# LINEAR COMBINATIONS OF TWO POLYGONAL NUMBERS THAT TAKE INFINITELY OFTEN A SQUARE VALUE

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**Abstract**

Using basic properties of the Pell equation and the theory of congruences, we investigate the question of when a linear combination of two different types of polygonal numbers is infinitely often a perfect square. Let $P_k(x)$ denote the $x$-th $k$-gonal numbers. We give sufficient conditions about $m, n$ such that the Diophantine equation

$$mP_p(x) + nP_q(y) = z^2$$

has infinitely many positive integer solutions $(x, y, z)$, where $p \geq 3,\ q \geq 3$.

## 1. Introduction

A polygonal number [3] is a positive number, corresponding to an arrangement of points on the plane, which forms a regular polygon. The $x$-th $k$-gonal number [3, p. 5] is

$$P_k(x) = \frac{x\big((k-2)(x-1)+2\big)}{2},$$

where $x \geq 1, k \geq 3$. There are many papers about the polygonal numbers and many properties of them have been studied, we can refer to the first chapter of [4] and D3 of [7].

In 2005, Bencze [1] raised a problem: determine all positive integers $n$ for which $1 + \frac{9}{2}n(n+1)$ is a perfect square. In 2007, Le [11] showed that all positive integers $n$ which make the form $1 + \frac{9}{2}n(n+1)$ to be a perfect square were given by

$$n = \frac{1}{2}\left(\frac{1}{6}\left(a^{2k+1} + b^{2k+1}\right) - 1\right),$$

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where $a = 3 + \sqrt{8}, b = 3 - \sqrt{8}$, and $k \in \mathbb{Z}^+$. In 2011, Guan [6] proved that all positive integers $n$ which make the form $1 + \frac{4n(n+1)s^2}{s^2-1}$ to be a perfect square were given by

$$n = \frac{1}{2}\left(\frac{1}{2s}\left(a^{2k+1} + b^{2k+1}\right) - 1\right),$$

where $a = s + \sqrt{s^2 - 1}, b = s - \sqrt{s^2 - 1}$, and $s$ is a positive odd integer with $s > 1$, $k \in \mathbb{Z}^+$. In 2013, Hu [8] used the theory of the Pell equation to study the positive integer solutions of the Diophantine equation

$$1 + nP_3(y - 1) = z^2,$$

where

$$n = \begin{cases} \dfrac{t^2 \pm 1}{2}, & t \equiv 1 \pmod 2, \quad t \geq 3, \\ \dfrac{t^2 \pm 2}{2}, & t \equiv 0 \pmod 2, \quad t \geq 2, \\ \dfrac{t(t-1)}{2}, & \qquad\qquad\qquad\quad t \geq 2. \end{cases}$$

In 2019, Peng [12] showed that if $2n$ is not a perfect square, then the Diophantine equation $1 + nP_3(y - 1) = z^2$ has infinitely many positive integer solutions. Meanwhile, she studied the Diophantine equation

$$mP_3(x - 1) + nP_3(y - 1) = z^2,$$

where $m, n \in \mathbb{Z}^+$, and proved that when $\frac{m(m+1)}{2} = u^2, n = 1$, there exist infinitely many pairs $(a, b)$ of integers such that $mP_3(x - 1) + nP_3(y - 1) = z^2$ has integer parametric solutions $(t, at + b, u(ct + d))$, where $t$ is a positive integer greater than 1. Moreover, she got two general results:

1) If $2(m + n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pellian equation

$$X^2 - 2(m + n)Z^2 = \left(\frac{m + n}{2}\right)^2 - r^2 mn$$

has a positive integer solution $(X_0, Z_0)$ satisfying

$$X_0 - rn + \frac{m + n}{2} \equiv 0 \pmod{m + n},$$

then the Diophantine equation $mP_3(x - 1) + nP_3(y - 1) = z^2$ has infinitely many positive integer solutions.

