A DIOPHANTINE PROBLEM RELATED TO A NIM-LIKE GAME

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Abstract

We study under which conditions a certain Nim-like game terminates after the very first round. This is related to a system of diophantine equations and inequalities.

1. Introduction

The game *Nim* is usually played by two players and three heaps of any number of objects. The two players take, one after the other, any number of objects from any single one of the heaps, and the goal is to be the last to take an object. The origin of *Nim* is probably the Chinese game *picking stones* which became known in the western world in the early 16th century. In the beginning of the twentieth century Charles L. Bouton (1869-1922) not only coined the name *Nim* for this and related games, he also gave a first and complete analysis in his treatise [2], which is nowadays considered as the birth of combinatorial game theory.

There is an interesting version of *Nim* due to the game theorist Richard A. Epstein[1]; a first discussion of Epstein's game can be found in the standard reference [1] of Elwyn R. Berlekamp et al. In Epstein's game there are again two players but just one heap of n objects. The two players take alternately m^2 objects from the heap or they add m^2 objects to the heap, where m^2 is the biggest square below the number n of objects in the heap. The goal is to be the last to take away a non-zero square.

[1]born in 1927; also known as *E.P. Stein*, a pseudonym which he uses, according to wikipedia, for writing popular works of fiction, historic and non-fictional books as well as screenplays for television and motion pictures

For obvious reasons the game should not be started with a square number of objects in the heap. If we start with a heap of 6 objects, for example, then the first player can take or can add 4 objects which leads to a heap consisting of 2 or 10 objects. And after the next round the number of objects can be 2 ± 1 or 10 ± 9 and so forth. If both players play optimal, then a game with initially 3 objects will turn into a loop consisting of alternating 3 and 2 objects (and there are further loops for other initial values of n). If they start with $n = 5$ objects, the game will terminate after the first move of the second player (since both, $5 - 2^2$ and $5 + 2^2$ are squares). There are, however, more complicated scenarios which make a complete analysis of the game difficult. We refer to Berlekamp et al. [1] for a graph illustrating the situation of 10 objects in the beginning and to Herbert Möller [6] for a further analysis of this and related Nim-like games.

In this short note we shall investigate whether there are further examples of n beyond 5 for which Epstein's game terminates after the first round.

The game starting with a heap consisting of exactly n objects will terminate after one round if either n is a perfect square (in which case the first player wins) or both $n - m^2$ and $n + m^2$ are positive integer squares (and then the second players wins). We shall consider only the more interesting second case, i.e.,

$$n - m^2 = x^2 \quad \& \quad n + m^2 = y^2 \tag{1}$$

for some positive integers x, y, where m^2 is the largest perfect square less than n. The latter condition can be rephrased by the inequalities

$$m^2 < n < (m + 1)^2. \tag{2}$$

If the system (1) of quadratic equations is solvable in positive integers under the restriction (2) on m^2 to be the largest square below n, then n will be called an *E.P.Stein number* and the triple n, m, x will be called a *solving triple*. If n, m, x are coprime, then the triple is already pairwise coprime and it will be called *primitive*.

Our main result is the following.

Theorem 1. *i) There is no perfect square amongst the E.P.Stein numbers.*
ii) There are infinitely many E.P.Stein numbers.

The proof of this theorem will be given in the following section. In the third and final section we apply a computer algebra package to list all *E.P.Stein numbers* below 10^5. It turns out that in this range there are 36 and their sequence starts with $5, 20, 45, 80, 145, 580, 949, \ldots$.

2. Solving a System of Quadratic Equations and Inequalities

In order to study those *E.P.Stein numbers* we begin with a hypothetical solving triple and multiply the corresponding equations (1). This leads to

$$n^2 = (m^2)^2 + (xy)^2, \tag{3}$$

which shows that solutions are related to pythagorean triples.

