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GAMES ON ARBITRARILY LARGE RATS AND PLAYABILITY

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Abstract

In 1973 Fraenkel discovered interesting sequences, dubbed the *rat sequences* (rat for rational numbers), that partition the positive integers. These sequences became famous, because of a related unsolved conjecture. Here we construct nim-type combinatorial games, called the *rat games*, for which these sequences constitute the losing positions for the current player. We define a notion of *playability* for classes of heap games and show that the rat games are playable. Moreover, we find new definitions of the rat sequences, including a variation of the classical *mex-rule*.

1. Introduction

In this paper we study so-called rational Beatty sequences, also known as *rat sequences*,[1] which partition the positive integers, and we build game rules for which these sequences constitute the losing positions (also known as P-positions) in perfect play. The games are acyclic 2-player impartial combinatorial games, with alternating play. They have perfect information, and we use the normal play convention: a player unable to move, loses [8, 11, 47]. It is thus well known that one can partition the game positions into previous player winning positions (P-positions) and next player winning positions (N-positions), and they can be computed recursively, starting with the terminal positions that are P-positions; see also Definition 2, where we define these notions in our setting.[2]

[1] A historical tour of these popular sequences is included in Section 13.

[2] For introductions to basic combinatorial game notions see [8], [1]. An entirely new fresh direction was initiated by Conway [11], beautifully popularized and illustrated by Knuth [35]. The authoritative graduate textbook by Siegel [47] fuses classical combinatorial game theory with the latest developments.

For an integer $d \geqslant 2$, we play *vector subtraction games* [29] on ordered d-tuples of nonnegative integers $\boldsymbol{x} = (x_1, \ldots, x_d)$. The *moves* are certain vectors $\boldsymbol{s} = (s_1, \ldots, s_d)$ and the move *options* are *vector subtractions* of the form $\boldsymbol{x} - \boldsymbol{s} = (x_1 - s_1, \ldots, x_d - s_d) \geqslant \mathbf{0}$. We give more details in Section 4 and later.

A standard question in combinatorial game theory is whether there is an *efficient winning strategy* for a set of game rules, and usually this boils down to two problems. Decide if a position is a P-position in polynomial time (in succinct input size). If not a P-position, then find a winning move in polynomial time. If we can do this, we claim to know a *solution* for the given game. More often, the answer is unknown, the best known algorithms being exponential.

On the other hand, famous sequences of vectors of non-negative integers (of course with already known complexity) can sometimes have interesting game rules associated with them, and such that the sequence constitutes the set of P-positions [37], [22]; see also, e.g., [13, 40, 39, 42, 17, 30, 25]. The problem is formalized in [12]. **This reverse problem is the topic of this study**; similar topics in classical game theory are called "Mechanism Design", so this study should fit under an umbrella of a kind of Mechanism Design for Combinatorial Games (MDCG).[3]

Trivial game rules can always be found, by letting each candidate N-position define a move to a terminal position, and forbid moves from all candidate P-positions (thus making them terminal P-positions). Such game rules are not very interesting; this motivated the introduction of so-called invariant rules [13], of which vector subtraction games [29] is an instance. The classical complexity problem for impartial games, becomes varied and interesting when we seek the rules of games, given a candidate set of solutions. See also Observation 1, where we show that invariant rules are not always possible to achieve.

MDCG is not just a computational problem, because we propose that humans be able to play these type of games (without computational aid). Although this is usually not a strict requirement, one typically assumes some recreational value of a combinatorial game/ruleset.

The early connection between Wythoff Nim [56, 14] and complementary sequences of modulus the golden ratio and its square respectively, recently led to research in finding game rules for any complementary pair of homogenous Beatty sequences of irrational modulus [5] (generalizing Wythoff's sequences) [13]. The solution [40] is appealing in a mathematical sense of the word (it introduced a new operator to combinatorial game theory) but unfortunately it is not known whether the rules of game even can be understood in polynomial time (in succinct input size), which should be minimal requirement for any notion of 'playability'.

We arrive at a motivation for this paper: rulesets for combinatorial games should be compact, playable, enjoyable and/or "suitable" [8]. Many games in the literature

[3]This was thankfully pointed out to the second author, by Prof. Rann Smorodinsky, at a Play-Time seminar at the Technion-Israel Institute of Tecnology in 2018.

have this property. To this end, in Section 4, we will define a concept of *playability* which fits any multi-pile heap game similar to the ones in this work.

The rat sequences have been studied before as candidate sets of P-positions [24], but only for small number of heaps, and where, even for the instance of games on three (unordered) heaps, the rules of games, although impressive in their statement, can be argued intractable for a human player. In this study we improve those rules. Here, we study the rat sequences as sequences of ordered d-tuples (integer vectors). See also a short discussion in the final paragraph of Section 13.

We will define two classes of games, *grandiose rules* and (playable) *compact rules*. Both settings are conveniently expressed via what the players cannot do, i.e. forbidden moves or *shortcuts*, and, as a main result, we will prove that the rules are the same although their expessions differ in complexity. Here is some more detailed overview:

- The class of sequences of interest are the *rat-vectors* ("rat" for rational modulus), and they are defined in Section 2. We show arithmetic periodicity along their 'columns'.
- Section 3 exemplifies the *grandiose rules*, as vector subtraction games via a certain matrix representation (revisited in later sections).
- Section 4 provides a general framework for maximal rulesets, defined via the notion of *shortcuts*, of which the *grandiose rules* is an instance.
- Section 5 uses this framework to show that the P-positions for the grandiose games (defined here) are the rat-vectors (Theorem 6).
- Section 6 defines the *compact rules* and the concept of *p*layability, and we show that the compact rules are playable.
- Section 7 defines the notion of *ternary recurrence*, which will help to understand the shortcuts in the setting of rat vectors. Moreover it is the tool to demonstrate equivalence of grandiose rules with compact rules; Theorem 10 verifies that the games in Sections 5 and 6 are the same.
- As a prerequisite for rats' matrix representations, in Section 8 we revisit the arithmetic periodic behavior of the rat vectors.
- Section 9 provides the *matrix representation* for the rat-vectors, and we exploit their connections with the standard binary counting system.
- Section 10 studies a 'mex-rule' of the rat-vectors, using also the matrix representation.
- Section 11 builds matrices for the *rats' shortcuts* (that connect the pairs of rat-vectors via subtraction).

- In Section 12, we find rats' Sprague-Grundy values, and we show that, in a specific sense, the rat games are close to the game of nim.
- We provide a historical rat tour in Section 13.

2. Fraenkel's Popular Rat Sequences

Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the positive- and non-negative integers, respectively. The *rat sequences* consist of vectors of the form $(\lfloor \frac{3n}{2} \rfloor, 3n-1)$, $(\lfloor \frac{7n}{4} \rfloor, \lfloor \frac{7n}{2} \rfloor - 1, 7n-3)$, $(\lfloor \frac{15n}{8} \rfloor, \lfloor \frac{15n}{4} \rfloor - 1, \lfloor \frac{15n}{2} \rfloor - 3, 15n-7)$, and so on, for $n \in \mathbb{N}$. Thus, for each dimension $d \geqslant 2$, we code the vectors by $\boldsymbol{r}(n) = (r_1(n), \ldots, r_d(n))$, $n \in \mathbb{N}$, where

$$r_i(n) = \left\lfloor \frac{(2^d-1)n}{2^{d-i}} \right\rfloor - 2^{i-1} + 1, \tag{1}$$

$i \in \{1, \ldots, d\}$. This representation will be referred to as the *standard form*, and, for each dimension $d \geqslant 1$, we can think of it as an infinite row-matrix on d columns, with rows (or columns) partitioning the positive integers (see Theorem 21 below). Note that for $d = 1$, the representation is simply $\mathbb{N}$, so henceforth we let $d \geqslant 2$. Moreover, for a given d, we let $\mathcal{R}(d) = \{\boldsymbol{r}(n) \mid n \in \mathbb{N}\}$. See Table 1 for some initial examples, and observe that in this study, the d-tuples are ordered (vectors).

Note that, for all dimensions d, for all rows n,

$$r_d(n) \equiv 2^{d-1} \pmod{2^d - 1}. \tag{2}$$

A sequence $x = (x_n)$ is *arithmetic periodic* with saltus s and period p if, for all $n \in \mathbb{N}$, $x_{n+p} = x_n + s$.[4] Suppose, for example that $x_1 = 1$, $x_2 = 3$, $p = 2$, and $s = 3$. This defines the arithmetic periodic sequence $x = (1, 3, 4, 6, \ldots)$, which is the meagre rat's ($d = 2$, see below) first column. In Table 1, we give the first few rows of the rat sequences for $d = 2, 3, 4$.

We call matrices such as in Table 1 *infinite rat matrices*. Let us make some initial observations about the columns of the infinite rat matrices.

Lemma 1. *Consider a number of heaps $d \geqslant 2$.*

- *For each column i, for all $t \in \mathbb{N}$, $r_i(n) \equiv r_i(n + t2^{d-i}) \pmod{2^d - 1}$.*
- *Each rat column i is* arithmetic periodic *with period 2^{d-i} and saltus $2^d - 1$. That is, for each column i, for all n, $r_i(n + 2^{d-i}) = r_i(n) + 2^d - 1$.*

[4]The notions "arithmetic periodic" and "saltus" in this context are due, to the best of our knowledge, to Winning Ways.