2) Let $u, v$ be integers with $u > \sqrt{2}v$, and $u$ a positive even integer. When $m = (u^2 - 2v^2)^2, n = 8u^2v^2$, then the Diophantine equation $mP_3(x-1) + nP_3(y-1) = z^2$ has infinitely many positive integer solutions.

In 2020, Jiang and Li [9] investigated the problem that the linear combination of two polygonal numbers is a perfect square, and they showed that if $k \geq 5$, $2(k-2)n$

is not a perfect square, and there is a positive integer solution $(Y', Z')$ of $Y^2 - 2(k-2)nZ^2 = (k-4)^2n^2 - 8(k-2)n$ satisfying

$$Y' + (k-4)n \equiv 0 \pmod{2(k-2)n}, \quad Z' \equiv 0 \pmod 2,$$

then the Diophantine equation $1 + nP_k(y) = z^2$ has infinitely many positive integer solutions $(y, z)$. Moreover, they studied the Diophantine equation

$$mP_k(x) + nP_k(y) = z^2,$$

where $m, n \in \mathbb{Z}^+$, and proved that when $\frac{ntr}{2}$ is a perfect square, where $t = r(k-2)-1$, there exist infinitely many pairs $(a, b)$ of positive integers such that $mP_k(x) + nP_k(y) = z^2$ has integer parametric solutions $(x, ax+b, u(cx+d))$, where $k \geq 5$. Further, they obtained two general results:

1) If $k \geq 5$, $2(k-2)(m+n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pellian equation

$$X^2 - 2(k-2)(m+n)Z^2 = (k-4)^2(m+n)^2 - 4(k-2)^2mnr^2$$

has a positive integer solution $(X_0, Z_0)$ satisfying

$$X_0 - 2(k-2)nr + (k-4)(m+n) \equiv 0 \pmod{2(k-2)(m+n)}, \quad Z_0 \equiv 0 \pmod 2,$$

then $mP_k(x) + nP_k(y) = z^2$ has infinitely many positive integer solutions.

2) Let $k \geq 5$, $m = 2(u^2-4u-4)^2$, $n = 2(u^2+4u-4)^2$. If $2(k-2)$ is not a perfect square, and the Pell equation $X^2 - 8(k-2)(u^2+4)^2Z^2 = 1$ has a positive integer solution $(U_0, V_0)$ satisfying $U_0 - 1 \equiv 0 \pmod{2(k-2)}$, then $mP_k(x) + nP_k(y) = z^2$ has infinitely many positive integer solutions.

In this paper, we continue the study of [9], and consider the positive integer solutions of the Diophantine equation

$$mP_p(x) + nP_q(y) = z^2, \tag{1}$$

where $m, n$ are positive integers and $p \geq 3$, $q \geq 3$. The main results are as follows.

**Theorem 1.** *Let $m = (q-4)^2(p-2)t$, $n = (p-4)^2(q-2)rt$, for the following two cases: 1) $p = 3$, $q \geq 5$, $r \in \mathbb{Z}^+$; 2) $p > 4$, $q > 4$, $r \equiv -1 \pmod{(p-2)}$. When $\frac{tr(r+1)}{2}$ is a perfect square, there exist infinitely many pairs $(a, b)$ of positive integers such that Equation (1) has integer parametric solutions $(au+b, (p-2)(q-4)cu, w(du+e))$, where $r, w$ are positive integers.*

**Theorem 2.** *If $p \geq 3$, $q \geq 3$ and $2((p-2)m + (q-2)n)$ is not a perfect square, $r \in \mathbb{Z}$, and the Pellian equation*

$$\begin{aligned} &X^2 - 2((p-2)m + (q-2)n)Z^2 \\ &= -4mn(p-2)(q-2)r^2 - 8mn(p-q)r + ((p-4)m + (q-4)n)^2 \end{aligned}$$

*has a positive integer solution* $(X_0, Z_0)$ *satisfying*

$$X_0 - 2(q-2)nr + ((p-4)m + (q-4)n) \equiv 0 \pmod{2((p-2)m + (q-2)n)},$$
$$Z_0 \equiv 0 \pmod 2,$$

