A NEW CLASS OF THE r-STIRLING NUMBERS AND THE GENERALIZED BERNOULLI POLYNOMIALS

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Abstract

The main object of this paper is to express the values at non-negative integers of the generalized Bernoulli polynomials by using a class of the Stirling numbers of the second kind.

1. Introduction

Recall that the r-Stirling number of the second kind, $\left\{ {n \atop k} \right\}_r$, counts the number of partitions of the set $[n] := \{1, 2, \ldots, n\}$ into k non-empty subsets such that the elements of the set $[r]$ are in different subsets [3]. These numbers are determined by its generating function to be:

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!} = \frac{1}{k!} \left(\exp(t) - 1 \right)^k \exp(rt),$$

where $\left\{ {n \atop k} \right\}_1 = \left\{ {n \atop k} \right\}_0 := \left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind.

In [11], the authors expressed $B_n^{(\alpha)}(\pm r)$ in terms of the r-Stirling numbers of both kinds. In [12], they expressed $B_n^{(\alpha)}\left(\pm \frac{r}{m}\right)$ in terms of the r-Whitney numbers of both kinds, where $B_n^{(\alpha)}(x)$ is the $n-$th order Bernoulli polynomial (see for example [8, 15]) defined by its exponential generating function to be

$$\sum_{n \geq 0} B_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{t}{\exp(t) - 1} \right)^{\alpha} \exp(xt),$$

where $B_n^{(1)}(x) = B_n(x)$ are the classical Bernoulli polynomials.

The generalized Bernoulli polynomials $B_n^{[s-1,\alpha]}(x)$ extend the polynomials introduced by Natalini and Bernardini [13] (see also [6, 2]), and are defined by Kurt [7] (see also [16]) as follows:

$$\sum_{n\geq 0} B_n^{[s-1,\alpha]}(x)\frac{t^n}{n!} = \left(\frac{\frac{t^s}{s!}}{\exp(t) - \sum_{j=0}^{s-1}\frac{t^j}{j!}}\right)^{\alpha} \exp(xt), \quad s \geq 1. \tag{1}$$

In order to give explicit formulas for these polynomials at non-negative integers, we introduce in this paper a class of the r-Stirling numbers of the second kind which can be viewed as a special case of those given in [10].

Definition 1. For $s \geq 1$, we define the *s-quasi-associated r-Stirling numbers of the second kind*, denoted by $\left\{ {n \atop k} \right\}_r^s$, by the number of partitions of an n-set into k blocks such that the first r elements are in different blocks, a block from the other $(k-r)$-blocks must be of cardinality greater than or equal to s.

Below, we show that the numbers $B_n^{[s-1,\alpha]}(r)$ are linked to these numbers (Theorems 2, 3) by

$$B_n^{[s-1,\alpha]}(r) = \sum_{j=0}^{n} \binom{n+sj}{n, s, \ldots, s}^{-1} \left\{ {n+sj+r \atop j+r} \right\}_r^{s+1} (-\alpha)_j,$$

$$B_n^{[s-1,\alpha]}(r) = \alpha\binom{\alpha+n}{n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{j!}{\alpha+j}\binom{n+sj}{n, s, \ldots, s}^{-1}\left\{ {n+sj+r \atop j+r} \right\}_r^{s},$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ for $n \geq 1$, $(x)_0 = 1$,

$$\binom{n+sj}{n, s, \ldots, s} := \frac{(n+sj)!}{n!\,(s!)^j}.$$

Before proving these identities, let us give some combinatorial properties of the s-quasi-associated r-Stirling numbers of the second kind defined above.

2. Combinatorial Properties

From the above definition, we may state that

$$\left\{ {n \atop k} \right\}_r^s = 0, \quad n < sk \ \text{ or } k < r,$$

$$\left\{ {0 \atop k} \right\}_r^s = \delta_{k,0}, \quad k \geq 0,$$

$$\left\{ {n \atop r} \right\}_r^s = r^{n-r}, \quad n \geq sr.$$

Using combinatorial arguments, we assert that these numbers admit an expression given by the following theorem.