$n\backslash d$	2		3			4			
1	1	2	1	2	4	1	2	4	8
2	3	5	3	6	11	3	6	12	23
3	4	8	5	9	18	5	10	19	38
4	6	11	7	13	25	7	14	27	53
5	7	14	8	16	32	9	17	34	68
6	9	17	10	20	39	11	21	42	83
7	10	20	12	23	46	13	25	49	98

Table 1: The first few rows of the rat sequences for $d = 2, 3, 4$.

Proof. We begin to show the first item. For all rows n, for all columns i,

$$\begin{aligned} r_i(n + 2^{d-i}) - r_i(n) &= \left\lfloor \frac{(2^d-1)n + (2^d-1)2^{d-i}}{2^{d-i}} \right\rfloor - \left\lfloor \frac{(2^d-1)n}{2^{d-i}} \right\rfloor \\ &= 2^d - 1. \end{aligned}$$

The second item is now obvious. □

3. Grandiose Rat Rules on Two and Three Heaps

Let us begin by exemplifying the *grandiose rules* in dimensions 2 and 3 (they will be defined in Definition 5 in Section 5). Motivated by Lemma 1, we will introduce a finite *matrix representation* (that will be defined in Section 9) for the rat sequences, and thus study games and sequences in suitable finite representations (compare with Section 2 where sequences have infinite representations).

3.1. The Meagre Rat

Beginning with $d = 2$, recall the standard form for the rat sequences $\boldsymbol{r}(n) = \left(\left\lfloor \frac{3n}{2} \right\rfloor, 3n-1\right)$, for $n \in \mathbb{N}$. The two heap *rat-matrix* is:

$$R^2 = \begin{pmatrix} 3t+1 & 6t+2 \\ 3t+3 & 6t+5 \end{pmatrix}$$

for any $t \in \mathbb{N}_0$. The interpretation is that, for $d = 2$, any given rat vector is a row, for some t. For example, $n = 3$ gives $\boldsymbol{r}(3) = (4, 8)$, and it is the first row with $t = 1$.

The meagre rat's *shortcut-matrix* consists of all (unordered) vector differences of the rows in R^2, with nonnegative entries, and it is:

$$F^2 = \begin{pmatrix} 3t & 6t \\ 3t+1 & 6t+3, \\ 3t+2 & 6t+3 \end{pmatrix}$$

for any $t \in \mathbb{N}_0$. For example $(11, 21)$ is a shortcut (row 3 and $t = 3$). In R^2 it is represented as the difference $(3 \cdot 3 + 3, 6 \cdot 3 + 5) - (1, 2)$, where we used row 2 with $t = 3$ and row 1 with $t = 0$.

3.2. The Not So Meagre Rat

The standard form for $d = 3$ is $\left(\left\lfloor \frac{7n}{4} \right\rfloor, \left\lfloor \frac{7n}{2} \right\rfloor - 1, 7n - 3\right)$, for $n \in \mathbb{N}$, and, for $t \in \mathbb{N}_0$, the not so meagre rat-matrix is

$$R^3 = \begin{pmatrix} 7t+1 & 14t+2 & 28t+4 \\ 7t+3 & 14t+6 & 28t+11 \\ 7t+5 & 14t+9 & 28t+18\text{'} \\ 7t+7 & 14t+13 & 28t+25 \end{pmatrix}$$

and the interpretation is that, for $d = 3$, any given rat vector is a row, for some t.

Its shortcut-matrix, for $t \in \mathbb{N}_0$, is

$$F^3 = \begin{pmatrix} 7t & 14t & 28t \\ 7t+1 & 14t+3 & 28t+7 \\ 7t+2 & 14t+3 & 28t+7 \\ 7t+2 & 14t+4 & 28t+7 \\ 7t+3 & 14t+7 & 28t+14 \\ 7t+4 & 14t+7 & 28t+14 \\ 7t+5 & 14t+10 & 28t+21 \\ 7t+5 & 14t+11 & 28t+21 \\ 7t+6 & 14t+11 & 28t+21 \end{pmatrix}.$$

The not so meagre rat's shortcut-matrix F^3 consists of all (unordered) vector differences of the rows in R^3.

Together R^3 and F^3 form the 3-dimensional *forbidden move matrices*. Any 3-tuple, which is not a row in R^3 of F^3 can be a move in the 3-dimensional rat-game. Intuitively, the current player can choose any vector which is not a forbidden move, and subtract it from the given position, provided that each coordinate remains non-negative. Such rules will be generalized in Section 4 and in the special case of arising from rat sequences, they will be called the *grandiose rules*, defined in Section 5. See also Example 3, for the case $d = 4$.

The shortcut-matrices grow very quickly, as the number of heaps grow, and may appear intractable at a first sight; see also Figure 1 of linear fractals which encodes the matrix for $d = 10$. One of our main results (Theorem 10) states that the *compact rules* described in Section 6 are equivalent with the grandiose rules.

4. A Maximal Ruleset Lemma

The first advantage of the maximal ruleset construction is that one of the directions, namely '$\mathcal{P} \nrightarrow \mathcal{P}$', is obvious. Another advantage is that not all candidate N-positions have to be checked, namely each N-position that is not a shortcut has already a move to the terminal 0-position. We state this basic observation as a 'maximal ruleset condition', Lemma 2 below, in a quite general setting. Intuitively this lemma is more efficient/useful for sparse candidate sets of P-positions such as the rat sequences (see Theorem 6).

Consider $(\mathcal{S}, +)$, a partially ordered semigroup $\mathcal{S}$ (closed under addition) with a unique smallest element zero, $\mathbf{0}$ (in particular every descending sequence of elements has a smallest element).

Definition 1 (Subtraction game). Let $\mathcal{M} \subseteq \mathcal{S}$ denote a *ruleset* of an $\mathcal{S}$-*subtraction game* $G(\mathcal{M})$, where $G(\mathcal{M})$ is an impartial normal play game defined as follows: there is a *move* from $\boldsymbol{x} \in \mathcal{S}$ to $\boldsymbol{y} \in \mathcal{S}$, if $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{M}$.

Definition 2 (P-position). The set $\mathcal{P} \subseteq \mathcal{S}$ is the set of P-positions of the game $G(\mathcal{M})$, with $\mathcal{M} \subseteq \mathcal{S} \setminus \{\mathbf{0}\}$, if

1. $\forall \boldsymbol{x} \notin \mathcal{P}\ \exists \boldsymbol{m} \in \mathcal{M} : \boldsymbol{x} - \boldsymbol{m} \in \mathcal{P}$
2. $\forall \boldsymbol{p} \in \mathcal{P}\ \forall \boldsymbol{m} \in \mathcal{M} : \boldsymbol{p} - \boldsymbol{m} \notin \mathcal{P}$

In this case, we write $\mathcal{P} = \mathcal{P}(\mathcal{M})$. Note that item 2 implies $\mathbf{0} \in \mathcal{P}$. If a position is not a P-position, then it is an N-position.

Definition 3 (Shortcut). For given $\mathcal{X} \subseteq \mathcal{S}$, let

$$\mathcal{F}_{\mathcal{X}} = \mathcal{X} - \mathcal{X} := \{\boldsymbol{p} - \boldsymbol{q} \mid \boldsymbol{p}, \boldsymbol{q} \in \mathcal{X}, \boldsymbol{p} \geqslant \boldsymbol{q}\}$$

denote the set of *shortcuts*, with respect to $\mathcal{X}$.

Note that, if $\mathcal{X} \neq \emptyset$, then $\mathbf{0} \in \mathcal{F}_{\mathcal{X}}$, and if $\mathbf{0} \in \mathcal{X}$, then $\mathcal{X} \subseteq \mathcal{F}$.

Definition 4 (Maximal ruleset). The set $\mathcal{M} = \mathcal{S} \setminus \mathcal{F}_{\mathcal{X}}$, of all elements except the shortcuts, is the *maximal ruleset*, with respect to the subset $\mathcal{X} \subseteq \mathcal{S}$.

Observation 1. Observe that it is not always the case that a maximal ruleset gives back the candidate set as a set of P-positions. Take for example a sequence of non-negative integers C that starts with the three numbers $\{0, 1, 3\} \subset C$. In fact, it is impossible to find a subtraction game $S \subset \mathbb{N}_0$ such that $P(S) = C$. Namely, the move from the position $2 \notin C$ has to go to either 0 or 1, but 1 is forbidden (because both 0 and 1 are in C), so 2 has to be a move. But this shortcuts 3 with 1.

Lemma 2 (Maximal ruleset condition). *Consider a set $\mathcal{X} \subseteq \mathcal{S}$, with $\mathbf{0} \in \mathcal{X}$. If, for all $\boldsymbol{f} \in \mathcal{F}_{\mathcal{X}} \setminus \mathcal{X}$, there exists an $\boldsymbol{x} \in \mathcal{X}$ such that*

$$\boldsymbol{f} - \boldsymbol{x} \in \mathcal{S} \setminus \mathcal{F}_{\mathcal{X}},$$

then $\mathcal{X} = \mathcal{P}(\mathcal{M})$ is the set of P-positions of the game $G(\mathcal{M})$ with maximal ruleset $\mathcal{M}$ with respect to $\mathcal{X}$, i.e. $\mathcal{M} = \mathcal{S} \setminus \mathcal{F}_{\mathcal{X}}$.

Proof. Notice that $\mathbf{0} \in \mathcal{F}_{\mathcal{X}} \cap \mathcal{X}$ and $\mathcal{X} \subseteq \mathcal{F}_{\mathcal{X}}$, with equality if and only if $\mathcal{X} = \{\mathbf{0}\}$. Item 2 in the definition of a P-position holds because for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, by definition of $\mathcal{F}_{\mathcal{X}}$ and maximal ruleset, $\boldsymbol{x} - \boldsymbol{y} \notin \mathcal{M}$.