*then Equation* (1) *has infinitely many positive integer solutions.*

**Theorem 3.** *For* $p \geq 3,\ q \geq 3$, *if* $m+n$ *is a perfect square, but* $2mn(p^2(q-2)m + q^2(p-2)n)$ *and* $2((p-2)m + (q-2)n)$ *are not perfect squares, then Equation* (1) *has infinitely many positive integer solutions.*

**Theorem 4.** *If* $p \geq 3,\ q \geq 3,\ p \neq 4,\ q \neq 4$ *and* $2(u^2(p-2)m + v^2(q-2)n)$ *is not a perfect square,* $u, v \in \mathbb{Z}$, *and the Pell equation*

$$U^2 - 2(u^2(p-2)m + v^2(q-2)n)V^2 = 1$$

*has a positive integer solution* $(U_0, V_0)$ *satisfying*

$$U_0 + 1 \equiv 0 \pmod{2(u^2(p-2)m + v^2(q-2)n)}, \quad V_0 \equiv 0 \pmod 2,$$

*then Equation* (1) *has infinitely many positive integer solutions.*

**Remark 1.** When $p = q$, this is the case studied by Jiang and Li [9].

**Remark 2.** When $p = 4,\ q = 4$, this corresponds to a linear combination of two square numbers. Cohen [2, Corollary 6.3.6.] studied the general case

$$Ax^2 + By^2 = Cz^2,$$

and gave the general solutions, i.e., "assume that $ABC \neq 0$, let $(x_0, y_0, z_0)$ be a particular nontrivial solution of $Ax^2 + By^2 = Cz^2$, and assume that $z_0 \neq 0$. The general solution in rational numbers to the equation is given by

$$\begin{cases} x = d\big(x_0(As^2 - Bt^2) + 2y_0Bst\big), \\ y = d\big(2x_0Ast - y_0(As^2 - Bt^2)\big), \\ z = dz_0(As^2 + Bt^2), \end{cases}$$

where $d \in \mathbb{Q}, s, t \in \mathbb{Z}$, and $\gcd(s,t) = 1$."

## 2. Preliminaries

To prove the above results, we give the following lemmas.

**Lemma 1** ([10])**.** *Let* $D$ *be a positive integer which is not a perfect square. Then the Pell equation* $x^2 - Dy^2 = 1$ *has infinitely many positive integer solutions. If*

*$(U, V)$ is the least positive integer solution of the Pell equation $x^2 - Dy^2 = 1$, then all positive integer solutions are given by*

$$x_s + y_s\sqrt{D} = (U + V\sqrt{D})^s,$$

*where $s$ is a positive integer.*

**Lemma 2** ([10])**.** *Let $D$ be a positive integer which is not a perfect square, $N$ be a nonzero integer, and $(U, V)$ be the least positive integer solution of $x^2 - Dy^2 = 1$. If $(x_0, y_0)$ is a positive integer solution of $x^2 - Dy^2 = N$, then an infinitude of positive integer solutions are given by*

$$x_s + y_s\sqrt{D} = (x_0 + y_0\sqrt{D})(U + V\sqrt{D})^s,$$

*where $s$ is a nonnegative integer.*

**Lemma 3** ([5])**.** *Let $D$ be a positive integer which is not a perfect square, $m_1, m_2$ be positive integers, and $N$ be a nonzero integer. If the Pellian equation $x^2 - Dy^2 = N$ has a positive integer solution $(u_0, v_0)$ satisfying*

$$u_0 \equiv a \pmod{m_1}, \quad v_0 \equiv b \pmod{m_2},$$

*then it has infinitely many positive integer solutions $(u, v)$ satisfying*

$$u \equiv a \pmod{m_1}, \quad v \equiv b \pmod{m_2}.$$

## 3. Proofs of the Theorems

*Proof of Theorem 1.* If $h(x, y) = mP_p(x) + nP_q(y)$, Equation (1) becomes $h(x, y) = z^2$. When

