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A FAMILY OF PARTITIONS EQUINUMEROUS WITH THE SET OF NODES OF A FAMILY OF TREES

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Abstract

In this paper we present an elementary bijection between a set of two-line matrices and the set of all partitions of n with the smallest part being at least c and the minimum distance between parts at least λ. We then describe a procedure to find the cardinality of the set of those two-line matrices. A special case that deserves to be highlighted is that we have a closed formula for the number of unrestricted partitions.

1. Introduction

Let $n, c \in \mathbb{N}$, with $c < n-1$, and $\lambda \in \mathbb{N} \cup \{0\}$. Let us define $\mathbb{M}(n, c, \lambda)$ to be the set of all two-line matrices

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ b_1 & b_2 & \cdots & b_s \end{pmatrix}, \tag{1}$$

such that $a_j, b_j \in \mathbb{N} \cup \{0\}$ and

$$a_s = c, \quad a_j = a_{j+1} + b_{j+1} + \lambda \quad \text{and} \quad \sum_{i=1}^{s}(a_i + b_i) = n.$$

The condition $c < n-1$ follows from the fact that

$$|\mathbb{M}(n, n, \lambda)| \;=\; |\mathbb{M}(n, n-1, \lambda)| \;=\; 1 \quad \text{and} \quad \mathbb{M}(n, n+t, \lambda) = \emptyset, \;\; \forall t \in \mathbb{N}.$$

There is a very strict relationship between two-line matrices and partitions. In fact, given $M \in \mathbb{M}(n, c, \lambda)$, written as (1), if we define $a_j + b_j = \mu_j$ we would have the partition of n

$$n \;=\; \mu_1 + \cdots + \mu_s,$$

where the least part $\mu_s \geq c$ and $\mu_j - \mu_{j-1} \geq \lambda$. On the other hand, given a partition $n \;=\; \mu_1 + \cdots + \mu_s$, with $\mu_s \geq c$ and $\mu_{j-1} - \mu_j \geq \lambda$, we can write

$$\begin{array}{rcl} \mu_s &=& c + b_s, \\ \mu_{s-1} &=& \mu_s + \lambda + b_{s-1} \quad = \quad a_{s-1} + b_{s-1}, \\ \mu_{s-2} &=& \mu_{s-1} + \lambda + b_{s-2} \quad = \quad a_{s-1} + b_{s-1} + \lambda + b_{s-2} \quad = \quad a_{s-2} + b_{s-2}, \end{array}$$

and continuing this process we obtain a matrix $M \in \mathbb{M}(n, c, \lambda)$ (see (1)). This establishes a bijection between the set $\mathbb{M}(n, c, \lambda)$ and the set $\mathbb{P}(n, c, \lambda)$ of all partitions of n with the smallest part being at least c and the minimum distance between parts being at least λ.

The relation between partitions and two-line matrices is not new and dates back to Frobenius [7] and Andrews [1]. The interpretation used here was introduced by Mondek-Ribeiro-Santos [11]. We refer the reader to [2], [3], [8] and [10] for more applications of this theory.

Our goal in this paper is to describe a *tree-like* structure to count the matrices in $\mathbb{M}(n, c, \lambda)$, and consequently obtain a formula to calculate the cardinality of $\mathbb{P}(n, c, \lambda)$. In particular we present a closed combinatorial formula for $p(n)$, the partition function.

There is in fact an extensive literature on the partition function, starting with the groundbreaking work of Hardy and Ramanujan [9] who were the first to have determined the asymptotic behaviour of $p(n)$. Later Rademacher [12] perfected their methods (known today as the *Circle Method*) to derive the first formula for $p(n)$. In 2013, Bruinier and Ono [4] presented a formula for $p(n)$ as a finite sum of algebraic numbers, and Dewar and Murty [6] used these ideas to derive the Hardy-Ramanujan asymptotic formula bypassing the Circle Method. Recently, Schneider [13] presented a way of determining $p(n)$ via computing the number of partitions having no part equal to one. Although he does not present a closed fromula, estimates on the number of these special partitions of n are given. All these formulas are derived via complex function theory, and in this scenario combinatorial formulas for $p(n)$ are a welcomed addition to the theory.