Hence it suffices to study item 1 in Definition 2, so assume $\boldsymbol{f} \notin \mathcal{X}$. If $\boldsymbol{f} \notin \mathcal{F}_{\mathcal{X}}$, then by definition of $\mathcal{M}$, there is a move to $\mathbf{0} \in \mathcal{P}$, so assume $\boldsymbol{f} \in \mathcal{F}_{\mathcal{X}} \setminus \mathcal{X}$. Since, by assumption, there exists an $\boldsymbol{x} \in \mathcal{X}$ such that $\boldsymbol{f} - \boldsymbol{x} \in \mathcal{S} \setminus \mathcal{F}_{\mathcal{X}} = \mathcal{M}$, there is an $\boldsymbol{m} \in \mathcal{M}$ such that $\boldsymbol{f} - \boldsymbol{m} = \boldsymbol{x}$. (Note that $\boldsymbol{x} \neq \mathbf{0}$, by the assumption.) $\square$

5. The Grandiose Rat Games

Let us define the grandiose games. In this section, and for the rest of the document, $\mathcal{S} = \mathbb{N}_0^d$ for some $d \in \mathbb{N}$, $d \geqslant 2$.

Definition 5. [Rats' grandiose game] The *grandiose game* $G(\mathcal{M})$ on d heaps is given by the maximal ruleset $\mathcal{M} = \mathbb{N}_0^d \setminus \mathcal{F}$ with respect to the forbidden moves $\mathcal{F} = \mathcal{F}_{\mathcal{R}(d)}$.

Next, we show that the rat sequences satisfy the condition of Lemma 2, and we study first the penultimate situations. We begin by showing that when we shift a divisor 2 inside the floor function, then the deviation is small.

Lemma 3. *For any number y,*

$$0 \leqslant \lfloor y \rfloor /2 - \left\lfloor \frac{y}{2} \right\rfloor \leqslant \frac{1}{2}. \tag{3}$$

Proof. Since $y \geqslant 2\lfloor \frac{y}{2} \rfloor$, for any y, then $\lfloor y \rfloor - 2\lfloor \frac{y}{2} \rfloor > (y-1) - y = -1$. Since the expression is an integer, by the strict inequality, the lower bound holds. The upper inequality follows by decompositioning into fractional parts. Put $\lfloor y \rfloor - 2\lfloor \frac{y}{2} \rfloor = y - (y - \lfloor y \rfloor) - 2(\frac{y}{2} - (\frac{y}{2} - \lfloor \frac{y}{2} \rfloor)) = (y - \lfloor y \rfloor) - 2(\frac{y}{2} + \lfloor \frac{y}{2} \rfloor)) \leqslant 1$. $\square$

The rat vectors define a total order, and the enumeration is given. The shortcut vectors are in general only partially ordered (for $d \leqslant 3$ they have a total order), but a natural ordering can be obtained by using a lexicographic ordering, with the first coordinate the smallest 'digit'. In particular one can prove that, for each d, we have a minimal vector, and it is of the form $(1, 3 \ldots, 2^d - 1)$.

Lemma 4. *Consider the rat sequences and their shortcuts.*

(a) *If $\boldsymbol{f} \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{R}$, then there exists an $\boldsymbol{r} \in \mathcal{R} \setminus \{\mathbf{0}\}$, such that $\boldsymbol{f} > \boldsymbol{r}$.*

(b) *If $\boldsymbol{r} \in \mathcal{R} \setminus \{\mathbf{0}, \boldsymbol{r}(1)\}$, then there exists an $\boldsymbol{f} \in \mathcal{F} \setminus \mathcal{R}$, such that $\boldsymbol{r} > \boldsymbol{f}$.*

Proof. We prove (a), and (b) is similar. It suffices to demonstrate that $\boldsymbol{r}(1) < \boldsymbol{r}(m) - \boldsymbol{r}(n)$, for any couple $n < m$, and so it suffices to show that the inequality holds for $m = n + 1$. Note that $r_d(1) = 2^{d-1} < 2^d - 1 = r_d(n+1) - r_d(n)$, for any $n \geqslant 1$. Note $r_{d-1}(1) = 2^{d-2}$. And, by Lemma 3,

$$
\begin{aligned}
2^{d-1} - 1 &= (2^d - 1)/2 - 1/2 && (4)\\
&\leqslant r_{d-1}(n+1) - r_{d-1}(n). && (5)
\end{aligned}
$$

Similarly, for all i, $r_{d-i}(1) = 2^{d-1-i} \leqslant 2^{d-i} - 1 \leqslant r_{d-i}(n+1) - r_{d-i}(n)$. In particular, by applying Lemma 3 $d-1$ times, for $i = d-1$, we get $r_1(1) = 2^0 = 2^1 - 1 \leqslant r_1(n+1) - r_1(n)$. □

We use the part (a) of the next lemma here, and part (b) later (for playability). Note in particular that part (a) is the condition in Lemma 2.

Lemma 5. *Consider the rat sequences and their shortcuts.*

(a) *For all $\boldsymbol{f} \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{R}$, there exists an $\boldsymbol{r} \in \mathcal{R}$ such that $\boldsymbol{f} - \boldsymbol{r} \in \mathbb{N}_0^d \setminus \mathcal{F}_{\mathcal{R}}$.*

(b) *For all $\boldsymbol{r} \in \mathcal{R} \setminus \{\mathbf{0}, \boldsymbol{r}(1)\}$, there exists an $\boldsymbol{f} \in \mathcal{F}$ such that $\boldsymbol{r} - \boldsymbol{f} \in \mathbb{N}_0^d \setminus \mathcal{F}_{\mathcal{R}}$.*

Proof. For (a), Lemma 4 shows that if $\boldsymbol{f} \in \mathcal{F}_{\mathcal{R}} \setminus \mathcal{R}$, then there exists an $\boldsymbol{r} \in \mathcal{R} \setminus \{\mathbf{0}\}$, such that $\boldsymbol{f} > \boldsymbol{r}$. This will be necessary for the remaining proof. We divide the rest of the proof into two cases: (i) We show $\boldsymbol{f} - \boldsymbol{r} \notin \mathcal{R}$. (ii) We show $\boldsymbol{f} - \boldsymbol{r} \notin \mathcal{F}_{\mathcal{R}} \setminus \mathcal{R}$. For (i), suppose for a contradiction that there exists an $\boldsymbol{f} \in \mathcal{F}_{\mathcal{R}}$ such that, for all m, there is an n, such that $\boldsymbol{f} - \boldsymbol{r}(m) = \boldsymbol{r}(n)$. But, for all i, j, $r_d(i) + r_d(j) \equiv 1 \pmod{2^d - 1}$ and for all i, j, $r_d(i) - r_d(j) \equiv 0 \pmod{2^d - 1}$. Hence, $f_d \neq r_d(n) + r_d(m)$, for any combination of $\boldsymbol{f}, n, m$ so (i) cannot hold. Similarly (ii) cannot hold, because if $\boldsymbol{f} - \boldsymbol{r} = \boldsymbol{g}$, for some $\boldsymbol{g} \in \mathcal{F} \setminus \mathcal{R}$, then $f_d - r_d \equiv g_d \pmod{2^d - 1}$, so $r_d \equiv 0 \pmod{2^d - 1}$, which is not true, because $r_d \equiv 2^{d-1} \pmod{2^d - 1}$ and both f_d and g_d are congruent to 0 modulo $2^d - 1$. Since both (i) and (ii) hold, we are done.

The proof of (b) is in analogy with that of (a). By using the same notation, note in particular that $f_d + g_d \not\equiv r_d \pmod{2^d - 1}$. □

Theorem 6. *For any given number of heaps $d \geqslant 2$, let $G(\mathcal{M})$ be the grandiose rat rules, i.e. $\mathcal{M} = \mathbb{N}_0^d \setminus \mathcal{F}_{\mathcal{R}}$ is the maximal ruleset on the rat sequences. Then $P(\mathcal{M}) = \mathcal{R}$.*

Proof. We combine Lemma 2 with Lemma 5 (a). Namely let $\mathcal{R} = \mathcal{P}$. Then the result follows. $\square$

Although the idea of the forbidden move matrices is intuitive and Theorem 6 is satisfactory, the Grandiose games do not appear a-priori playable. However, we will show that the rules are equivalent with playable *compact rules*, the topic of Section 6 and onwards.

6. Rats' Compact Rules

The rules of our *compact games* are as follows. Let $d \geqslant 2$ be an integer. We play on ordered d-tuples (vectors) of non-negative integers $\boldsymbol{x} = (x_1, \ldots, x_d)$.

The move options are *vector subtractions*, and any vector subtraction $\boldsymbol{x} - \boldsymbol{s} = (x_1 - s_1, \ldots, x_d - s_d) \geqslant \mathbf{0}$ is allowed, with $\boldsymbol{s} = (s_1, \ldots, s_d)$, except if it satisfies either of the following two properties, a or b :

a(i) $s_d \equiv 2^{d-1} \pmod{2^d - 1}$ AND

a(ii) for all $i \in \{2, \ldots, d\}$, $s_i - 1 \leqslant 2s_{i-1} \leqslant s_i$,

OR

b(i) $s_d \equiv 0 \pmod{2^d - 1}$ AND

b(ii) for all $i \in \{2, \ldots, d\}$, $s_i - 1 \leqslant 2s_{i-1} \leqslant s_i + 1$.