$$m = (q-4)^2(p-2)t, \quad n = (p-4)^2(q-2)rt,$$

let

$$x = au + b, \quad y = (p-2)(q-4)cu.$$

Then

$$\begin{aligned} &h(au+b, (p-2)(q-4)cu) \\ &\quad = \frac{1}{2}(q-4)^2(p-2)^2t(a^2 + rc^2(q-2)^2(p-4)^2)u^2 \\ &\quad\quad + \frac{1}{2}(q-4)^2(p-2)t((2(p-2)b - (p-4))a - (p-4)^2(q-2)rc)u \\ &\quad\quad + \frac{1}{2}(q-4)^2(p-2)tb((p-2)b - (p-4)). \end{aligned}$$

Consider

$$g(u) = h(au+b, (p-2)(q-4)cu)$$

as a quadratic polynomial of $u$. If $g(u) = 0$ has multiple roots, then the discriminant of $g(u)$ is zero, i.e.,

$$\begin{aligned}&a^2 - 2cr(q-2)(2(p-2)b-(p-4))a\\&-rc^2(q-2)^2(4(p-2)^2b^2-4(p-2)(p-4)b-(p-4)^2r) = 0.\end{aligned}$$

It implies

$$a = c(q-2)(2(p-2)br-(p-4)r+2\sqrt{\Delta}), \tag{2}$$

where $\Delta = (p-2)r(r+1)b((p-2)b-(p-4))$.

To find $a \in \mathbb{Z}^+$, we take $\Delta = W^2$. Then

$$\left(\frac{2(p-2)b-(p-4)}{|p-4|}\right)^2 - r(r+1)\left(\frac{2W}{|p-4|r(r+1)}\right)^2 = 1.$$

Letting

$$X = \frac{2(p-2)b-(p-4)}{|p-4|}, \quad Y = \frac{2W}{|p-4|r(r+1)}, \tag{3}$$

we obtain the Pell equation

$$X^2 - r(r+1)Y^2 = 1. \tag{4}$$

It is easy to see that the pair $(X_0, Y_0) = (2r+1, 2)$ is the fundamental solution of Equation (4). So an infinitude of positive integer solutions of Equation (4) are given by

$$X_s + Y_s\sqrt{r(r+1)} = \left(2r+1+2\sqrt{r(r+1)}\right)^{s+1}, \ s \geq 0.$$

Thus,

$$\begin{cases} X_s = 2(2r+1)X_{s-1} - X_{s-2}, & X_0 = 2r+1, \ X_1 = 8r^2+8r+1,\\ Y_s = 2(2r+1)Y_{s-1} - Y_{s-2}, & Y_0 = 2, \ Y_1 = 4(2r+1).\end{cases}$$

According to the above recurrence relations, we have

$$X_s - 1 \equiv 0 \pmod 2, \quad Y_s \equiv 0 \pmod 2.$$

From (2), we get $a_s = cr(q-2)(p-4)(X_s + (r+1)Y_s)$. Hence, $a_s$ is a positive integer.

1) When $p = 3, \ q \geq 5$, from (3), we obtain

$$b_s = \frac{X_s - 1}{2}, \quad W_s = \frac{r(r+1)Y_s}{2}.$$

It is obvious that $b_s$ and $W_s$ are integers.

2) When $p > 4,\ q > 4$, we have

$$b_s = \frac{(p-4)(X_s+1)}{2(p-2)}, \quad W_s = \frac{(p-4)r(r+1)Y_s}{2}.$$

In view of $b_s$ is an integer, we need $X_s + 1 \equiv 0 \pmod{2(p-2)}$. If $r \equiv -1 \pmod{(p-2)}$, then $X_0 + 1 \equiv 0 \pmod{2(p-2)}$. According to the above recurrence relation, it is easy to prove that

$$X_s \equiv \begin{cases} -1 & \pmod{2(p-2)}, \quad s \equiv 0 \pmod 2, \\ 1 & \pmod{2(p-2)}, \quad s \equiv 1 \pmod 2. \end{cases}$$

Hence, when $s \equiv 0 \pmod 2$, we have

$$X_s + 1 \equiv 0 \pmod{2(p-2)},$$

then $b_s$ is a positive integer.