In 2016, Choliy and Sills [5] also presented a closed combinatorial formula for $p(n)$, based upon the determination of the number of partitions of n with Durfee square of order $k \leq \sqrt{n}$. Although we have different starting points, our formulas bear some interesting similarities to each other.

2. Special Matrices

We start with some notation that will be helpful for the understanding of the procedure. Given any pair of two-line matrices A and B, let us define the juxtaposition operation $A \uplus B$ as

$$\begin{pmatrix} a_1 & \cdots & c_s \\ b_1 & \cdots & d_s \end{pmatrix} \uplus \begin{pmatrix} c_1 & \cdots & c_t \\ d_1 & \cdots & d_t \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_s & c_1 & \cdots & c_t \\ b_1 & \cdots & b_s & d_1 & \cdots & d_t \end{pmatrix}. \quad (2)$$

The next matrices are essential to our work. Let

$$A(t_1, t_2) = \begin{pmatrix} c + t_1\lambda & c + (t_1 - 1)\lambda & \cdots & c + \lambda & c \\ 1 + t_2 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$D(t_1, t_2, t_3) = \begin{pmatrix} c + (t_1 + 1)\lambda + 1 + t_2 \\ t_3 \end{pmatrix}.$$

Now define recursively, for $\ell \geq 4$,

$$D(t_1, \ldots, t_\ell) = \begin{pmatrix} c + (t_1 + 1)\lambda + 1 + (\ell - 3)\lambda + \sum_{i=2}^{\ell-1} t_i \\ t_\ell \end{pmatrix} \uplus D(t_1, \ldots, t_{\ell-1}).$$

Example 1. For $c = 3$ and $\lambda = 2$ we have

$$\begin{aligned}
A(t_1, t_2) &= \begin{pmatrix} 2t_1 + 3 & 2(t_1 - 1) + 3 & \cdots & 5 & 3 \\ 1 + t_2 & 0 & \cdots & 0 & 0 \end{pmatrix} \\
D(t_1, t_2, t_3) &= \begin{pmatrix} 2t_1 + t_2 + 6 \\ t_3 \end{pmatrix} \\
D(t_1, t_2, t_3, t_4) &= \begin{pmatrix} 2t_1 + t_2 + t_3 + 8 & 2t_1 + t_2 + 6 \\ t_4 & t_3 \end{pmatrix} \\
D(t_1, t_2, t_3, t_4, t_5) &= \begin{pmatrix} 2t_1 + \sum_{i=2}^{4} t_i + 10 & 2t_1 + t_2 + t_3 + 8 & 2t_1 + t_2 + 6 \\ t_5 & t_4 & t_3 \end{pmatrix}.
\end{aligned}$$

In particular,

$$D(2, 3, 3, 4, 4, 1, 1) \uplus A(2, 3) = \begin{pmatrix} 33 & 30 & 24 & 18 & 13 & 7 & 5 & 3 \\ 1 & 1 & 4 & 4 & 3 & 4 & 0 & 0 \end{pmatrix}.$$

Notation 2. For convenience sake, let us define

$$L(x) = c + (x + 1)\lambda + 1 \quad \text{and} \quad Q(x) = c(x + 1) + \lambda \frac{x(x + 1)}{2}. \quad (3)$$

Lemma 1. *Let A be any two-line matrix and define $\sigma(A)$ to be equal to the sum of all entries of A. Then (see* (3)*)*

$$\sigma(A(t_1, t_2)) = Q(t_1) + 1 + t_2.$$

and

$$\sigma(D(t_1, \dots, t_l)) = (l-2)L(t_1) + \lambda\frac{(l-2)(l-3)}{2} + (l-2)t_2 + \sum_{j=1}^{l-2} j \cdot t_{l-j+1}.$$

Proof. The first equality follows directly from the definition of $A(t_1, t_2)$. Note that

$$\sigma(D(t_1, t_2, \dots, t_l)) = (L(t_1) + (l-3)\lambda + (t_2 + \cdots + t_l) + \sigma(D(t_1, t_2, \dots, t_{l-1})),$$

and in particular

$$\begin{aligned} \sigma(D(t_1, t_2, t_3, t_4)) &= L(t_1) + \lambda + t_2 + t_3 + t_4 + \sigma(D(t_1, t_2, t_3)) \\ &= 2L(t_1) + \lambda + 2(t_2 + t_3) + t_4. \end{aligned}$$