For $x \in \mathbb{N}$, let $\varphi(x) = \left\lceil \frac{x}{2} \right\rceil$. Note that, coded in binary digits, then $\varphi(x)$ is the right shift of $x + 1$, where the least significant digit is dropped. For example with $x+1 = 8 = 1000$, then $\varphi(x) = 4 = 100$, and $x+1 = 9 = 1001$, then $\varphi(x) = 5 = 101$.

Lemma 7. *If s is a rat vector, then item* a *is satisfied with, for all i, $\varphi(s_i) = s_{i-1}$. If s is a shortcut vector, then item* b *is satisfied with $\varphi(s_i) \in \{s_{i-1} - 1, s_{i-1}\}$*

Example 1. Consider $d = 3$ with the starting position $(1, 3, 7)$. There is no move to $\mathbf{0}$, since condition b is satisfied by $s_3 = 7 = 2^3 - 1$ and since $2s_{i-1} + 1 = s_i$, for $i = 2, 3$; $2 \times 1 + 1 = 3$ and $2 \times 3 + 1 = 7$. However, there is a move to $(1, 2, 4)$, since $(0, 1, 3)$ satisfies neither a nor b. But $r_1 = (1, 2, 4)$ is the smallest (using lexicographic order) position of the forms a or b, which implies that the next move will be a losing move. Hence position $(1, 3, 7)$ is an N-position, a winning position for the current player.

A nice property for a ruleset defined on any number of heaps is that the description on how to move does not increase too fast when the number of heaps grows. Since the modulus is rational, our winning strategies and games are *arithmetic periodic*, and we will use matrix representations to hightlight this fact. For each dimension d, and all $n \in \mathbb{N}$, we have that

- the number of rat-vectors with coordinates less than or equal to $2^{d-1}(2^d-1)n$ is $2^{d-1}n$,
- the number of proper shortcuts with coordinates less than or equal to $2^{d-1}(2^d-1)n$ is $(3^{d-1})n < 2\binom{2^{d-1}}{2}n$ (many get canceled, and we show this in Section 11).

So, by arithmetic periodicity, if we give the job to the previous player to refute any suggested vector not in $\mathcal{M}$, then, by exhaustive search the number $3^{d-1}+2^{d-1}$ is an upper bound. This number is constant in the heap sizes, but still exponential in the number of heaps. By this alone, the grandiose games do not appear playable for a large number of heaps. Here, we prove that there is a much faster way to refute a move, namely, by the compact rules—a linear procedure in the number of heaps (Theorem 8). And, in Section 7 we show that the two rulesets are the same, so the compact rules make the grandiose games *playable* (Definition 7), for any number of heaps.

Note that, the quintessence of impartial combinatorial games, Nim, has a linear time procedure in the number of heaps to decide whether a given vector is a move[5] (if heap sizes are bounded) and so does of course Moore's Nim (but some other variations are slightly more complex, such as Fraenkel's multi-pile generalization of Wythoff Nim [21], with accompanying conjecture). Not all game rules are defined over an arbitrarily finite number of heaps, but for those classes, where it is applicable, we suggest the following terminology.

Definition 6 (Linear test). Consider $a, b, c, p, q \in \mathbb{Q}$. An $(a, b, c; p, q)-$*linear test* on an ordered d-tuple of integers $x = (x_1, \ldots, x_d)$ is a double inequality of the form, for all $i \in \{2, \ldots, d\}$, $x_i + a \leqslant bx_{i-1} \leqslant x_i + c$, together with a congruence $x_d \equiv p \pmod{q}$.[6] The linear test *accepts* if all conditions hold.

For example, the compact rules item a, for $d = 4$, is a $(-1, 2, 0; 8, 15)$-linear test. In given contexts we say simply *linear test*.

Definition 7 (Playability). The rules of a heap game are *playable* if

- (for bounded heap sizes) they depend on a finite number of linear tests on the heaps, where accept means that a vector is a move;
- (for a bounded number of heaps) they are described by at most a log-linear procedure in the heap sizes;[7]

[5] Of course, for Nim we must disallow any vector subtraction if more than one coordinate is positive (or if a positive coordinate is larger than the corresponding heap size).

[6] One could consider a variety of linear test conditions apart from the ones we choose here, as long as the linear check across the vector satisfies reasonable simple conditions in the sense of the notion of playability in Definition 7.

[7] This is trivially required since, for example, we must check that move coordinates are not larger than heap sizes. Apart from this requirement, we will see that, in our case, the complexity of the rules is *constant in the size of the heaps.*

- there is a constant $c \geqslant 1$ that does not depend on the number of heaps, such that, for all but c non-terminal P-positions, there is a move to an N-position that does not have a terminal position as an option.[8]

We begin by proving that the compact games are playable.

Theorem 8. *The compact rules are playable.*

Proof. The first part of playability is immediate by Definitions 6 and 7. For the second part, for any $d \geqslant 2$, Lemma 5 (b) shows that all but the penultimate P-position has a move to a forbidden move position (so we may take $c = 1$ in Definition 7). $\square$

7. Ternary Recurrence, Proper Shortcuts and the Main Playability Result

In this section, we connect the Grandiose rules with the Compact rules, and thus show that the maximal move-set construction gives playable rat games.

Definition 8. A d-tuple $\boldsymbol{x} = (x_1, \ldots, x_d)$ has a *ternary recurrence* if it satisfies $x_d \equiv 0 \pmod{2^d - 1}$, and, for all $i \in \{2, \ldots, d\}$, $x_{i-1} \in \left\{ \left\lfloor \frac{x_i}{2} \right\rfloor, \left\lceil \frac{x_i}{2} \right\rceil \right\}$.

The word ternary reflects that, for given x_{i-1}, there are exactly three possibilities of x_is for which membership is satisfied. Note the similarity with the compact game rules. The "if" part of the the following result depends on Theorem 17 in Section 11; however, the "only if" direction is independent of later results and, as we will see in Theorem 10, it implies the connection between the compact rules and the grandiose games.

Lemma 9. *A vector $\boldsymbol{x}$ is a proper shortcut if and only if it has a ternary recurrence.*

Proof. By definition, the vector $\boldsymbol{x}$ is a shortcut, if, for all i, for some $k > 0$, $x_i = r_i(n + k) - r_i(n)$, with r_i defined as in (1). This gives $x_d = k(2^d - 1)$, so the congruence part holds.

Next, we prove that $0 \leqslant x_{i-1} - \left\lfloor \frac{x_i}{2} \right\rfloor \leqslant 1$, for all i, if $\boldsymbol{x}$ is a shortcut. Let $\varphi = x_{i-1} - \left\lfloor \frac{x_i}{2} \right\rfloor$.

If $\boldsymbol{x}$ is a shortcut, then

$$\varphi = r_{i-1}(n + k) - r_{i-1}(n) - \left\lfloor \frac{r_i(n + k) - r_i(n)}{2} \right\rfloor .$$

How much does the second term differ from the first? Note that, if we *shift* the divisor in the second term inside the inner floor functions, then we get $r_{i-1}(n +$

[8]Of course penultimate P-positions do not satisfy this condition, which motivates the lower bound on c.

$k) - r_{i-1}(n) - \lfloor r_{i-1}(n+k) - r_{i-1}(n) \rfloor = 0$, because r is an integer valued function. Hence, by applying Lemma 3 in shifting back the divisor 2 outside the inner most nested floor functions, we get $-1 \leqslant \varphi \leqslant 1$.

Since the expression is an integer, it suffices to exclude the case $\varphi = -1$, so, let us assume, for a contradiction, that

$$1 + r_{i-1}(n+k) - r_{i-1}(n) = \left\lfloor \frac{r_i(n+k) - r_i(n)}{2} \right\rfloor.$$

Again, by using Lemma 3, the expression inside the right-hand side floor function has increased at most a half, by moving the divisor 2 outside the inner floor functions. Since the expression was an integer before this operation, the possible increase of a half will be canceled by the outer floor function. Thus the left-hand side is too large.

We have proved that, if $\boldsymbol{x}$ is not of the form in the second part of the theorem, then $\boldsymbol{x}$ is a move.

For the other direction, suppose that

$$x_d \equiv 0 \pmod{2^d - 1}$$

and, for all $i \in \{2, \ldots, d\}$,

$$x_{i-1} \in \left\{ \left\lfloor \frac{x_i}{2} \right\rfloor, \left\lceil \frac{x_i}{2} \right\rceil \right\}.$$

We have to demonstrate that there exist n and k such that $\boldsymbol{x} = \boldsymbol{r}(n+k) - \boldsymbol{r}(n)$. We use the shortcut-matrix defined in Section 11. Since each ternary vector defines uniquely each row, and starting with x_d, the existence is clear. □

We have the following main result on playability, implied by Lemma 9.

Theorem 10 (Playability). *For a fixed $d \geqslant 2$, the compact game (from Section 6) is the grandiose game $\mathcal{M}$. That is, the compact game rules suffice to play the grandiose game. Hence, for both games there is a constant time (in the heap sizes) and linear time (in the number of heaps) procedure to decide whether a given d-tuple is a move.*

Proof. This follows from Lemma 9, since the compact rules have ternary recurrence. □

Therefore both games have the rat sequences as its set of P-positions. Let us give another play example, here with $d = 4$.