Therefore, Equation (1) becomes

$$\frac{tr(r+1)}{2}(du+e)^2 = z^2.$$

If $\frac{tr(r+1)}{2}$ is a perfect square, there exist infinitely many pairs $(a, b)$ of positive integers such that Equation (1) has integer parametric solutions $(au+b, (p-2)(q-4)cu, w(du+e))$, where $c, w$ are positive integers. □

**Example 5.** When $p = 6, q = 5,\ r = 3,\ t = 6,\ \frac{tr(r+1)}{2} = 6^2$ is a perfect square, then $m = 24,\ n = 216$ and $a_0 = 270c,\ b_0 = 2$. Hence, Equation (1) has integer parametric solutions $(270cu+2, 4cu, 12(156cu+1))$, where $c, u$ are positive integers.

*Proof of Theorem 2.* Let $y = x + r, r \in \mathbb{Z}$, Equation (1) becomes

$$\begin{aligned}&(2((p-2)m+(q-2)n)x - ((p-4)m - (2(q-2)r-(q-4))n))^2\\&-2((p-2)m+(q-2)n)(2z)^2\\&= -4mn(p-2)(q-2)r^2 - 8mn(p-q)r + ((p-4)m+(q-4)n)^2.\end{aligned}$$

Take $X = 2((p-2)m+(q-2)n)x - ((p-4)m-(2(q-2)r-(q-4))n)$ and $Z = 2z$, we get

$$\begin{aligned}&X^2 - 2((p-2)m+(q-2)n)Z^2\\&= -4mn(p-2)(q-2)r^2 - 8mn(p-q)r + ((p-4)m+(q-4)n)^2.\end{aligned} \tag{5}$$

By Lemma 1, if $2((p-2)m + (q-2)n)$ is not a perfect square, the Pell equation

$$X^2 - 2((p-2)m+(q-2)n)Z^2 = 1$$

has infinitely many positive integer solutions. By Lemma 2, if Equation (5) has a positive integer solution, it has infinitely many positive integer solutions. Assume that Equation (5) has a positive integer solution $(X_0, Z_0)$ satisfying

$$\begin{aligned}&X_0 - 2(q-2)nr + ((p-4)m + (q-4)n) \equiv 0 \pmod{2((p-2)m + (q-2)n)},\\ &Z_0 \equiv 0 \pmod{2}.\end{aligned}$$

By Lemma 3, Equation (5) has infinitely many positive integer solutions $(X, Z)$ satisfying the above condition, which leads to infinitely many $x, z \in \mathbb{Z}^+$. Then there are infinitely many $y = x + r \in \mathbb{Z}^+$. Hence, Equation (1) has infinitely many positive integer solutions $(x, x + r, z)$. □

**Example 6.** When $p = 6,\ q = 5,\ r = 1,\ m = 2,\ n = 1$, Equation (5) becomes

$$X^2 - 22Z^2 = -87. \tag{6}$$

It has a positive integer solution $(X_0, Z_0) = (1651, 352)$ satisfying

$$X_0 - 1 \equiv 0 \pmod{22}, \quad Z_0 \equiv 0 \pmod{2}.$$

Note that $(u, v) = (197, 42)$ is the least positive integer solution of $X^2 - 22Z^2 = 1$. By Lemma 3, Equation (6) has infinitely many positive integer solutions $(X, Z)$ satisfying the above condition, which leads to infinitely many $x, z \in \mathbb{Z}^+$. Then there are infinitely many $y = x+1 \in \mathbb{Z}^+$. Hence, Equation (1) has infinitely many positive integer solutions $(x, x + 1, z)$.