Now the second equality follows from an inductive argument. □

3. The Blocks $B(t_1)$ and Their Generations

Let us call *blocks* the following matrices in $\mathbb{M}(n, c, \lambda)$:

$$\begin{gathered} B(0) = \begin{pmatrix} c \\ m(0) \end{pmatrix}, \quad B(1) = \begin{pmatrix} c+\lambda & c \\ m(1) & 0 \end{pmatrix}, \quad \cdots \\ \cdots, \quad B(r) = \begin{pmatrix} c + r\lambda & \cdots & c+\lambda & c \\ m(r) & \cdots & 0 & 0 \end{pmatrix}, \end{gathered} \tag{4}$$

where (see (3))

$$m(0) = n - c, \quad m(1) = n - (2c + \lambda), \quad \dots \quad , m(r) = n - Q(r). \tag{5}$$

It is easy to see that, $B(j) = A(j, m(j) - 1)$, but due to their importance in all that follows, we have decided to write them explicitly. Let us denote by $\mathbb{B}(n, c, \lambda)$ the set of all blocks in $\mathbb{M}(n, c, \lambda)$.

Observe that the block $B(r)$ exists if and only if $m(r) \geq 0$, that is, if we have $n \geq Q(r)$. On the other hand, since by definition $n > c + 1$ and $\lambda \geq 0$, we have that the quadratic equation (or linear if $\lambda = 0$)

$$Q(x) - n = \frac{1}{2}(\lambda x^2 + x(2c + \lambda) + 2(c - n)) = 0 \tag{6}$$

has two distinct non-zero roots, of opposite signs if $\lambda \neq 0$ and one root if $\lambda = 0$.

Lemma 2. *Let us denote by x_1 the positive root of* (6) *and $r_1 = \lfloor x_1 \rfloor$. Then r_1+1 is the number of blocks in $\mathbb{M}(n,c,\lambda)$, that is $|\mathbb{B}(n,c,\lambda)| = r_1+1$.*

Proof. It follows directly from (6) that $m(r_1) \geq 0$ and $m(r_1+1) < 0$. $\square$

Example 3. Let $n = 25$, $c = 3$ and $\lambda = 2$. It follows from (6) and Lemma 2 that $Q(x) - n = x^2 + 4x - 22$ and $r_1 = \lfloor x_1 \rfloor = 3$. Hence there are four blocks in $\mathbb{M}(25,3,2)$ and they are:

$$B(0) = \begin{pmatrix} 3 \\ 22 \end{pmatrix}, \qquad B(1) = \begin{pmatrix} 5 & 3 \\ 17 & 0 \end{pmatrix},$$

$$B(2) = \begin{pmatrix} 7 & 5 & 3 \\ 10 & 0 & 0 \end{pmatrix}, \quad B(3) = \begin{pmatrix} 9 & 7 & 5 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Given a block $B(t_1)$, we present here a list of two-line matrices that shall be called the *descendants of* $B(t_1)$. These descendants will be separated according to their kinship level. The definition is given recursively, and we will end this section preseting examples of generations of blocks.

Notation 4. For simplicity, let us adopt the following notation:

$$\begin{aligned} F_\ell &= F(t_1,\ldots,t_\ell), \quad F_\ell^* = F(t_1,\ldots,t_{\ell-1},0), \\ D_\ell &= D(t_1,\ldots,t_\ell), \quad D_\ell^* = D(t_1,\ldots,t_{\ell-1},0), \\ m_\ell &= m(t_1,\ldots,t_\ell), \quad m_\ell^* = m(t_1,\ldots,t_{\ell-1},0). \end{aligned} \tag{7}$$