Example 2. Let $\boldsymbol{x} = (4, 7, 15, 29)$. Then $x_4 \not\equiv 0 \pmod{2^4 - 1}$. So $\boldsymbol{x}$ is a move. Let $\boldsymbol{x} = (4, 7, 15, 30)$. Then $x_4 \equiv 0 \pmod{2^4 - 1}$. In addition $30/2 = 15$, $\lfloor 15/2 \rfloor = 7$ and $\lceil 7/2 \rceil = 4$, so $\boldsymbol{x} \in \mathcal{R} - \mathcal{R}$ is a shortcut.

How do you move from $(3,6,12,23)+(4,7,15,30)=(7,13,27,53)$? The first vector is a P-position, but the second is a shortcut. Is there any other attainable P-position? We must find a move of the form $(7-\left\lfloor\frac{15n}{8}\right\rfloor, 14-\left\lfloor\frac{15n}{4}\right\rfloor, 30-\left\lfloor\frac{15n}{2}\right\rfloor, 60-15n)$. The forth coordinate is correct, so we proceed by dividing by 2 and applying the floor function, and thus verify the third coordinate (for $n=1,2,3$). For the second coordinate: is there an n such that

$$14-\left\lfloor\frac{15n}{4}\right\rfloor \in \left\{\left\lfloor\frac{30-\left\lfloor\frac{15n}{2}\right\rfloor}{2}\right\rfloor, \left\lfloor\frac{30-\left\lfloor\frac{15n}{2}\right\rfloor}{2}\right\rfloor+1\right\}?$$

It turns out that $n=1$ also gives a shortcut, but for $n=3$, the move $(2,3,8,15)$ takes you to the P-position $(5,10,19,38)$.

8. The Rats' Binary Wheel

This section expands on the matrix representation examples in Section 3. We have a general observation on the floor function.

Lemma 11. *For any $x,y\in\mathbb{N}$, $\left\lfloor\frac{x}{y}\right\rfloor-\left\lfloor\frac{x-1}{y}\right\rfloor=1$ if $x\equiv 0 \pmod y$, and otherwise $\left\lfloor\frac{x}{y}\right\rfloor-\left\lfloor\frac{x-1}{y}\right\rfloor=0$.*

Proof. Write $x=\alpha y+\beta$, with $0\leqslant\beta<y$. This gives $\left\lfloor\frac{x}{y}\right\rfloor-\left\lfloor\frac{x-1}{y}\right\rfloor=\left\lfloor\frac{\alpha y+\beta}{y}\right\rfloor-\left\lfloor\frac{\alpha y+\beta-1}{y}\right\rfloor=\left\lfloor\frac{\beta}{y}\right\rfloor-\left\lfloor\frac{\beta-1}{y}\right\rfloor=1$ if and only if $\beta=0$. □

Notation 1. For each column $j\in\{1,\ldots,d\}$, the *gap* between the rows $n\geqslant 2$ and $n-1$ is $\Delta_j(n):=r_j(n)-r_j(n-1)$.

Lemma 12 (Rat wheel). *For all n,j, $\Delta_j(n)=2^j$, unless $n\equiv 0 \pmod{2^{d-j}}$, in which case $\Delta_j(n)=2^j-1$.*

Proof. For all n,j, $\Delta_j(n)=\left\lfloor\frac{(2^d-1)n}{2^{d-j}}\right\rfloor-\left\lfloor\frac{(2^d-1)(n-1)}{2^{d-j}}\right\rfloor=2^j+\left\lfloor\frac{n-1}{2^{d-j}}\right\rfloor-\left\lfloor\frac{n}{2^{d-j}}\right\rfloor$. The result follows by Lemma 11. □

We have the following identity between consecutive rows in the infinite rat matrix. Note how it relates the binary counting system with the rat vectors, to be exploited next in Section 9.

Lemma 13 (A rat-gap identity). *For any $d\geqslant 2$, and any $n\geqslant 2$,*

$$\sum_{j\in\{2,\ldots,d\}} 2^{d-j+1}\Delta_{j-1}(n)-2^{d-j}\Delta_j(n)=1. \qquad (6)$$

Proof. For each row $n \geqslant 2$, there is a smallest indexed column γ, such that $n \equiv 0 \pmod{2}^{d-j}$, and so, by Lemma 12, $\Delta_\gamma(n) = 2^\gamma - 1$ (note, for all rows, $\Delta_d(n) = 2^d - 1$). It follows that $\Delta_\rho(n) = 2^\rho - 1$, for all $\rho \geqslant \gamma$. By Lemma 12, in the expression (6), the powers of 2 will get cancelled, so we are only concerned with the part $-2^{d-\gamma}(-1) + 2^{d-\gamma}(-1) - 2^{d-\gamma-1}(-1) + 2^{d-\gamma-1}(-1) - 2^{d-\gamma-2}(-1) + \ldots + 2^{d-d+1}(-1) - 2^{d-d}(-1) = 1$. □

9. The Anatomy of Rats: Matrix Representations

The standard form is not too convenient to work with, mainly because of the floor function. We extend the examples in Section 3 by reviewing the idea of the matrix representation for $d = 4$.

Example 3. The case $d = 4$ was dubbed 'fat rat' [24]. Recall the standard form of the P-positions without $\mathbf{0}$, $n \geqslant 1$,

$$\boldsymbol{r}(n) = \left(\left\lfloor \frac{15}{8} n \right\rfloor, \left\lfloor \frac{15}{4} n \right\rfloor - 1, \left\lfloor \frac{15}{2} n \right\rfloor - 3, 15n - 7 \right). \tag{7}$$

Let us list the first 11 expansions of the standard form, and using $t = \lfloor (n-1)/8 \rfloor \geqslant 0$, with $n \geqslant 1$,

n	$r_1(n)$	$r_2(n)$	$r_3(n)$	$r_4(n)$
1	$15t+1$	$30t+2$	$60t+4$	$120t+8$
2	$15t+3$	$30t+6$	$60t+12$	$120t+23$
3	$15t+5$	$30t+10$	$60t+19$	$120t+38$
4	$15t+7$	$30t+14$	$60t+27$	$120t+53$
5	$15t+9$	$30t+17$	$60t+34$	$120t+68$
6	$15t+11$	$30t+21$	$60t+42$	$120t+83$
7	$15t+13$	$30t+25$	$60t+49$	$120t+98$
8	$15t+15$	$30t+29$	$60t+57$	$120t+113$
9	$15t+1$	$30t+2$	$60t+4$	$120t+8$
10	$15t+3$	$30t+6$	$60t+12$	$120t+23$
11	$15t+5$	$30t+10$	$60t+19$	$120t+38$.

Notice the periodicity after the first 8 rows, modulus $15j$, $j = 1, 2, 3, 4$ in the respective columns.[9]

By the column-wise arithmetic periodic behavior of the rat sequences, for any given number of heaps, it is convenient to represent them in matrix notation. When we code them modulo $2^d - 1$, and given the first column, there is a simple bijection

[9]The reader is encouraged to check that the values of $\boldsymbol{r}(n)$, as n ranges from 1 to 11, are identical to the 11 rows of the matrix. For example, for $n = 6$, the value of (7) is $(11, 21, 42, 83)$, the same as the line $n = 6$, $t = \lfloor 6/8 \rfloor = 0$ of $\mathcal{R}_4$. For $n = 9$, (7 yields $(16, 32, 64, 128)$, same as row 9 of $\mathcal{R}_4$ with $t = \lfloor 9/8 \rfloor = 1$.

with the binary numeration system, which is proved in Theorem 14, in this section. This can be seen for any dimension d by studying the saltus and period of the system, as observed in Section 2. Here, we give a proof of independent interest, using Lemma 13.

Definition 9 (Binary matrices). We denote the entry in the ith row and the jth column of the rat-matrix by $R_{i,j}(t)$, where $t \in \mathbb{N}$ is a variable (see Example 3). Denote the ith row, $i \in \{0, \ldots, 2^{d-1} - 1\}$, of the *rat-matrix* by $R_i(t)$, $n \in \mathbb{N}_0$. Let $R_{i,1}(t) = (2^d - 1)t + 2i + 1$, and for $j \in \{2, \ldots, d\}$, $R_{i,j}(t) = 2R_{i,j-1}(t) - b_{i,d-j}$, where $b_i = b_{i,d-2} \cdots b_{i,0}$ is the number i represented in binary.

Note that, by using the binary representation, it is natural to index the rows of the rat-matrix by $i \in \{0, \ldots, 2^{d-1} - 1\}$ (but in the standard form, we follow the tradition, and start the indexing of rows with $n = 1, 2, \ldots$).

The following result gives the translation between rats' standard form and the (binary) matrix representation. Moreover it will be used in Section 10 to show that the rats' mex-rule is equivalent with the rat-vectors.

Theorem 14 (Rats are binary). *Let $d \geqslant 2$. For all $n \in \mathbb{N}$,*

$$\boldsymbol{r}(n) = R_{n-1 \pmod{2^{d-1}}}\left(\left\lfloor \frac{n-1}{2^{d-1}} \right\rfloor\right).$$

Proof. For each $n \in \mathbb{N}$, we must verify that, with $t = \left\lfloor \frac{n-1}{2^{d-1}} \right\rfloor$, row $n-1$ in the rat-matrix corresponds with row n in the standard form: recall, for all columns j,

$$r_j(n) = \left\lfloor \frac{(2^d - 1)n}{2^{d-j}} \right\rfloor - 2^{j-1} + 1.$$

We study the gaps of the entries in the columns of the respective forms. First we show that they correspond within a rat-matrix, and then we demonstrate that the glueing of matrices gives back the infinite form. Recall that $\Delta_j(n) = r_j(n) - r_j(n-1)$.