Assume the solving triple is not primitive, i.e., n and m are not coprime. Then every prime factor of $\gcd(n, m)$ divides x and y too. Thus, the greatest common divisor of n and m is a square, $\gcd(n, m) =: d^2$ say. Writing $n = d^2 N$ and $m^2 = d^2 M^2$, division by d^2 leads to a smaller triple (although the letters are capitals), namely

$$N - M^2 = (x/d)^2 \quad \& \quad N + M^2 = (y/d)^2,$$

where both, x/d and y/d are positive integers; here M^2 is the largest square below N since otherwise $(M + 1)^2 \leq N$ would imply

$$(m + d)^2 = m^2 + 2d^2M + d^2 = (dM + d)^2 = d^2(M + 1)^2 \leq d^2N = n.$$

Hence, every non-primitive solving triple n, m, x comes from a primitive solving triple $N, M, x/d$. The converse is not true in general as follows from the *E.P.Stein number* $n = 5$. The number $N = 125 = 5^2 \cdot n$ is not an *E.P.Stein number* since the largest square below 125 is $121 = 11^2$ but $125 + 121 = 246$ is not a square.

Now imagine that n, m, x are coprime. Then m^2, xy, n is a primitive pythagorean triple and it follows from Euclid's parametrization that either

$$n = a^2 + b^2 \quad \& \quad m^2 = 2ab \quad \& \quad xy = a^2 - b^2 \tag{4}$$

or

$$n = a^2 + b^2 \quad \& \quad m^2 = a^2 - b^2 \quad \& \quad xy = 2ab,$$

where a and b are coprime positive integers of different parity satisfying $a > b$ (see [4], §13.2). If $m^2 = a^2 - b^2$, then $x^2 = n - m^2 = 2b^2$, which is impossible since $\sqrt{2}$ is irrational. Hence, we may assume that (4) holds. This implies $x = a - b$ and $y = a + b$.

In view of $m^2 = 2ab$ with $\gcd(a, b) = 1$ it follows from the unique prime factorization of the integers that either

$$a = 2u^2 \quad \& \quad b = v^2 \tag{5}$$

or

$$a = u^2 \quad \& \quad b = 2v^2 \tag{6}$$

with $\gcd(u, v) = 1$. Substituting any of those pairs of quadratic equations in $n = a^2 + b^2$, leads to $n = z^4 + 4w^4$. Applying Fermat's method of infinite descent, one can show that the latter equation has no solution n within the set of perfect squares. This is intimately related to the non-solvability of the biquadratic Fermat equation which was already demonstrated by Fermat and Euler (see [7], p. 38). We thus conclude that *there is no perfect square amongst the E.P.Stein numbers.* This proves the first assertion of the theorem.

In view of (5) and (6) we have $m^2 = 2ab = 4u^2v^2$ which shall be the largest square below $n = a^2 + b^2$. Therefore, we make the ansatz $a - b = c$ with some fixed positive integer c. This leads to the equation

$$u^2 - 2v^2 = \pm c. \tag{7}$$

We begin with the case $c = 1$ and the well-known Pell equation:

$$u^2 - 2v^2 = \pm 1 \tag{8}$$

(which is named after the 17th century mathematician John Pell who had nothing to do with it; see [5], p. 4). For each sign $\pm$ there exist infinitely many solutions in positive integers $(u_j, v_j)_j$, and they are all generated by powers of the so-called fundamental solution $(u_1, v_1) = (1, 1)$ as follows. Taking $1 + \sqrt{2}$ to the power j, leads, in view of the irrationality of $\sqrt{2}$, to some unique integers u_j, v_j:

$$(1 + \sqrt{2})^j = u_j + v_j\sqrt{2}$$

(see [4], §14.5); e.g., $(1+\sqrt{2})^1 = 1+1\cdot\sqrt{2}$, which yields $a = 2, b = 1$ and the *E.P.Stein number* $n = 2^2 + 1^2 = 5$. Another example results from $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$ which leads to the values $a = 8, b = 9$, giving the *E.P.Stein number* $n = 8^2 + 9^2 = 145$ with $m^2 = 12^2$; this continues with $(1 + \sqrt{2})^3 = 7 + 5\sqrt{2}$ corresponding to the *E.P.Stein number* $n = 50^2 + 49^2 = 4901$ with $m^2 = 70^2$.