For $t_1 \in \{0,1,\ldots,r_1\}$, we define the *second generation* of $B(t_1)$ as the set $\mathbb{S}_2(t_1)$ of all matrices $F(t_1,t_2)$, called the *first descendants*. These matrices are defined as follows:

$$\begin{aligned} F(t_1,0) &= \begin{pmatrix} L(t_1) & c+t_1\lambda & c+(t_1-1)\lambda & \cdots & c+\lambda & c \\ m(t_1,0) & 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} L(t_1) \\ m(t_1,0) \end{pmatrix} \uplus A(t_1,0), \end{aligned} \tag{8}$$

and for $t_2 = 1,\ldots,\lfloor \frac{m(t_1,0)}{2} \rfloor$

$$\begin{aligned} F(t_1,t_2) &= \begin{pmatrix} L(t_1)+t_2 & c+t_1\lambda & c+(t_1-1)\lambda & \cdots & c+\lambda & c \\ m(t_1,t_2) & 1+t_2 & 0 & \cdots & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} L(t_1)+t_2 \\ m(t_1,t_2) \end{pmatrix} \uplus A(t_1,t_2). \end{aligned} \tag{9}$$

Hence $|\mathbb{S}_2(t_1)| = s_2 = \lfloor \frac{m(t_1,0)}{2} \rfloor + 1$, and for $t_2 = 0, 1, \ldots, s_2$, we have

$$m(t_1, t_2) = m(t_1, 0) - 2t_2 \quad \text{and} \quad m(t_1, 0) = m(t_1) - L(t_1) - 1. \tag{10}$$

For $t_1 \in \{0, 1, \ldots, r_1\}$ and $t_2 \in \{0, 1, \ldots, s_2\}$, we define the *third generation of* $B(t_1)$ as the set $\mathbb{S}_3(t_1)$ of all matrices $F(t_1, t_2, t_3)$ (the *descendants* of the matrices $F(t_1, t_2)$). These matrices are defined as:

$$\begin{aligned} F(t_1, t_2, 0) &= \begin{pmatrix} L(t_1) + \lambda + t_2 & L(t_1) + t_2 & c + t_1\lambda & \cdots & c \\ m(t_1, t_2, 0) & 0 & 1 + t_2 & \cdots & 0 \end{pmatrix} \\ &= \begin{pmatrix} L(t_1) + \lambda + t_2 \\ m(t_1, t_2, 0) \end{pmatrix} \uplus D(t_1, t_2, 0) \uplus A(t_1, t_2), \end{aligned} \tag{11}$$

and for $t_3 = 1, \ldots, \lfloor \frac{m(t_1,t_2,0)}{2} \rfloor$ define

$$\begin{aligned} F(t_1, t_2, t_3) &= \begin{pmatrix} L(t_1) + \lambda + t_2 + t_3 & L(t_1) + t_2 & c + t_1\lambda & \cdots & c \\ m(t_1, t_2, t_3) & t_3 & 1 + t_2 & \cdots & 0 \end{pmatrix} \\ &= \begin{pmatrix} L(t_1) + \lambda + t_2 + t_3 \\ m(t_1, t_2, t_3) \end{pmatrix} \uplus D(t_1, t_2, t_3) \uplus A(t_1, t_2), \end{aligned} \tag{12}$$

where

$$m(t_1, t_2, t_3) = m(t_1, t_2, 0) - 2t_3 \quad \text{and} \quad m(t_1, t_2, 0) = m(t_1, t_2) - L(t_1) - \lambda - t_2. \tag{13}$$

For $\ell \geq 4$ we define recursively the *ℓ-th generation* of $B(t_1)$ as the set $\mathbb{S}_\ell(t_1)$ of all matrices $F(t_1, \ldots, t_\ell)$ (the descendants of $F(t_1, \ldots, t_{\ell-1})$). These matrices are defined as (see (7)):

$$F_\ell^* = \begin{pmatrix} L(t_1) + (\ell - 2)\lambda + \sum_{i=2}^{\ell-1} t_i \\ m_\ell^* \end{pmatrix} \uplus D_\ell^* \uplus A(t_1, t_2), \tag{14}$$

and for $t_\ell = 0, 1, \ldots, \lfloor \frac{m_\ell^*}{2} \rfloor$ define

$$F_\ell = \begin{pmatrix} L(t_1) + (\ell - 2)\lambda + \sum_{i=2}^{\ell} t_i \\ m_\ell \end{pmatrix} \uplus D_\ell \uplus A(t_1, t_2), \tag{15}$$

where (see (7))

$$\begin{aligned} m_\ell &= m_\ell^* - 2t_\ell \quad \text{and} \\ m_\ell^* &= m_{\ell-1} - L(t_1) - (\ell - 2)\lambda - \textstyle\sum_{i=2}^{\ell-1} t_i. \end{aligned} \tag{16}$$

This is a formal definition, for these generations only exist if $m(t_1, \ldots, t_\ell) \geq 0$. Before addressing this question, let us present two examples that will make all these concepts clearer.