Let us begin by showing that row 0 in the rat-matrix corresponds to the first row in the standard form. Since $R_{0,1}(t) = (2^d - 1)t + 1$ and, by $b_0 = \mathbf{0}$, then, for all $j \in \{1, \ldots, d\}$, $R_{0,j}(0) = 2^{j-1}$. Also

$$\begin{aligned}
r_j(1) &= \left\lfloor \frac{2^d - 1}{2^{d-j}} \right\rfloor - 2^{j-1} + 1 && (8)\\
&= \left\lfloor \frac{2^d}{2^{d-j}} \right\rfloor + \left\lfloor \frac{-1}{2^{d-j}} \right\rfloor - 2^{j-1} + 1 && (9)\\
&= \frac{2^d}{2^{d-j}} - 1 - 2^{j-1} + 1 && (10)\\
&= 2^{j-1}, && (11)
\end{aligned}$$

for all j.

We want to show that for any row n, $\boldsymbol{r}(n) = R_i(\lfloor \frac{n-1}{2^{d-1}} \rfloor)$, with $n-1 = \alpha 2^{d-1} + i$, for some non-negative integer α, and where

$$0 \leqslant i < 2^{d-1}. \tag{12}$$

We begin by showing that the first entries correspond, and note that the last equality in both simplifications follow by (12):

$$\begin{aligned}
r_1(n) &= \left\lfloor \frac{(2^d-1)n}{2^{d-1}} \right\rfloor \\
&= \left\lfloor \frac{(2^d-1)(\alpha 2^{d-1}+i+1)}{2^{d-1}} \right\rfloor \\
&= \alpha(2^d-1) + \left\lfloor \frac{(2^d-1)(i+1)}{2^{d-1}} \right\rfloor \\
&= \alpha(2^d-1) + 2i + 2 + \left\lfloor \frac{-i-1}{2^{d-1}} \right\rfloor \\
&= \alpha(2^d-1) + 2i + 2 - 1;
\end{aligned}$$

$$\begin{aligned}
R_{i,1}\left(\left\lfloor \frac{t-1}{2^{d-1}} \right\rfloor\right) &= (2^d-1)\left\lfloor \frac{t-1}{2^{d-1}} \right\rfloor + 2i + 1 \\
&= (2^d-1)\left\lfloor \frac{\alpha 2^{d-1}+i}{2^{d-1}} \right\rfloor + 2i + 1 \\
&= (2^d-1)\left(\alpha + \left\lfloor \frac{i}{2^{d-1}} \right\rfloor\right) + 2i + 1 \\
&= \alpha(2^d-1) + \left\lfloor \frac{i}{2^{d-1}} \right\rfloor (2^d-1) + 2i + 1 \\
&= \alpha(2^d-1) + 2i + 1.
\end{aligned}$$

Next, for all $j \in \{2, \ldots, d\}$, we show that

$$b_{i,d-j} = 2R_{i,j-1}(t) - R_{i,j}(t) = 2r_{j-1}(n+1) - r_j(n+1).$$

Of course

$$b_i - b_{i-1} = \sum 2^{d-j} b_{i,d-j} - \sum 2^{d-j} b_{i-1,d-j} = 1.$$

Hence, it suffices to show that

$$\sum 2^{d-j}(2r_{j-1}(n+1) - r_j(n+1) - (2r_{j-1}(n) - r_j(n))) = 1,$$

that is, that

$$\sum 2^{d-j+1}(r_{j-1}(n+1) - r_{j-1}(n)) + 2^{d-j}(r_j(n) - r_j(n+1)) = 1,$$

that is, that

$$\sum 2^{d-j+1}\Delta_{j-1}(n+1) - 2^{d-j}\Delta_j(n+1) = 1.$$

This follows by Lemma 13. □

10. A Mex-binary Rule Interpretation

Let $B_{d-1} = (b_{i,j})$ denote the matrix of numbers in binary where the rows i represent the numbers from 0 to $2^{d-1} - 1$. For example,

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We index this matrix in a natural way with respect to the binary representations of the numbers, from right to left, starting with 0 and ending with $d-1$. Thus, for example, $b_3 = 011$, $b_{3,0} = b_{3,1} = 1$ and $b_{3,2} = 0$. In contrast, the columns of the rat matrices are indexed from 1 to d, left to right.

Let $d \geqslant 3$ be a fixed integer. For $n \geqslant 1$, $i \in \{2, \dots, d\}$, let $\Xi_{i,i-1}(n) := r_i(n) - 2r_{i-1}(n)$.

Lemma 15. *(i) Let s be a positive odd integer, x a real number, $0 < x \leqslant 1/2$. Then $2\lfloor (s/2) - x\rfloor = s - 1$.*

(ii) For all $n \in \mathbb{N}, \Xi_{i,i-1}(n) = \begin{cases} 0, & \text{if } \exists k \in \mathbb{N}_0 : k2^{d-i+1} < n \leqslant k2^{d-i+1} + 2^{d-i}, \\ -1, & \text{otherwise.} \end{cases}$

Proof. (i) Write $s = 2v - 1$, where v is a positive integer. Then $2\lfloor (s/2) - x\rfloor = 2\lfloor (2v-1)/2 - x\rfloor = 2v + 2\lfloor (-1/2) - x\rfloor = 2v - 2 = s - 1$.

(ii) For the offset differences, $(-2^{i-1} + 1) - 2(-2^{i-2} + 1) = -1$ for all n.

Let $k \geqslant 0$ be fixed. Let $n = k2^{d-i+1} + t, \quad 0 < t \leqslant 2^{d-i}$. Then

$$\begin{aligned}
\Xi_{i,i-1}(n) &= \left\lfloor \frac{(2^d-1)(k2^{d-i+1}+t)}{2^{d-i}} \right\rfloor - 2\left\lfloor \frac{(2^d-1)(k2^{d-i+1}+t)}{2^{d-i+1}} \right\rfloor - 1 \\
&= \left\lfloor \frac{(2^d-1)t}{2^{d-i}} \right\rfloor - 2\left\lfloor \frac{(2^d-1)t}{2^{d-i+1}} \right\rfloor - 1 \\
&= \left\lfloor \frac{-t}{2^{d-i}} \right\rfloor - 2\left\lfloor \frac{-t}{2^{d-i+1}} \right\rfloor - 1 = -1+2-1 = 0.
\end{aligned}$$

Now let $n = k2^{d-i+1} + 2^{d-i} + t, \quad 0 < t \leqslant 2^{d-i}$. Then

$$\begin{aligned}
\Xi_{i,i-1}(n) &= \left\lfloor \frac{(2^d-1)(k2^{d-i+1}+2^{d-i}+t)}{2^{d-i}} \right\rfloor - 2\left\lfloor \frac{(2^d-1)(k2^{d-i+1}+2^{d-i}+t)}{2^{d-i+1}} \right\rfloor - 1 \\
&= 2^d - 1 + 2^i t - 2\left\lfloor \frac{(2^d-1)2^{d-i}+(2^d-1)t)}{2^{d-i+1}} \right\rfloor - 2 \\
&= 2^d - 1 - 2\left\lfloor \frac{(2^d-1)}{2} - \frac{t}{2^{d-i+1}} \right\rfloor - 2 \\
&= (2^d-1) - (2^d-2) - 2 \quad \text{(by part (i))} \\
&= -1.
\end{aligned}$$

□

Definition 10. For $X \subset \mathbb{N}$, let $\text{mex}X = \min(\mathbb{N} \setminus X)$, the smallest positive integer not in X.

In particular, mex of the empty set is 1. We represent the rat vectors via the mex-rule as follows.

Definition 11 (Rats' mex rule). For $n \geqslant 1$, let

$$\begin{aligned}
u_{n,1} &= \text{mex}\{u_{m,1}, \ldots, u_{m,d} : 1 \leqslant m < n\}, \\
u_{n,i} &= 2u_{n,i-1} - b_{n-1,d-i}, \quad 2 \leqslant i \leqslant d.
\end{aligned}$$

For $n \geqslant 1$, $i \in \{2, \ldots, d\}$, let $\delta_{i,i-1}(n) := u_{n,i} - 2u_{n,i-1}$.

Theorem 16. *For all $n \in \mathbb{N}$, and all $i \in \{1, \ldots, d\}$,* $\quad u_{n,i} = r_i(n)$.

Proof. For every fixed $d \geqslant 2$, we use induction on n. Notice that $u_{1,1} = r_1(1) = 1$. Moreover, it follows by the definitions that $\delta_{i,i-1}(1) = \Xi_{i,i-1}(1) = 0$, for all $i \in \{2, \ldots, d\}$. This implies that, for all i, $u_{1,i} = r_i(1)$. Assume that for some $n > 1$, $u_{m,i} = r_i(m)$ for all $1 \leqslant m < n$. Let a be the smallest positive integer not in $r_i(m)$ for any $i \in \{1, \ldots, d\}$, $m < n$. It must appear somewhere, since the rat sequences are complementary. Therefore $r_1(n) = a$. Indeed, every $r_i(n)$ is an increasing sequence, so a could not appear in $r_1(m)$ for any $m > n$. Moreover,

$r_i(n) < r_{i+1}(n)$ for all i, n, so a cannot appear in $r_i(m)$ for any $i > 1$, $m > n$. Now a is also the smallest positive integer not in $u_{m,i}$ for any $i \in \{1, \ldots, d\}$, $m < n$ by the induction hypothesis. Then by the mex property, $u_{n,1} = a$. The binary representation in Theorem 14, or equivalently Lemma 15 (ii), implies $u_{n,i} = r_i(n)$ for all i, completing the induction. □

11. The Rats' Ternary Shortcuts

Similar to R, which, for each d, is a finite matrix, we define the (proper) *shortcut-matrix* F (F for forbidden subtractions), which will also be finite, with d columns. We will prove that it contains 3^{d-1} rows, using the natural ternary representations, obtained as a consequence of the binary representation of the rat-matrix.[10]

A d-dimensional vector $\boldsymbol{t}$ is *ternary* if, for all $0 \leqslant j \leqslant d-1$, $t_j \in \{0, 1, 2\}$.