Theorem 1. *For $n \geq sk \geq sr \geq 1$, the s-quasi-associated r-Stirling numbers of the second kind can be written as*

$$\left\{ {n \atop k} \right\}_r^s = \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k=n-r-s(k-r)} \frac{1}{n_1! \cdots n_r! \, (n_{r+1}+s)! \cdots (n_k+s)!}.$$

Proof. To partition a set $[n]$ into k blocks $B_1, \ldots, B_k$ such that every block does not contain any element of $[r]$, must be of cardinality greater than or equal to s. The first r elements are in different blocks (of cardinalities not less than 1), let the elements of $[r]$ be in different blocks $B_1, \ldots, B_r$. So, there are $\frac{1}{(k-r)!}\binom{n-r}{n_1,\ldots,n_k} := \frac{(n-r)!}{(k-r)!}\frac{1}{n_1!\cdots n_k!}$ ways to choose $n_1, \ldots, n_k$ in $[n] \setminus [r]$ such that

- $n_1 \geq 0, \ldots, n_r \geq 0 : n_1, \ldots, n_r$ are, respectively, in $B_1, \ldots, B_r$,

- $n_{r+1} \geq s, \ldots, n_k \geq s : n_{r+1}, \ldots, n_k$ are, respectively, in $B_{r+1}, \ldots, B_k$.

Then the total number of these partitions is:

$$\begin{aligned} \left\{ {n \atop k} \right\}_r^s &= \frac{1}{(k-r)!} \sum_{n_1+\cdots+n_k=n-r,\ n_{r+1}\geq s,\ldots,n_k\geq s} \binom{n-r}{n_1,\ldots,n_k} \\ &= \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k=n-r-s(k-r)} \frac{1}{n_1! \cdots n_r! \, (n_{r+1}+s)! \cdots (n_k+s)!}. \end{aligned}$$

□

By a simple manipulation of Theorem 1, we may state the following.

Corollary 1. *The s-quasi-associated r-Stirling numbers of the second kind have generating function*

$$\sum_{n\geq k} \left\{ {n+r \atop k+r} \right\}_r^s \frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{i\geq s}\frac{t^i}{i!}\right)^k \exp(rt).$$

Using combinatorial arguments, we give below three recurrence relations.

Proposition 1. *For $n > sr \geq 1$ we have*

$$\left\{ {n \atop k} \right\}_r^s = k\left\{ {n-1 \atop k} \right\}_r^s + \binom{n-r-1}{s-1}\left\{ {n-s \atop k-1} \right\}_r^s.$$

Proof. To partition the set $[n]$ into k blocks such that every block from the $[n]\setminus[r]$ must be of cardinality greater than or equal to s and the elements of $[r]$ must be in different blocks, we separate the element n and proceed as follows:
(a) If n is in a block intersecting $[r]$, there are $\left\{ {n-1 \atop k} \right\}_r^s$ ways to partition the set $[n-1]$ into k blocks with the same conditions. The element n (not really used) can be inserted in the r blocks which intersect $[r]$, so we count $r\left\{ {n-1 \atop k} \right\}_r^s$ ways.
(b) If n is in a block of cardinality s and does not intersect $[r]$, there are $\binom{n-r-1}{s-1}$ ways to choose $s-1$ elements to be with this element in the same block. The remaining $n-s$ elements can be partitioned into $k-1$ blocks in $\left\{ {n-s \atop k-1} \right\}_r^s$ ways. So, the number of ways in this case must be $\binom{n-r-1}{s-1}\left\{ {n-s \atop k-1} \right\}_r^s$.
(c) If n is in a block of cardinality $\geq s+1$ and does not intersect $[r]$, there are $(k-r)\left\{ {n-1 \atop k} \right\}_r^s$ ways. Thus, the number of all partitions is given by $\left\{ {n \atop k} \right\}_r^s = r\left\{ {n-1 \atop k} \right\}_r^s + \binom{n-r-1}{s-1}\left\{ {n-s \atop k-1} \right\}_r^s + (k-r)\left\{ {n-1 \atop k} \right\}_r^s$. □

Proposition 2. *For $r \geq 1$ we have*

$$\left\{ {n \atop k} \right\}_r^s = \sum_{j\geq 0} \binom{n-r}{j} \left\{ {n-1-j \atop k-1} \right\}_{r-1}^s .$$