Example 5. Assuming $n = 25$, $c = 3$ and $\lambda = 2$, we have in Example 3 the list of the four blocks. Let us now describe the list of the first descendants of these blocks, observing that now $L(x) = 2x + 6$. It follows from (5) that (see Example 3)

$$m(0) = 22, \;\; m(1) = 17, \;\; m(2) = 10 \;\text{ and }\; m(3) = 1.$$

Hence (see (10))

$$m(0,0) = m(0) - 6 - 1 = 15 \;\text{ and }\; m(1,0) = 8.$$

The values of $m(2,0)$ and $m(3,0)$ are negative, hence the blocks $B(2)$ and $B(3)$ have no descendants. Since

$$m(0,t_2) = 15 - 2t_2 \;\text{ and }\; A(0,t_2) = \begin{pmatrix} 3 \\ 1+t_2 \end{pmatrix},$$

we obtain the following list of first descendants of $B(0)$ (see (8) and (9)):

$$F(0,0) = \begin{pmatrix} 6 & 3 \\ 15 & 1 \end{pmatrix}, \quad F(0,1) = \begin{pmatrix} 7 & 3 \\ 13 & 2 \end{pmatrix}, \quad F(0,2) = \begin{pmatrix} 8 & 3 \\ 11 & 3 \end{pmatrix},$$

$$F(0,3) = \begin{pmatrix} 9 & 3 \\ 9 & 4 \end{pmatrix}, \quad F(0,4) = \begin{pmatrix} 10 & 3 \\ 7 & 5 \end{pmatrix}, \quad F(0,5) = \begin{pmatrix} 11 & 3 \\ 5 & 6 \end{pmatrix}$$

$$F(0,6) = \begin{pmatrix} 12 & 3 \\ 3 & 7 \end{pmatrix}, \quad F(0,7) = \begin{pmatrix} 13 & 3 \\ 1 & 8 \end{pmatrix}.$$

Hence $|\mathbb{S}_2(0)| = 8$. Since

$$m(1,t_2) = 8 - 2t_2 \;\text{ and }\; A(1,t_2) = \begin{pmatrix} 5 & 3 \\ 1+t_2 & 0 \end{pmatrix},$$

we obtain the list of first descendants of $B(1)$:

$$F(1,0) = \begin{pmatrix} 8 & 5 & 3 \\ 8 & 1 & 0 \end{pmatrix}, \quad F(1,1) = \begin{pmatrix} 9 & 5 & 3 \\ 6 & 2 & 0 \end{pmatrix}, \quad F(1,2) = \begin{pmatrix} 10 & 5 & 3 \\ 4 & 3 & 0 \end{pmatrix},$$

$$F(1,3) = \begin{pmatrix} 11 & 5 & 3 \\ 2 & 4 & 0 \end{pmatrix}, \quad F(1,4) = \begin{pmatrix} 12 & 5 & 3 \\ 0 & 5 & 0 \end{pmatrix}.$$

Hence $|\mathbb{S}_2(1)| = 5$.

Example 6. Assuming $n = 25$, $c = 3$ and $\lambda = 2$, we want the list of the descendants of $F(t_1, t_2)$. Now (see Examples 3 and 5) we have $m(0,0,0) = m(0,0) - 6 - 2 = 7$ and $m(1,0,0) = m(1,0) - 8 - 2 < 0$, hence only the first block has descendants in the third generation. Since the only positive values of $m(0,t_2,0)$ are (see (13))

$$m(0,0,0) = 15 - 8 = 7, \;\; m(0,1,0) = 13 - 9 = 4, \;\text{ and }\; m(0,2,0) = 11 - 10 = 1,$$

then only $F(0,0)$, $F(0,1)$ and $F(0,2)$ have descendants. Now (see Example 1)

$$m(0,0,t_3) = 7-2t_3, \quad A(0,0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \text{ and } D(0,0,t_3) = \begin{pmatrix} 6 \\ t_3 \end{pmatrix},$$

$$m(0,1,t_3) = 4-2t_3, \quad A(0,1) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \text{ and } D(0,1,t_3) = \begin{pmatrix} 7 \\ t_3 \end{pmatrix},$$

$$m(0,2,t_3) = 1-2t_3, \quad A(0,2) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \text{ and } D(0,2,t_3) = \begin{pmatrix} 8 \\ t_3 \end{pmatrix}.$$