*Proof of Theorem 3.* By Theorem 2, it is sufficient to find a positive integer solution $(X_0, Z_0)$ to Equation (5) satisfying

$$\begin{aligned}&X_0 - 2(q-2)nr + ((p-4)m + (q-4)n) \equiv 0 \pmod{2((p-2)m + (q-2)n)},\\ &Z_0 \equiv 0 \pmod{2}.\end{aligned}$$

Suppose that

$$\begin{aligned}X_0 &= \ pm + qn + 4mnt(p-q)((p-2)m + (q-2)n),\\ r &= -(pm + qn)t((p-2)m + (q-2)n),\end{aligned}$$

then $Z_0$ satisfies

$$Z_0^2 = 2mn(p^2(q-2)m + q^2(p-2)n)((p-2)m + (q-2)n)^2t^2 + 4(m+n). \tag{7}$$

If $m + n$ is a perfect square but $2mn(p^2(q-2)m + q^2(p-2)n)$ is not a perfect square, from Lemma 2, Equation (7) has infinitely many positive integer solutions $(Z_0, t)$. And suppose $(Z_0', t_0)$ is an arbitrary positive integer solution of Equation (7).

If $2((p-2)m+(q-2)n)$ is not a perfect square, by Lemma 1, the Pell equation $X^2-2((p-2)m+(q-2)n)Z^2=1$ has infinitely many positive integer solutions. Put $(U_0, V_0)$ be the least positive integer solution of $X^2-2((p-2)m+(q-2)n)Z^2=1$. And Equation (5) becomes

$$\begin{aligned}
&X^2-2((p-2)m+(q-2)n)Z^2\\
&=-4nm(q-2)(p-2)(mp+nq)^2((p-2)m+n(q-2))^2t_0^2\\
&+8nm(p-q)(mp+nq)((p-2)m+n(q-2))t_0+((p-4)m+n(q-4))^2,
\end{aligned}\tag{8}$$

which has a positive integer solution

$$(X_0, Z_0)=(pm+qn+4mn(p-q)((p-2)m+(q-2)n)t_0, Z_0')$$

satisfying

$$\begin{aligned}
X_0+2n(q-2)(mp+nq)((p-2)m+n(q-2))t_0&\\
+(p-4)m+n(q-4)\equiv&\ 0 \pmod{2((p-2)m+(q-2)n)},\\
Z_0\equiv&\ 0 \pmod{2}.
\end{aligned}$$

By Lemma 2, an infinitude of positive integer solutions of Equation (8) are given by

$$\begin{aligned}
X_s+Z_s\sqrt{2((p-2)m+(q-2)n)}=&\left(X_0+Z_0\sqrt{2((p-2)m+(q-2)n)}\right)\\
&\times\left(U_0+V_0\sqrt{2((p-2)m+(q-2)n)}\right)^s,\ s\geq 0.
\end{aligned}$$

From some calculations, we have

$$\begin{cases}X_{2s+2}= & 2(2U_0^2-1)X_{2s}-X_{2s-2},\\ Z_{2s+2}= & 2(2U_0^2-1)Z_{2s}-Z_{2s-2},\end{cases}$$

where

$$\begin{aligned}
X_0=&\ pm+qn+4mn(p-q)((p-2)m+(q-2)n)t_0,\\
X_2=&\ (2U_0^2-1)X_0+4((p-2)m+(q-2)n)U_0V_0Z_0,\\
Z_0=&\ Z_0',\\
Z_2=&\ (2U_0^2-1)Z_0+2U_0V_0X_0.
\end{aligned}$$

From $X=2((p-2)m+(q-2)n)x-((p-4)m-(2(q-2)r-(q-4))n)$ and $Z=2z$, we have

$$x=\frac{X+(p-4)m+n(q-4)}{2((p-2)m+(q-2)n)}+n(q-2)(mp+nq)t_0, \quad z=\frac{Z}{2}.$$