Taking the continued fraction expansion of the quadratic irrationality $\sqrt{2} = [1, \overline{2}]$ into account, one observes that the solutions to (8) form the convergents u_j/v_j to $\sqrt{2} = [1, \overline{2}]$, which implies directly the recursion formula

$$u_{j+1} = us_j + u_{j-1} \qquad \& \qquad v_{j+1} = 2v_j + v_{j-1}.$$

Solving this recursion yields the explicit representations

$$u_j = \tfrac{1}{2}(\alpha^{j+1} + (-\alpha)^{-j-1}) \qquad \& \qquad v_j = \tfrac{1}{2\sqrt{2}}(\alpha^{j+1} - (-\alpha)^{-j-1})$$

where $\alpha = 1 + \sqrt{2}$ (similar to Binet's formula for the Fibonacci numbers; see again [4], §10.14 & 14.5). This leads to $u_j \approx \frac{1}{2}\alpha^{j+1}, t_j \approx \frac{1}{2\sqrt{2}}\alpha^{j+1}$ and $a \approx \frac{1}{4}\alpha^{2j+2} \approx b$, resp. $n \approx \frac{1}{8}\alpha^{2(j+1)} \approx \frac{1}{8}(3 + 2\sqrt{2})^{j+1}$ (which matches the data of the *E.P.Stein numbers* below 10^5 in §3).

Thus, the solutions to the Pell equation (8) lead to solutions of the system (1) of quadratic equations, however, it remains to check whether also inequality (2) is satisfied. In view of (7) we begin with $a - b = c$ and rewrite (2) via (5) or (6) equivalently as

$$(2uv)^2 < u^2 + 4v^2 < (2uv + 1)^2;$$

the first inequality is trivially fulfilled, while the second one is equivalent to $(u^2 - 2v^2)^2 < 4uv + 1$ or

$$c^2 < 1 + 4uv. \tag{9}$$

For $c = 1$ and $c = 2$ this inequality holds for all positive integers u, v. Hence, we conclude that *there exist infinitely many E.P.Stein numbers*. This proves the second assertion of the theorem.

In view of the latter inequality, one could imagine to find *E.P.Stein numbers* among solutions to (7) with $c \neq 1$. And this is indeed true: for example, for $c = 7$ we have $5^2 - 2\cdot 3^2 = 7$ (with $u = 5, v = 3$ satisfying (9)), giving the *E.P.Stein number* $n = 949$ with $m^2 = 900$ (and $n - m^2 = 7^2, n + m^2 = 1849 = 43^2$). Unfortunately, not every solution of such a Pell-type equation leads to an *E.P.Stein number* as the example

$$11^2 - 2 \cdot 7^2 = 23 \tag{10}$$

shows: here (9) is not satisfied (since $c^2 = 529 \geq 308 = 1 + 4uv$); indeed, the corresponding values $a = 121$ and $b = 98$ lead to $n = 24\,245$ and this is not a solution (since $24\,245 - 115^2 = 220$ is not a square). However, we may use (10) in order to find an *E.P.Stein number* by the following reasoning. We can combine an arbitrary integer solution (u, v) to (7) with a solution (u_j, v_j) of (8) by setting $U = uu_j + 2vv_j$ and $V = uv_j + u_jv$, and we obtain another solution to (7). This follows from a straightforward computation:

$$\begin{aligned} U^2 - 2V^2 &= (uu_j + 2vv_j)^2 - 2(uv_j + u_jv)^2 \\ &= (u_j - 2v_j^2)(u - 2v^2) \\ &= \pm 1 \cdot (\pm c) \\ &= \pm c. \end{aligned}$$

For the expert reader we shall mention here that this is related to a group law on hyperbolas and the norm equation in the quadratic number field $\mathbb{Q}(\sqrt{2})$ (see [5], Chapter 4). If the solution (U, V) now satisfies (9), then the corresponding n is an *E.P.Stein number*. We illustrate this with an example: combining $(u, v) = (11, 7)$ with $(u_1, v_1) = (1, 1)$ (both from above), we find $(U, V) = (25, 18)$ (solving $U^2 - 2V^2 = -23$ and satisfying (9) since $c^2 = 529 < 1801 = 1 + 4UV$) as well as $a = 25^2, b = 18^2$; this gives $n = a^2 + b^2 = 949$, which we already know to be an *E.P.Stein number* by the solution of another Pell-type equation.