Therefore, the first descendants of $F(0,0)$ are (see (11) and (12))

$$F(0,0,0) = \begin{pmatrix} 8 & 6 & 3 \\ 7 & 0 & 1 \end{pmatrix}, \qquad F(0,0,1) = \begin{pmatrix} 9 & 6 & 3 \\ 5 & 1 & 1 \end{pmatrix},$$

$$F(0,0,2) = \begin{pmatrix} 10 & 6 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad F(0,0,3) = \begin{pmatrix} 11 & 6 & 3 \\ 1 & 3 & 1 \end{pmatrix};$$

the first descendants of $F(0,1)$ are

$$F(0,1,0) = \begin{pmatrix} 9 & 7 & 3 \\ 4 & 0 & 2 \end{pmatrix}, \qquad F(0,0,1) = \begin{pmatrix} 10 & 7 & 3 \\ 2 & 1 & 2 \end{pmatrix},$$

$$F(0,0,2) = \begin{pmatrix} 11 & 7 & 3 \\ 0 & 2 & 2 \end{pmatrix};$$

and the first descendant of $F(0,2)$ is

$$F(0,2,0) = \begin{pmatrix} 10 & 8 & 3 \\ 1 & 0 & 3 \end{pmatrix}.$$

Hence $|\mathbb{S}_3(0)| = 8$.

As can be seen from the construction above, matrices that are first descendants of a two-line matrix have exactly one more column.

4. Properties of the Descendants

Lemma 3. *Let $\ell \geq 2$. For $F(t_1, \ldots, t_\ell) \in \mathbb{S}_\ell(t_1)$ we have*

$$m(t_1, \ldots, t_\ell) = m(t_1) - (\ell-1)L(t_1) \; - \; \lambda\frac{(\ell-1)(\ell-2)}{2} \; - \; \sum_{j=2}^{\ell} j \cdot t_{\ell-j+2} \; - 1. \quad (17)$$

Proof. From (10) and (13) we have

$$m_2 = m(t_1) - L(t_1) - 1 - 2t_2 \quad \text{and} \quad m_3 = m_2 - L(t_1) - \lambda - t_2 - 2t_3,$$

hence

$$m_3 = m(t_1) - 2L(t_1) - \lambda - 3t_2 - 2t_3 - 1,$$

and these cases confirm (17) for $\ell = 2$ and $\ell = 3$. Since from (16) we have

$$m_\ell = m_{\ell-1} - L(t_1) - (\ell-2)\lambda - \sum_{i=2}^{\ell-1} t_i - 2t_\ell,$$

the result follows from an inductive argument. □

Lemma 4. *If $B(t_1) \in \mathbb{M}(n, c, \lambda)$ then any descendant of $B(t_1)$ is also in $\mathbb{M}(n, c, \lambda)$.*

Proof. Let $F(t_1, \dots, t_l)$ be a descendant of $B(t_1)$. We have to check whether $F(t_1, \dots, t_l)$ satisfies all the properties of $\mathbb{M}(n, c, \lambda)$. By construction, the matrix $A(t_1, t_2)$ satisfies the conditions

$$c_s = c, \quad \text{and} \quad c_j = c_{j+1} + d_{j+1} + \lambda.$$

The matrix $D(t_1, \dots, t_l)$, also by construction, satisfies the condition

$$c_j = c_{j+1} + d_{j+1} + \lambda$$

(see Example 1), and observe that the matrix $D(t_1, \dots, t_l) \uplus A(t_1, t_2)$ satisfies both conditions

$$c_s = c, \quad \text{and} \quad c_j = c_{j+1} + d_{j+1} + \lambda,$$

since the last column of $D(t_1, \dots, t_l)$ and the first column of $A(t_1, t_2)$ are, respectively,

$$\begin{pmatrix} c + (t_1+1)\lambda + 1 + t_2 \\ t_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c + t_1\lambda \\ 1 + t_2 \end{pmatrix}.$$