Proof. To partition the set $[n]$ into k blocks such that the first r-elements are in diffrent blocks and every block does not intersect $[r]$ must be of cardinality greater than or equal to s, we operate as follows:
The element r can be in a block of cardinality $j+1$ in $\binom{n-r}{j}\left\{ {n-1-j \atop k-1} \right\}_{r-1}^s$ ways, so we illustrate two cases,
(a) The number of ways for choosing the j elements between $(n-1)-(r-1)$ elements of $[n]\setminus[r]$ to be in the same block with the element r is $\binom{n-r}{j}$,
(b) The number of ways to partition the remaining $n-(j+1) = n-1-j$ elements into $k-1$ blocks such that every block does not intersect $[r]$, must be of cardinality greater than or equal to s, and the elements of $[r-1]$ are in different blocks is $\left\{ {n-1-j \atop k-1} \right\}_{r-1}^s$. □

3. Application to the Generalized Bernoulli Polynomials

We give in this section two expressions in terms of the s-quasi-associated r-Stirling numbers of the second kind for $B_n^{[s-1,\alpha]}(r)$. The following theorem gives a simplified expression for $B_n^{[s-1,\alpha]}(r)$ for all non-negative integers r.

Theorem 2. *We have*

$$B_n^{[s-1,\alpha]}(r) = \sum_{j=0}^{n} \binom{n+sj}{n, s, \ldots, s}^{-1} \left\{ {n+sj+r \atop j+r} \right\}_r^{s+1} (-\alpha)_j$$

for a non-positive integer α*, where* $\alpha = -k$*. We also have*

$$B_n^{[s-1,-k]}(r) = k!\binom{n+sk}{n,s,\ldots,s}^{-1}\left\{ {n+sk+r \atop k+r} \right\}_r^s.$$

Proof. From the definition of $B_n^{[s-1,\alpha]}(x)$, we get

$$\sum_{n\geq 0} B_n^{[s-1,\alpha]}(r)\frac{t^n}{n!} = \left(\sum_{j=0}^{\infty}\binom{j+s}{s}^{-1}\frac{t^j}{j!}\right)^{-\alpha}\exp(rt) = \exp(rt)\sum_{n\geq 0} f_n(-\alpha)\frac{t^n}{n!},$$

which gives $B_n^{[s-1,\alpha]}(r) = \sum\limits_{k=0}^{n}\binom{n}{k} r^{n-k} f_k(-\alpha)$, where $(f_n(x))$ is a sequence of binomial type with $f_n(1) = \binom{n+s}{s}^{-1}$. Use the known relation (see [14])

$$f_n(-\alpha) = \sum_{j=0}^{n} B_{n,j}\left(\binom{i+s}{s}^{-1}\right)(-\alpha)_j$$

to obtain

$$B_n^{[s-1,\alpha]}(r) = \sum_{j=0}^{n}(-\alpha)_j\sum_{k=j}^{n}\binom{n}{k} r^{n-k} B_{k,j}\left(\binom{i+s}{s}^{-1}\right),$$

where $B_{n,k}(x_i) := B_{n,k}(x_1, x_2, \ldots)$ is the partial Bell polynomial, see [1, 4, 9].

Now, the exponential generating function of

$$A(n,j) := \sum_{k=j}^{n}\binom{n}{k} r^{n-k} B_{k,j}\left(\binom{i+s}{s}^{-1}\right)$$

must be

$$\begin{aligned}
\sum_{n\geq j} A(n,j)\frac{t^n}{n!} &= \exp(rt)\sum_{k\geq j} B_{k,j}\left(\binom{i+s}{s}^{-1}\right)\frac{t^k}{k!} \\
&= \frac{1}{j!}\left(\sum_{i\geq 1}\frac{s!t^i}{(i+s)!}\right)^j \exp(rt) \\
&= \frac{t^{-sj}}{j!}\left(s!\sum_{i\geq s+1}\frac{t^i}{i!}\right)^j \exp(rt) \\
&= \sum_{n\geq j}\frac{(s!)^j\, n!}{(n+sj)!}\left\{ {n+sj+r \atop j+r} \right\}_r^{s+1}\frac{t^n}{n!}.
\end{aligned}$$