We thus conclude that *any solution of (7) in positive integers, leads to an E.P.Stein number.* In view of the classical case $c = 1$, thus *there exist infinitely many E.P.Stein numbers.* It might be difficult to characterize those coming from other values of $\pm c$ in (7). By means of algebraic number theory one can study which integers are values taken by the indefinite quadratic form $(x, y) \to x^2 - 2y^2$ (see [5], §16.3), however, we shall not follow this line of inquiry here.

3. A Computer Search and Only One Prime

Using a computer algebra package one can easily find all *E.P.Stein numbers* below a reasonable quantity. A possible program code for Mathematica is listed below:

```
NN := 100000
PerfectSquareQ[n_] := IntegerQ[Sqrt[n]]
For[n = 1, n ≤ NN, n++, m := Floor[Sqrt[n]];
  If[PerfectSquareQ[n − m^2] == True, If[PerfectSquareQ[n + m^2] == True,
    Print["n =", n, " ", "m =", m, ";" FactorInteger[n]]]]]
```

Besides the *E.P.Stein number* n, the corresponding m (according to our notation above) is printed as well as the prime factorization of n. Choosing a larger value for NN one can extend the computations.

We shall provide a list of all *E.P.Stein numbers* below 100 000. We begin with the small ones below 500 and have a closer look on their prime factorization:

$$\begin{array}{rcl} 5, & & \\ 20 & = & 2^2 \cdot \mathbf{5}, \\ 45 & = & 3^2 \cdot \mathbf{5}, \\ 80 & = & (2^2)^2 \cdot \mathbf{5}, \\ 145 & = & 5 \cdot 29. \end{array}$$

We observe that all *E.P.Stein numbers* so far are multiples of 5. The numbers $n = 20, 45$ and 80 result from the very first one by multiplication with a square; however, as we have already noticed in the previous section, multiplying with the square 5^2 would lead to 125 which is not an *E.P.Stein number*. Actually, whenever we begin with an *E.P.Stein number* n, multiplying the solving triple n, m, x with a sufficiently big square d^2 will lead to a solution nd^2, md, xd for the system (1), where the additional condition (2) does not hold; more precisely, when

$$d^2(n - m^2) \geq 2dm + 1. \tag{11}$$

The next *E.P.Stein number* is 145, and this as well as 5 correspond to the first solutions of the Pell equation (8).

We extend our list of *E.P.Stein numbers* up to 10 000:

$$\begin{aligned}
580 &= 2^2 \cdot \mathbf{5} \cdot \mathbf{29}, \\
949 &= 13 \cdot 73, \\
1305 &= 3^2 \cdot \mathbf{5} \cdot \mathbf{29}, \\
1649 &= 17 \cdot 97, \\
2320 &= (2^2)^2 \cdot \mathbf{5} \cdot \mathbf{29}, \\
3625 &= 5^2 \cdot \mathbf{5} \cdot \mathbf{29}, \\
4901 &= 13^2 \cdot 29, \\
5220 &= (2 \cdot 3)^2 \cdot \mathbf{5} \cdot \mathbf{29}, \\
7105 &= 7^2 \cdot \mathbf{5} \cdot \mathbf{29}, \\
9280 &= (2^3)^2 \cdot \mathbf{5} \cdot \mathbf{29}.
\end{aligned}$$