Definition 12. Index the vectors in the set $\{\boldsymbol{r} - \boldsymbol{r}' \notin \mathcal{R} \mid \boldsymbol{r}, \boldsymbol{r}' \in R\}$ in increasing right-to-left lexicographic order[11], and let F denote the *shortcut-matrix* where $\boldsymbol{f}_i$ is the ith row from the top, and starting with row 0.

The following example is relevant to the proof of Theorem 17.

Example 4. Let $d = 3$. See Section 3. The vectors of F^3 gives the $3^2 = 9$ ternary pairs starting with top row: $11, 22, 02, 10, 21, 01, 12, 20, 00$. (Column-wise, divide by two, subtract with previous column and finally shift result by 1.) This representation gives rise to pictures such as Figure 1, explained in the appendix of a previous version of this paper [27].)

Theorem 17. *The shortcut-matrix F contains exactly 3^{d-1} distinct rows.*

Proof. We show that each $(d-1)$-dimensional ternary vector t, describes precisely one row in the matrix (see also Example 4), and then the result follows. For all $i, k \in \{1, \ldots, 2^{d-1}\}$ and $j \in \{1, \ldots, d\}$, we have that

$$R_{i+1,j} - R_{k+1,j} = 2(r_j(i) - R_{k,j}) + b_{i,j} - b_{k,j}, \tag{13}$$

where $t_j := b_{i,j} - b_{k,j} + 1 \in \{0, 1, 2\}$. Hence, for each pair of rows i, k, we define the ternary vector $\boldsymbol{t} = b_i - b_k + 1$. Not that, given any ternary vector $\boldsymbol{t}$, it is easy to find two binary vectors such that their difference $+1$ is $\boldsymbol{t}$. Now, for each $\boldsymbol{t}$, there is an equivalence class of pairs of binary vectors, and it is given by 2^u, where u is the

[10] Thus the proof gives a bit more information than the statement, and we use our understanding of the structure to find the natural order of the rows in F. In this section, we abuse notation and say 'shortcut-matrix' instead of the somewhat lengthy 'proper shortcut-matrix'.

[11] Row f_i is before row f_j if column k is the rightmost column where they differ, and then $f_{i,k} < f_{j,k}$

number of 1s in $\boldsymbol{t}$. Suppose now that we produce the same ternary vector $\boldsymbol{t}$ in two different ways, say by finding rows such that

$$\boldsymbol{t} = b_i - b_j = b_k - b_\ell. \tag{14}$$

We must show that the two ways to obtain $\boldsymbol{t}$ results in the same row in F, and to this purpose it suffices to show that the last entries $R_{i+1,d} - R_{j+1,d}$ and $R_{k+1,d} - R_{\ell+1,d}$, in the two representations will be the same. Observe that $i - j > 0$ if and only if $k - \ell > 0$. This gives that the respective differences in the first colums will be the same. Then, because of (14), then by (13), we get the claim for the last column. Thus, the definition of $\boldsymbol{t}$ gives a unique row vector in the shortcut-matrix, and so the number of rows is correct. □

We note that the construction in Lemma 9 suggests a similar definition of the shortcut-matrix. We will not use the following definition and observation in this paper, but we include it for completeness.

Definition 13. Let $d \geqslant 2$. For each $i \in \{1, \ldots, 2^{d-1}\}$, $j \in \{1, \ldots, d\}$, we construct a tree-structure of depth d, where the root has label $(j, x) = (d, i(2^d - 1))$. If x is even, then the node (j, x) has one child, labeled $(j - 1, x/2)$, and otherwise it has two children labeled $(j - 1, (x - 1)/2)$ (to the left) and $(j - 1, (x + 1)/2)$ (to the right). Let T^d denote the family of all such trees, and let $T^d(n)$ denote the same family, but where each label (j, x) has been replaced with $2^{j-1}(2^d - 1)n + x$.

Observation 2. Each path in $T^d(n)$, from a leaf to the root, represents a unique row in the shortcut-matrix. We obtain the lexicographic order of the rows in the shortcut-matrix by reading the paths left to right and starting with $i = 1$, etc.

The last column of the shortcut-matrix appears to satisfy a regular behavior, for increasing dimensions d.

Conjecture 18. Consider the shortcut-matrix for a given d. The number of entries of $k(2^d - 1)$, for $k \in \mathbb{N}$, is represented by a sequence of vectors $(\sigma^d)_{d \geqslant 2}$ of lengths 2^{d-2}:

$$(2), (3, 2), (4, 3, 5, 2), (5, 4, 7, 3, 8, 5, 7, 2),$$
$$(6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2), \ldots$$

The entries of σ^d are defined recursively by $\sigma_1^2 = 2$ and, for $d > 2$,

$$\sigma_1^d = \sigma_1^{d-1} + 1.$$

For all $1 \leqslant j \leqslant 2^d$,

$$\sigma_{2j}^{d+1} = \sigma_j^d,$$

and for all $1 \leqslant j < 2^d$,

$$\sigma_{2j+1}^{d+1} = \sigma_j^d + \sigma_{j+1}^d.$$

In an online extension of this manuscript [27, Appendix], using figures, data, and conjectures, we show that the structures of the shortcut matrices, for increasing d, satisfy interesting Cantor-like line fractals, with apparent complex behavior; exemplified here in Figure 1. Disregarding this apparent complexity, we have here proved that the compact games are the same as the grandiose games.

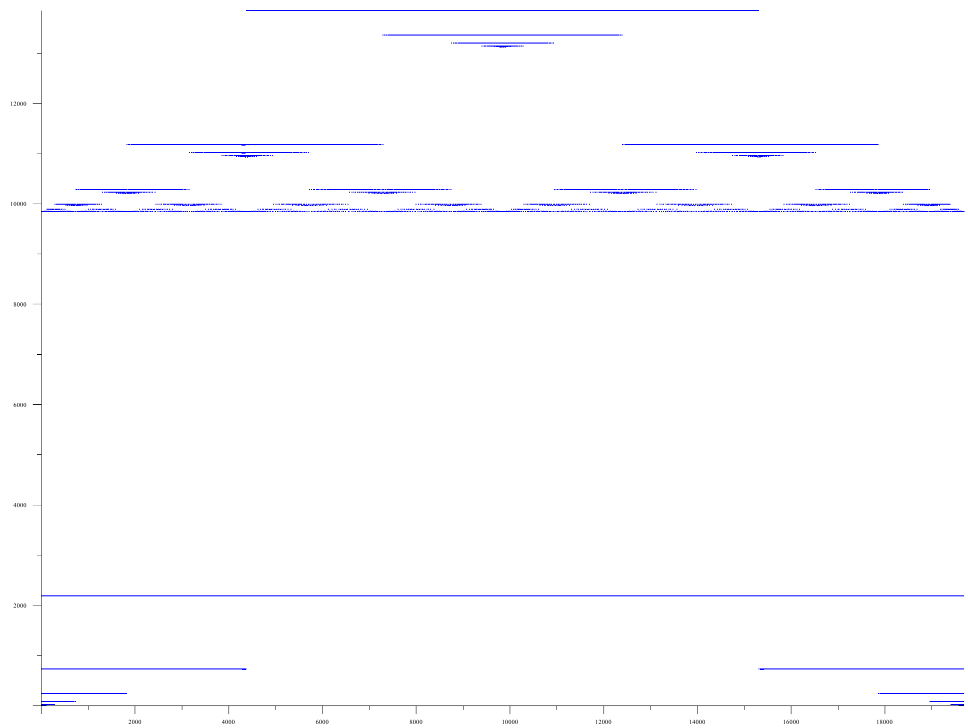

Figure 1: Line fractals code the shortcut-matrix for $d = 10$ heaps. See [27, Appendix] for an explanation of the patterns.

12. Rat Games are Approximately Nim

The game of nim is probably the most famous impartial combinatorial game. It has the property that every normal play impartial game G is equivalent to a heap of nim; the size of the nim heap is its nim value (a.k.a Sprague-Grundy value). We say that an impartial heap game Γ (on d heaps) is *almost nim* if the total number of objects in the heaps is almost always its nim value. Precisely, let $\gamma(n)$ denote the number of game positions with a total number of n objects, for which the nim-value of Γ is not n. Then Γ is *almost-nim-heap* if

$$\lim_{n\to\infty}\frac{\gamma(n)}{n}=0.$$

Moreover, if the nim value 0 is the only nim value which differs from the total number of objects in the d heaps, then we call Γ an *approximate-nim-heap.*

In Table 2, we give as example the Sprague-Grundy values of the rat game on 2 heaps. By the following result, the Sprague-Grundy functions of the rat-games are completely understood.

