & \langle (a+1)\ a \rangle & \langle (a+1)^2\ a \rangle & & \langle (a+1)^{a-1}\ a \rangle
\end{tikzcd}$$

From Lemmas 3 and 6, the values on the leaves in the tree are all 0 and we may then calculate the values on the other nodes.

$$\begin{tikzcd}
1 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 1 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & 2 \arrow[r, dash] \arrow[d, dash] & 1 \arrow[d, dash] \\
0 & 0 & 0 & & 0 & 0
\end{tikzcd}$$

Therefore, the value of the game is 1.

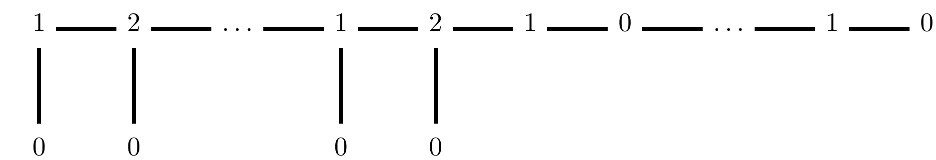
#### 5.1.3. $a$ is Even

Suppose that a game begins with $a$ moves of $a+1$ spaces. From this position, by Lemma 5, a move of $a$ spaces is not possible so, from this point on, each player with be forced to move $a+1$ spaces. The game will end after $2a-1$ moves of $a+1$ spaces have been made and all spaces will have been visited. Therefore, the game tree has the following form.

$$\begin{tikzcd}
\langle \emptyset \rangle \arrow[r, dash] \arrow[d, dash] & \langle a+1 \rangle \arrow[r, dash] \arrow[d, dash] & \langle (a+1)^2 \rangle \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \langle (a+1)^{a-1} \rangle \arrow[d, dash] \\
\langle a \rangle & \langle (a+1)\ a \rangle & \langle (a+1)^2\ a \rangle & & \langle (a+1)^{a-1}\ a \rangle
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, dash] & \langle (a+1)^a \rangle \arrow[r, dash] & \langle (a+1)^{a+1} \rangle \arrow[r, dash] & \cdots \arrow[r, dash] & \langle (a+1)^{2a-1} \rangle
\end{tikzcd}$$

The leaf node in which all spaces have been visited has value 0. The other leaf nodes also have value 0 by Lemmas 3 and 6. We may now calculate the values on the other nodes.




Therefore, the value of the game is 1.

### 5.2. The Case $n = kd - 1, k \geq 2$

The following two lemmas will be used to prove Theorem 5 in this case.

**Lemma 7.** *The position $\langle (a+1)\ a \rangle$ is a $\mathcal{P}$-position.*

*Proof.* The winning strategy for player $B$ is to employ the diamond strategy, advancing the token $2a + 1 = d$ spaces beyond his last move. After the position $\langle (a+1)\ a \rangle$ is reached, the token is on space $d$. On the first pass around the circle, $B$ will move to $2d, 3d, \ldots, (k-1)d$ and then begin the second pass by moving to $kd \equiv 1 \pmod{n}$.

From space 1, $A$ must advance the token $a + 1$ spaces to $a + 2$ since $a + 1$ has already been visited. Player $B$'s response is to move $a$ spaces to $2a+2 = d+1$, once again using the diamond strategy on this second pass around the circle to move to $2d + 1, 3d + 1, \ldots, (k-1)d + 1$ and then to $kd + 1 \equiv 2$ to begin the third pass.

In the case $a = 2$, the game is now over. Player $A$ has no move since $2 + a$ and $2 + a + 1 = 2a + 1$ have already been played. However, if $a \geq 3$ then, from space 2, $A$ is forced to move to $a + 3$, $B$'s reply is to $(a + 3) + a = d + 2$ and the third pass around the circle is underway.

Play continues in this way with $B$ eventually completing the $(a-1)$st pass around the circle by moving the token to $a - 1$. Player $A$ is forced to move $a + 1$ spaces to $2a$ (since he moved to $2a - 1$ on the previous pass) and $B$'s response is to $2a + a = d + a - 1$. On this final pass around the circle, $B$ moves to $2d + a - 1, 3d + a - 1, \ldots, (k-1)d + a - 1$ and then wins by moving finally to $a$. At this point, $A$ cannot move since neither $2a$ nor $d = 2a + 1$ is available.