In a similar way as above, the *E.P.Stein number* $145 = 5 \cdot 29$ leads to further *E.P.Stein numbers* by multiplication with suitable squares. Therefore, we shall distinguish between *primary* and *induced E.P.Stein numbers*. We observe examples of primary *E.P.Stein numbers* (not arising from multiplication of a smaller one with a square), namely, $949, 1649$ and 4901. The smallest of those, 949, already appeared in the context of the special solutions to Pell-type equations above, namely, $5^2 - 2 \cdot 3^2 = 7$ and $25^2 - 2 \cdot 18^2 = -23$ in the previous section. The number 1649 is related to $5^2 - 2 \cdot 4^2 = -7$ or $5^2 - 2 \cdot 3^2 = +7$, whereas 4901 is linked with $99^2 - 2 \cdot 70^2 = 1$ as already mentioned above, so this is a primary *E.P.Stein number*. It is interesting that 4901 is divisible by the square 13^2, however, their quotient 29 is not an *E.P.Stein number* although it appears as prime factor in many other *E.P.Stein numbers*.

It is worth having a closer look at the prime factorization of *E.P.Stein numbers*. In view of (1), every *E.P.Stein number* n is a sum of two integer squares and therefore all prime factors $p \equiv 3 \bmod 4$ in the prime factorization of n appear with an even exponent (which, for example, rules out $13 \cdot 29$ to be listed). This is related to the fact that prime numbers $p \equiv 3 \bmod 4$ cannot be written as a sum of two squares (as follows from squares being congruent to 0 or 1 mod 4) and Fermat's two square theorem which states that primes $p \equiv 1 \bmod 4$ can always be represented as a sum of two squares (which is a consequence of the arithmetic in $\mathbb{Z}[\sqrt{-1}]$ although Fermat's reasoning had been different; see [4, Section 15.1 & 20.3]).

It appears that 5 *is the only prime E.P.Stein number.* In fact, it follows from (1) that an *E.P.Stein number* n can be written as a difference of two integer squares or a product of two integers, namely,

$$n = y^2 - m^2 = (y + m)(y - m).$$

If n is prime, then $y - m$ has to be equal to 1. Thus, substituting $y = m + 1$ in (1) and (2) leads to $n = (m+1)^2 - m^2 = 2m+1$ and $m^2 < n = 2m+1$, which is solvable only for $m = 1$ or $m = 2$, where the first leads to a prime $n = 2 \cdot 1 + 1 \equiv 3 \bmod 4$, which is impossible by the just mentioned result on sums of two squares, and $m = 2$ corresponds to the prime $n = 5$.

We continue with a list of all 36 *E.P.Stein numbers* below 10^5. We indicate the primary *E.P.Stein numbers* boldfaced and list the induced ones behind;

$$\begin{aligned}
\mathbf{5} &\rightsquigarrow 20, 45, 80;\\
\mathbf{145} &\rightsquigarrow 580, 1305, 2320, 3625, 5220, 7105, 9280, 11\,745, 14\,500, 17\,545,\\
& 20\,880, 24\,505, 28\,420, 32\,625, 37\,120, 41\,905, 46\,980, 52\,345,\\
& 58\,000, 63\,945, 70\,180, 76\,705, 83\,520;\\
\mathbf{949};&\\
\mathbf{1649};&\\
\mathbf{4901} &\rightsquigarrow 19\,604, 44\,109, 78\,416,\\
\mathbf{31\,025},&\\
\mathbf{54\,805}.&
\end{aligned}$$

Notice that some primary *E.P.Stein numbers* induce (finitely) many others whereas some (e.g. 949) remain alone in their branch. The reason behind this fact is that for an induced solving triple nd^2, md, xd inequality (11) has to be satisfied which, for example, in the case $n = 949$ imposes the condition $49d^2 < 1 + 60d$ on d. It is easy to compute that there is no induced *E.P.Stein number* in the branch of 145 beyond 83 520 (which we indicated with a semicolon above). The *E.P.Stein number* 31 025, however, will induce further not listed *E.P.Stein numbers*.

It appears that there are 15 *E.P.Stein numbers* below 10^4 and 61 below 10^6. Therefore, we may expect that their number grows logarithmically (as the solutions to Pell equations). Concerning the winning positions in Epstein's game, one can find a not unrelated open research problem (E26) in Richard Guy's collection [3] of unsolved problems in number theory.

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