Theorem 19. *The grandiose rat games are almost-nim-heaps. In fact, they are approximate-nim-heaps. Their Sprague-Grundy values are given by the total number of tokens in the heaps, except for the rat vectors, which have Sprague-Grundy value 0.*

Proof. By construction, the rat vectors have nim value 0. Study an arbitrary position $\boldsymbol{x}$, and assume that all smaller positions have the prescribed Sprague-Grundy values. Now, if $\boldsymbol{x}$ is a rat vector, then previous results give the value 0, so consider $\boldsymbol{x}$ not a rat vector. If $\boldsymbol{x}$ is not a forbidden move, then there is a move to the terminal 0. If $\boldsymbol{x}$ is a forbidden move vector, then there is a move to the smallest non-zero rat vector, because $x_d - r_d \equiv 2^{d-1} - 1 \pmod{2^3 - 1}$, so $\boldsymbol{x} - \boldsymbol{r}$ is not a shortcut.

For all the other Sprague-Grundy values, observe that if $\boldsymbol{y} < \boldsymbol{x}$ has a desired total heap size but the S-G value is not available because $\boldsymbol{y}$ is a rat vector, then by shifting the move vector in two coordinates by subtracting 1 from one and adding 1 to the other, say to exchange $\boldsymbol{y}$ for $\boldsymbol{z}$, $\boldsymbol{z}$ cannot be a rat vector. If $\boldsymbol{x} - \boldsymbol{z}$ is a shortcut, then increase one coordinate and decrease another to violate the ternary recurrence. □

0	1	2	3	4	5	6
1	2	3	4	5	6	7
2	0	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	0	9	10	11
6	7	8	9	10	11	12
7	8	9	10	11	12	13
8	9	10	11	0	13	14

Table 2: The Sprague-Grundy values for the rat game on 2 heaps. The North-West corner is the terminal position $(0, 0)$.

That is, if you play a grandiose rat game in disjunctive sum with another game, then you can play approximately as if the game were nim, just keep in mind the exception that the rat vectors have value zero.[12] This result obviously also holds for the compact rat games, since we proved that the moves are the same.

Example 5. Let $\mathcal{G} = (1,4,5)_{\text{NIM}}$ be a game of NIM, and let $\mathcal{H} = (1,4,5)_{\text{RAT}}$ be a RAT-game. In the game $\mathcal{G}+\mathcal{H}$, a winning move is to $\mathcal{G}+(1,2,4)_{\text{RAT}}$, since the nim-value of each component game is 0, and since $\mathcal{H}-(1,2,4)_{\text{RAT}} = (0,2,1)_{\text{RAT}} \notin \mathcal{R}$.

Let $\mathcal{G} = (1,2,5,8)_{\text{NIM}}$ be a game of NIM, and let $\mathcal{H} = (3,4,5,6)_{\text{RAT}}$ be a RAT-game. In the game $\mathcal{G}+\mathcal{H}$, a winning move is to $\mathcal{G}+(3,0,5,6)_{\text{RAT}}$, since the nim-value of each component game is 14.

Let $\mathcal{G} = (1,2,5,8)_{\text{NIM}}$ be a game of NIM, and let $\mathcal{H} = (11,21,42,83)_{\text{RAT}}$ be a RAT-game. In the game $\mathcal{G}+\mathcal{H}$, a winning move is to $(1,2,5,6)_{\text{NIM}} + (11,21,42,83)_{\text{RAT}}$, since the nim-value of each component game is 0. Indeed, $83 \equiv 2^3 \pmod{2^4-1}$, and the recursive word is binary, namely $b_5 = 101$.

13. Rat History

Let us include some material to show the historical and mathematical value of the rat-vectors. This history provides some motivation for this paper.

So-called Beatty sequences [5, 6] are normally associated with irrational *moduli* α, β. Recent studies deal with rational moduli α, β. Clearly if $a/b \neq g/h$ are rational, then the sequences $\{\lfloor na/b \rfloor\}$ and $\{\lfloor ng/h \rfloor\}$ cannot be complementary, since $kbg \times a/b = kha \times g/h = kag$ for all $k \geqslant 1$. Also the former sequence is missing the integers $ka-1$ and the latter $kg-1$, so both are missing the integers $kag-1$ for all $k \geqslant 1$. However, complementarity can be maintained for the *nonhomogeneous* case: in [18, 46], necessary and sufficient conditions on α, γ, β, δ are given so that the sequences $\{\lfloor n\alpha+\gamma \rfloor\}$ and $\{\lfloor n\beta+\delta \rfloor\}$ are complementary—for both irrational moduli and rational moduli. We are not aware of any previous work in this direction, except that in Bang [3], necessary and sufficient conditions are given for $\{\lfloor n\alpha \rfloor\} \supseteq \{\lfloor n\beta \rfloor\}$ to hold, both for the case α, β irrational and the case α, β rational. Results of this sort also appear in Niven [45], for the homogeneous case only. In Skolem [52] and Skolem [53] the homogeneous and nonhomogeneous cases are studied, but only for α and β irrational. Fraenkel formulated the following conjecture for two or more sequences:[13]

Conjecture 20. If the vectors $(\lfloor n\alpha_i + \gamma_i \rfloor)_{i=1}^{d}$, $n \in \mathbb{N}$ split the positive integers

[12] Note that Singmaster's result [51], that almost no positions in an impartial game are P-positions, implies that any approximate nim-heap is also almost-heap.

[13] Erdős and Graham mention the conjecture in [16] (p. 19), as well as Graham et. al. in [34]. It is also a research problem in 'Concrete Mathematics' by Graham, Knuth, Patashnik [32] (ch. 3), and is mentioned by Tijdeman [54], [55]. In [19] a weaker conjecture, implied by the full conjecture, is formulated and proved for special cases: If $d \geqslant 3$, then there are always two distinct moduli with integral ratio. Simpson proved it when one of the moduli (and hence the only one) is $\leqslant 2$ [49].

with $d \geqslant 3$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_d$, then

$$\alpha_i = (2^d - 1)/2^{d-i},\ i = 1, \ldots, d. \tag{15}$$

Fraenkel [19] proved that this system of vectors splits (i.e. partitions) the positive integers with explicit values for $\gamma_i = -2^{i-1} + 1$ as in (1).

Theorem 21 (Fraenkel 1973). *For any dimension $d \geqslant 2$, the rat vectors $\boldsymbol{r}(n)$ (as in Section 2 partition the positive integers.*

It is well-known that if all the α_i in Conjecture 20 are integers with $d \geqslant 2$ and $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_d$, then $\alpha_{d-1} = \alpha_d$. A generating function proof using a primitive root of unity was given by Mirsky, Newman, Davenport and Rado; see Erdős [15]. A first elementary proof was given independently in [7] and by Simpson [48]. Graham [31] showed that if one of the d moduli is irrational then all are irrational, and if $d \geqslant 3$, then two moduli are equal. Thus distinct integer moduli or distinct irrational moduli cannot exist for $d \geqslant 2$ or $d \geqslant 3$ respectively in a splitting system. For irrational α with $1/\alpha + 1/\beta = 1$, $\{\lfloor n\alpha \rfloor\}, \{\lfloor n\beta \rfloor\}$ is a splitting system. Now the multiplier n can be split into two or more splitting systems—for example $\{\lfloor n\alpha \rfloor\} = \{\lfloor 2n\alpha \rfloor\} \cup \{\lfloor (2n+1)\alpha \rfloor\}$. Graham [31] proved that the splitting of the multipliers n is the *only* way an irrational splitting system can exist, and hence $d \geqslant 3$. Notice that usually, if at all, the rationals stick with the integers and the irrationals are "orphans". Here the irrationals stick with the integers, and the rationals seem to be stubborn "orphans" according to the Fraenkel conjecture.

The conjecture was proved for $d = 3$ by Morikawa [43], $d = 4$ by Altman et al. [2], for all $3 \leqslant d \leqslant 6$ by Tijdeman [55] and for $d = 7$ by Barát and Varjú [4] and was generalized by Graham and O'Bryant [33]. Other partial results were given by Morikawa [44] and Simpson [50]. Many others have contributed partial results – see Tijdeman [54] for a detailed history. The conjecture has some applications in job scheduling and related industrial engineering areas, in particular: 'Just-In-Time' systems, see, e.g., Altman et al. [2], Brauner and Jost [10], Brauner and Crama [9]. However, the conjecture itself has not been settled. So this is a problem that has been solved for the integers, has been solved for the irrationals, and is wide open for the rationals!

At last, we mention some recent rat history related to games. The conjecture, with accompanying Theorem 21, induced the "rat game" and its associates the "mouse game" [24] (rat – rational), played on 3 and 2 piles of tokens respectively, whose P-positions are the cases $d = 2, 3$ of definition (1) respectively, together with $\mathbf{0}$. Those games were defined on unordered tuples of integers. However, the rules are lengthy and for the case $d = 3$, already quite complicated (we omit a presentation here); arguably they are only intended for players with a degree in mathematics or computer science, and they cannot be described as *vector subtraction games*—a natural notion, which includes many classical games, introduced by Golomb [29] (extended in, e.g., [41]). Apart from the ending condition, the moves of a vector subtraction game are independent of the sizes of the heaps; and this property was generalized to the notion of "invariant moves" by Duchêne and Rigo in [13]. In response to the heap size dependency of the mouse game, a vector subtraction game on two heaps, dubbed *the mouse trap* [36] was developed, using the so-called $\star$-operator [40]. Indeed, the inaccessibility of those rules, and the difficulty of a generalization, further motivates our approach. By the complexity of unordered d-tuples (the number of permutations is $d!$), when d increases, we have devoted this work to a compact game solution in the case of ordered d-tuples (vectors).

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