So, we obtain

$$A(n,j)=\frac{n!\,(s!)^j}{(n+sj)!}\left\{ {n+sj+r \atop j+r} \right\}_r^{s+1} \quad \text{and} \quad B_n^{[s-1,\alpha]}(r)=\sum_{j=0}^{n} A(n,j)\,(-\alpha)_j\,.$$

The second part of the theorem follows from the expansion

$$\sum_{n\geq 0} B_n^{[s-1,-k]}(r)\,\frac{t^n}{n!}=t^{-sk}\,(s!)^k\left(\sum_{i\geq s}\frac{t^i}{i!}\right)^k \exp(rt)=k!t^{-sk}\sum_{n\geq sk}\left\{ {n+r \atop k+r} \right\}_r^s \frac{t^n}{n!}.$$

□

The next corollary is a particular case from Theorem 2.

Corollary 2. *We have*

$$B_n^{(\alpha)}(r):=B_n^{[0,\alpha]}(r)=\sum_{j=0}^{n}\frac{n!}{(n+j)!}\left\{ {n+j+r \atop j+r} \right\}_r^2 (-\alpha)_j\,.$$

So, the values of the Bernoulli polynomials at non-negative integers are given by

$$B_n(r):=B_n^{(1)}(r)=\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}^{-1}\left\{ {n+j+r \atop j+r} \right\}_r^2,$$

and from the known identity $B_n^{(n+1)}(x)=(x-1)^{\underline{n}}$, we may state that for $\alpha=n+1$ we have

$$\sum_{j=0}^{n}(-1)^j\left\{ {n+j+r \atop j+r} \right\}_r^2=(r-1)^{\underline{n}}\,.$$

Other expressions in terms of the s-quasi-associated r-Stirling numbers of the second kind for $B_n^{[s-1,\alpha]}(r)$ are given as follows.

Theorem 3. *Let $p,\ r,\ n,\ s$ be non-negative integers such that $s\geq 1$, $p\geq n$. Then we have*

$$B_n^{[s-1,\alpha]}(r)=\alpha\binom{\alpha+p}{p}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\frac{j!}{\alpha+j}\binom{n+sj}{n,s,\ldots,s}^{-1}\left\{ {n+sj+r \atop j+r} \right\}_r^s,$$

where

$$\binom{x}{k}:=\frac{(x)_k}{k!}.$$

Proof. For any polynomial f of degree $n\leq p$, Melzak's formula [5] gives

$$f(x+\alpha)=\alpha\binom{\alpha+p}{p}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\frac{f(x-j)}{\alpha+j},$$

where $\binom{x}{n}=\frac{x^{\underline{n}}}{n!}$. By Theorem 2, we deduce that $B_n^{[s-1,\alpha]}(x)$ is a polynomial in α of degree at most n. Then by setting $f(x)=B_n^{[s-1,x]}(r)$ in Melzak's Formula we obtain

$$B_n^{[s-1,\alpha]}(r)=\alpha\binom{\alpha+p}{p}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\frac{B_n^{[s-1,-j]}(r)}{\alpha+j}.$$

The desired identity follows by using the second identity of Theorem 2, $\square$

Where $p=n$ in Theorem 3 we get the following corollary,

Corollary 3. *We have*

$$B_n^{[s-1,\alpha]}(r)=\alpha\binom{\alpha+n}{n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{j!}{\alpha+j}\binom{n+sj}{n,s,\ldots,s}^{-1}\left\{ {n+sj+r \atop j+r} \right\}_r^s.$$

This gives the values of the high order Bernoulli polynomials at non-negative integers to be

$$B_n^{(\alpha)}(r):=B_n^{[0,\alpha]}(r)=\alpha\binom{\alpha+n}{n}\sum_{j=0}^{n}\frac{(-1)^j}{\alpha+j}\frac{\binom{n}{j}}{\binom{n+j}{j}}\left\{ {n+j+r \atop j+r} \right\}_r.$$

So, the values of the Bernoulli polynomials at non-negative integers are given by

$$B_n(r)=\sum_{j=0}^{n}(-1)^j\frac{\binom{n+1}{j+1}}{\binom{n+j}{j}}\left\{ {n+j+r \atop j+r} \right\}_r.$$

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