Finally, observe that the matrix $F(t_1, \dots, t_l)$ also satisfies both conditions since the first column of $F(t_1, \dots, t_l)$ and the first column of $D(t_1, \dots, t_l)$ are, respectively,

$$\begin{pmatrix} L(t_1) + (l-2)\lambda + \sum_{i=2}^{l} t_i \\ m(t_1, \dots, t_l) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L(t_1) + (l-3)\lambda + \sum_{i=2}^{l-1} t_i \\ t_l \end{pmatrix}.$$

The only thing left to be proven is that the sum of all entries of $F(t_1, \dots, t_l)$ is equal to n. From the definition we have (see (7))

$$\sigma(F_\ell) = L(t_1) + (\ell-2)\lambda + \sum_{i=2}^{\ell} t_i + m_\ell + \sigma(D_\ell) + \sigma(A(t_1, t_2)).$$

By Lemmas 1 and 3, and by (5), we have

$$\begin{aligned}\sigma(F_\ell) &= (\ell-1)L(t_1)+\lambda\tfrac{(\ell-1)(\ell-2)}{2}+\textstyle\sum_{j=2}^{\ell} j\cdot t_{\ell-j+2}+m_\ell \\ &= m(t_1)+Q(t_1)=n,\end{aligned}$$

completing the proof. □

Lemma 5. *Let $M \in \mathbb{M}(n,c,\lambda)$. Then there exists a $t_1 \in \{0,1,\ldots,r_1\}$ such that $M \in \mathbb{S}_\ell(t_1)$, for some $\ell \in \mathbb{N}$.*

Proof. Let us write M as

$$M=\begin{pmatrix} a_1 & a_2 & \cdots & a_{s-1} & c \\ b_1 & b_2 & \cdots & b_{s-1} & b_s \end{pmatrix},$$

where $a_j = a_{j+1}+b_{j+1}+\lambda$ and $\sigma(M)=n$ (see Lemma 1). Hence (see (3))

$$n \;=\; \sigma(M) \;=\; Q(s-1)+\sum_{j=1}^{s} j\cdot b_j.$$

It follows from Lemma 2 (see also (6)) that $n < Q(x_1+1)$, hence $s \le \lfloor x_1 \rfloor + 1$, for $b_j \in \mathbb{N}\cup\{0\}$, for any j. Therefore the maximum number of columns of M is at most r_1+1. Let i_0 be the biggest value such that $b_s = b_{s-1} = \cdots = b_{s-i_0} = 0$, in this case we have

$$M=\begin{pmatrix} a_1 & a_2 & \cdots & c+i_0\lambda & c+(i_0-1)\lambda & \cdots & c+\lambda & c \\ b_1 & b_2 & \cdots & b_s-(i_0+1) & 0 & \cdots & 0 & 0 \end{pmatrix},$$

that is, $M \in \mathbb{S}_\ell(i_0)$, completing the proof. □

5. Conditions for the Existence of $\mathbb{S}_{\ell}(t_1)$

The previous section presents a formal tree-graph, with the blocks $B(t_j)$ as the root-vertices, and the descendants are the branches distributed according to its kinship level. This is a formal definition. The condition for the existence of each branch-vertex is given in the next lemma.

Lemma 6. *The matrix $F(t_1,\ldots,t_\ell)$ is in $\mathbb{S}_\ell(t_1)$ if and only if*

$$n \;\ge\; Q(t_1)+(\ell-1)L(t_1) \;+\; \lambda\frac{(\ell-1)(\ell-2)}{2} \;+\; \sum_{j=2}^{\ell} j\cdot t_{\ell-j+2}+1. \qquad (18)$$