Then

$$\begin{cases} x_{2s+2} = 2(2U_0^2-1)x_{2s} - x_{2s-2} - 8n(q-2)(mp+nq)((p-2)m+n(q-2))V_0^2t_0 \\ \qquad\qquad + 4V_0^2((p-4)m+n(q-4)), \\ y_{2s+2} = x_{2s+2} - (mp+nq)t_0((p-2)m+n(q-2)), \\ z_{2s+2} = 2(2U_0^2-1)z_{2s} - z_{2s-2}, \end{cases}$$

where

$$\begin{aligned} x_0 =& 1 + nq((p-2)m+n(q-2))t_0, \quad x_2 = 2V_0^2X_0 + 2U_0V_0Z_0 + x_0, \\ y_0 =& 1 - mp((p-2)m+n(q-2))t_0, \quad y_2 = 2V_0^2X_0 + 2U_0V_0Z_0 + y_0, \\ z_0 =& \frac{Z_0}{2}, \quad z_2 = \frac{(2U_0^2-1)Z_0 + 2U_0V_0X_0}{2}. \end{aligned}$$

Therefore, if $m+n$ is a perfect square, but $2mn(p^2(q-2)m+q^2(p-2)n)$ and $2((p-2)m+(q-2)n)$ are not perfect squares, then Equation (1) has infinitely many positive integer solutions $(x_{2s+2}, y_{2s+2}, z_{2s+2})$, where $s \geq 0$. $\square$

**Example 7.** When $p=6,\ q=5,\ m=3,\ n=1\ (m+n=2^2)$, Equation (7) becomes

$$Z^2 = 572400t^2 + 16,$$

which has a positive integer solution $(Z_0, t_0) = (5296, 7)$. And Equation (5) becomes

$$X^2 - 30Z^2 = -839782391,$$

which has a positive integer solution

$$(X_0, Z_0) = (1283, 5296)$$

satisfying

$$X_0 + 14497 \equiv 0 \pmod{30}, \quad Z_0 \equiv 0 \pmod{2}.$$

Note that $(U_0, V_0) = (11, 2)$ is the least positive integer solution of $X^2 - 30Z^2 = 1$. Hence, we have

$$\begin{cases} x_{2s+2} = 482x_{2s} - x_{2s-2} - 231952, & x_0 = 526,\ x_2 = 243814, \\ y_{2s+2} = x_{2s+2} - 2415, & y_0 = -1889,\ y_2 = 241399, \\ z_{2s+2} = 482z_{2s} - z_{2s-2}, & z_0 = 2648,\ z_2 = 666394. \end{cases}$$

Thus, Equation (1) has infinitely many positive integer solutions $(x_{2s+2}, y_{2s+2}, z_{2s+2})$, where $s \geq 0$.

*Proof of Theorem 4.* When $x = ut,\ y = vt, u, v, t \in \mathbb{Z}^+$, Equation (1) becomes

$$\begin{aligned} &(2(u^2(p-2)m + v^2(q-2)n)t - (u(p-4)m + v(q-4)n))^2 \\ &- 2(u^2(p-2)m + v^2(q-2)n)(2z)^2 = (u(p-4)m + v(q-4)n)^2. \end{aligned}$$

Take

$$X = 2(u^2(p-2)m + v^2(q-2)n)t - (u(p-4)m + v(q-4)n), \quad Z = 2z, \tag{9}$$

we get

$$X^2 - 2(u^2(p-2)m + v^2(q-2)n)Z^2 = (u(p-4)m + v(q-4)n)^2. \tag{10}$$

By Lemma 1, if $2(u^2(p-2)m+v^2(q-2)n)$ is not a perfect square, the Pell equation

$$U^2 - 2(u^2(p-2)m + v^2(q-2)n)V^2 = 1 \tag{11}$$

has infinitely many positive integer solutions. Therefore, Equation (10) has infinitely many positive integer solutions. Suppose $(U, V)$ is a positive integer solution of Equation (11), then $(|u(p-4)m + v(q-4)n|U, |u(p-4)m + v(q-4)n|V)$ is a positive integer solution of Equation (10). From (9), we have

$$t = \frac{|u(p-4)m + v(q-4)n|(U+1)}{2(u^2(p-2)m + v^2(q-2)n)}, \quad z = \frac{|u(p-4)m + v(q-4)n|V}{2}. \tag{12}$$