It remains to show that all of $B$'s moves are possible. Since $B$ always moves $d$ spaces beyond his last move, we have $B_i = (i+1)d \pmod{n}$ for $i = 1, 2, \ldots, ka-1$. If $B_i = B_j$ for some $1 \leq i < j \leq ka - 1$ then, since $d$ and $n$ are coprime, we must have $n \mid (j - i)$ which is impossible. Player $A$'s moves are $A_i = id + a_i \pmod{n}$ for $1 \leq i \leq ka-1$ where $a_i \in \{a, a+1\}$ for each $i$. If $A_i = B_j$ for some $1 \leq i, j \leq ka-1$ then $(j + 1 - i)d \equiv a_i \pmod{n}$. Now $kd \equiv 1 \pmod{n}$, so $ka_i d \equiv a_i \pmod{n}$ and thus $ka_i d \equiv (j + 1 - 1)d \pmod{n}$. Since $d$ and $n$ are coprime, this implies that $j+1-i$ and $ka_i$ differ by a multiple of $n$. But this is impossible as $ka_i - (j+1-i) \geq ka_i - (ka - 1) \geq 1$ and also $ka_i - (j + 1 - i) \leq ka_i + ka - 3 \leq k(2a + 1) - 3 < n$. Therefore, $B$ is never blocked by a previous move of $A$. An example of this lemma is shown in Figure 4. $\square$

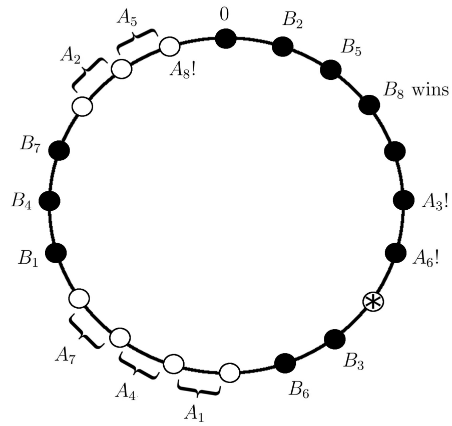



Figure 4: Example of Lemma 7 with $a = 3, k = 3, n = 20$

**Lemma 8.** *The position* $\langle (a+1)^2 \rangle$ *is a* $\mathcal{P}$*-position.*

*Proof.* Play begins with the token on space $2a+2 = d+1$ having previously visited 0 and $a+1$. Player $B$, using the diamond strategy, advances the token to spaces $2d+1, 3d+1, \ldots, (k-1)d+1$.

Player $A$ may now move $a$ or $a+1$ spaces, except in the case $a = 2$ where a move of $a+1 = 3$ is not possible. Suppose that $A$ moves $a$ spaces to $(k-1)d+1+a$. The situation in which $A$ moves $a+1$ spaces is similar and is considered below.

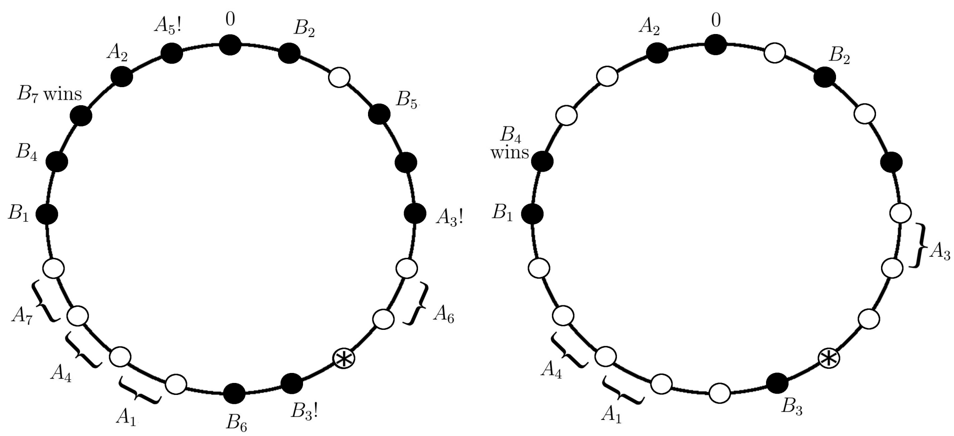
Player $B$ responds by moving $a$ spaces to $(k-1)d+1+2a = kd \equiv 1$ and the second pass around the circle has begun. From space 1, $A$ is forced to move to $a+2$ and then $B$ is forced to reply to $2a+3 = d+2$.

Once again, using the diamond strategy, $B$ moves to $2d+2, 3d+2, \ldots, (k-1)d+2$. If $a = 2$ then the game ends here as $A$ has no possible move. Otherwise, $A$ completes the second pass by moving to $(k-1)d+a+2$ or $(k-1)d+a+3$. $B$'s response to $(k-1)d+2+d = kd+2 \equiv 3$ begins the third pass around the circle.

Finally, on the $a$th pass around the circle, $B$ moves to $a, d+a, 2d+a, \ldots, (k-1)d+a$ and now $A$ is blocked since both $(k-1)d+a+a = kd-1 \equiv 0$ and $(k-1)d+a+a+1 = kd \equiv 1$ have been visited previously.

It remains to show that all of $B$'s moves are possible. On the $i$th pass around the circle, $B$ moves to spaces that are congruent to $i$ modulo $d$ so $B$ never gets in his own way. (The only exception to this is on the second pass where $B$ moves to 1 but that space is available to him at that time.) Thus, every move of $B$ is to a space of the form $b+md$ where $1 \leq b \leq a$. Moreover, every move by $A$ on the $i$th pass around the circle is to a space that is congruent to $a+i$ or $a+i+1$ modulo $d$. Therefore, from pass 1 through to pass $a$, player $A$ moves to spaces on the circle that are of the form $c+md$ where $a+1 \leq c \leq 2a+1 = d$ and so $B$ is never blocked by a previous move of $A$. An example of this case is shown in Figure 5a.

Finally, we consider the case in which $A$ moves $a+1$ spaces to $(k-1)d+a+2$ to complete the first pass. In this situation, $B$ responds by moving $a$ spaces to $(k-1)d+2a+2 = kd+1 \equiv 2$. Now $A$ may move to either $a+2$ or $a+3$ and, in either event, $B$ responds by moving to $2a+3 = d+2$. The analysis now proceeds as above, except that the game ends earlier, on the $(a-1)$st pass. An example of this case is shown in Figure 5b. □




(a) $A$ moves 3 spaces to complete first pass (b) $A$ moves 4 spaces to complete first pass

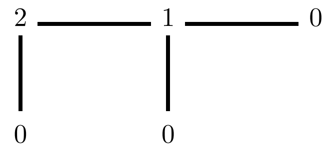
Figure 5: Example of Lemma 8 with $a = 3, k = 3, n = 20$

#### 5.2.1. Proof of Theorem 5 in the Case $k \geq 2$

The game tree is

$$\begin{array}{ccccc} \langle \emptyset \rangle & \text{———} & \langle a+1 \rangle] & \text{———} & \langle (a+1)^2 \rangle \\ | & & | & & \\ \langle a \rangle & & \langle (a+1)\ a \rangle. & & \end{array}$$

From Lemmas 3, 7, and 8, the values on the leaves in the tree are all 0. We may then determine that the value of the game is 2.




## 6. Conclusion

This paper has presented some results on Sprague-Grundy values of Modular Nim. Much more, however, remains to be discovered about this intruiging game. We offer the following suggestions for further research.

- Our investigation of the Line Game was limited to the move set $\{1, 2\}$. It would be of interest to consider the Line Game played with other move sets, and then to see if any results obtained could be applied to Modular Nim. In particular, for the move set $\{1, 2, 3\}$, we could assume that the forbidden set does not contain three consecutive spaces and perhaps the analysis would be similar to that in Section 2.

- Apart from those considered in this paper, there are other examples of Modular Nim for which the outcome classes have been determined. For example, Theorem 3.1 in [3] characterizes the outcome classes for Modular Nim using the move set $\{1, 4\}$. Perhaps the Sprague-Grundy values could be determined for this game, and others, such as those studied in [2] and [3].

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