If Equation (11) has a positive integer solution $(U_0, V_0)$ satisfying

$$U_0 + 1 \equiv 0 \pmod{2(u^2(p-2)m + v^2(q-2)n)}, \quad V_0 \equiv 0 \pmod{2},$$

then, by Lemma 3, Equation (11) has infinitely many positive integer solutions $(U, V)$ satisfying the above condition, which leads to infinitely many $t, z \in \mathbb{Z}^+$. Hence, Equation (1) has infinitely many positive integer solutions $(x, y, z)$ satisfying $x = ut,\ y = vt$, where $u, v \in \mathbb{Z}^+$. $\square$

**Example 8.** When $m = n = 1$, $p = 6,\ q = 5$, $u = 2,\ v = 1$, Equation (11) becomes

$$U^2 - 38V^2 = 1.$$

By the theory of Pell equation, an infinitude of positive integer solutions of the above Pell equation are given by

$$U_s + V_s\sqrt{38} = \left(37 + 6\sqrt{38}\right)^{s+1},\ s \geq 0.$$

Thus,

$$\begin{cases} U_s = 74U_{s-1} - U_{s-2}, \quad U_0 = 37,\ U_1 = 2737, \\ V_s = 74V_{s-1} - V_{s-2}, \quad V_0 = 6,\ V_1 = 444. \end{cases}$$

From (12), we have

$$t_s = \frac{5(U_s+1)}{38}, \quad z_s = \frac{5V_s}{2}.$$

When $s \equiv 0 \pmod 2$, it is easy to see that

$$U_s + 1 \equiv 0 \pmod{38}, \quad V_s \equiv 0 \pmod 2,$$

which leads to $t_s, z_s$ are all positive integers. Hence, Equation (1) has infinitely many positive integer solutions $(x_s, y_s, z_s)$ satisfying $x_s = 2t_s,\ y_s = t_s$, where $s \equiv 0 \pmod 2$.

**Remark 3.** When $m = M^2,\ n = 1$, for $p = 4,\ q \geq 3$, we can get some parametric solutions.

1) For $p = 4,\ q = 3$, Equation (1) becomes

$$(z - Mx)(z + Mx) = \frac{y(y+1)}{2}.$$

Hence, we can take

$$\begin{cases} z - Mx = \dfrac{y+1}{4Mr}, \\ z + Mx = 2Mry, \end{cases}$$

where $r$ is a positive integer.

It leads to

$$x = \frac{8M^2r^2y - y - 1}{8M^2r}, \quad z = \frac{8M^2r^2y + y + 1}{8Mr}.$$

Take $y = 8M^2rt - 1$, we get

$$x = 8M^2r^2t - r - t, \quad z = M(8M^2r^2t - r + t).$$

2) For $p = 4,\ q = 4$, Equation (1) becomes

$$M^2x^2 + y^2 = z^2.$$

By the Pythagorean theorem, we have

$$x = k(u^2 - v^2), \quad y = 2Mkuv, \quad z = Mk(u^2 + v^2),$$

where $u > v$.

3) For $p = 4,\ q \geq 3$, Equation (1) becomes

$$(z - Mx)(z + Mx) = \frac{y\big((q-2)(y-1) + 2\big)}{2}.$$

Then we can take

$$\begin{cases} z - Mx = \dfrac{y}{4Mr}, \\ z + Mx = 2Mr((q-2)(y-1) + 2), \end{cases}$$

where $r$ is a positive integer.

It leads to

$$x = \frac{(8M^2r^2(q-2)-1)y}{8M^2r} - (q-4)r, \quad z = \frac{(8M^2r^2(q-2)+1)y}{8Mr} - (q-4)Mr.$$

Take $y = 8M^2rt$, we have

$$x = 8(q-2)M^2r^2t - (q-4)r - t, \quad z = M(8(q-2)M^2r^2t - (q-4)r + t).$$

Hence, when $m = M^2,\ n = 1$, for $p = 4,\ q \geq 3$, Equation (1) has infinitely many positive integer solutions.

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