Proof. The condition for the existence of the matrix $F(t_1, \ldots, t_\ell)$ is that $m_\ell \geq 0$. From Lemma 3, (5), (7) and (10) we have

$$\begin{aligned} m_\ell &= m(t_1) - (\ell-1)L(t_1) \;-\; \lambda\tfrac{(\ell-1)(\ell-2)}{2} \;-\; \sum_{j=2}^{\ell} j \cdot t_{\ell-j+2} - 1, \\ &= n - Q(t_1) - (\ell-1)L(t_1) \;-\; \lambda\tfrac{(\ell-1)(\ell-2)}{2} \;-\; \sum_{j=2}^{\ell} j \cdot t_{\ell-j+2} - 1, \end{aligned} \tag{19}$$

completing the proof. □

Notation 7. Let us denote by $\mathcal{L}_\ell$ the polynomial (see (18))

$$\begin{aligned} \mathcal{L}_\ell(t_1, \ldots, t_\ell) &= Q(t_1) + (\ell-1)L(t_1) \;+\; \lambda\tfrac{(\ell-1)(\ell-2)}{2} \;+\; \sum_{j=2}^{\ell} j \cdot t_{\ell-j+2} + 1 \\ &= \mathcal{L}_\ell(t_1, 0, \ldots, 0) \;+\; \sum_{j=2}^{\ell} j \cdot t_{\ell-j+2}. \end{aligned} \tag{20}$$

Lemma 7. *For $t_1 \in \{0, 1, \ldots, r_1\}$ we have:*

(a) *the set* $\mathbb{S}_\ell(t_1) \neq \emptyset$ *if and only if* $m(t_1, 0, \ldots, 0) \geq 0$;
(b) *the set* $\mathbb{S}_\ell(0) \neq \emptyset$ *whenever the set* $\mathbb{S}_\ell(t_1) \neq \emptyset$.

Proof. Clearly if $m(t_1, 0, \ldots, 0) \geq 0$, then $F(t_1, 0, \ldots, 0) \in \mathbb{S}_\ell(t_1)$. Conversely assume $F(t_1, \ldots, t_\ell) \in \mathbb{S}_\ell(t_1)$. It follows from Lemma 3 that

$$m(t_1, \ldots, t_\ell) = m(t_1, 0, \ldots, 0) - \sum_{j=2}^{\ell} j \cdot t_{\ell-j+2}\ , \tag{21}$$

hence, if $m(t_1, \ldots, t_\ell) \neq 0$ then $m(t_1, 0, \ldots, 0) \neq 0$. This proves part (a).

It follows from part (a) that we may assume $m(t_1, 0, \ldots, 0) \geq 0$, which gives

$$n \geq \mathcal{L}_\ell(t_1, 0, \ldots, 0) \;=\; Q(t_1) + (\ell-1)L(t_1) + \lambda\frac{(\ell-1)(\ell-2)}{2} + 1,$$

by (18). According to (3) we have (recall that $c \in \mathbb{N}$ and $\lambda \in \mathbb{N} \cup \{0\}$),

$$Q(t_1) = Q(0) + ct_1 + \lambda\tfrac{t_1(t_1+1)}{2} \qquad \text{and} \qquad L(t_1) = L(0) + \lambda t_1,$$

hence, if $n \geq \mathcal{L}_\ell(t_1, 0, \ldots, 0)$, then necessarily we have $n \geq \mathcal{L}_\ell(0, 0, \ldots, 0)$, completing the proof of part (b). □

Lemma 8. *Let x_0 be the positive root of the polynomial (see (3))*

$$\mathcal{P}_0(x) = n - Q(x) - (x+1) = -\frac{\lambda}{2}x^2 - x(c + \frac{\lambda}{2} + 1) - c - 1 + n. \qquad (22)$$

Then $\ell_o = \lfloor x_0 \rfloor + 1$ is the maximum value of ℓ such that $\mathbb{S}_\ell(0) \neq \emptyset$. In particular, for any $t_1 \in \{0, 1, \ldots, r_1\}$, if $\mathbb{S}_\ell(t_1) \neq \emptyset$ then $\ell \leq \ell_o$.

Proof. We want the maximum value of ℓ such that

$$n \geq \mathcal{L}_\ell(0, \ldots, 0) = Q(0) + (\ell - 1)L(t_1) + \lambda\frac{(\ell-1)(\ell-2)}{2} + 1.$$

Now (see (3))

$$\begin{aligned}
\mathcal{L}_\ell(0, \ldots, 0) &= c + (\ell-1)(c+1+\lambda) + \lambda\tfrac{(\ell-1)(\ell-2)}{2} + 1 \\
&= c((\ell-1)+1) + \lambda\tfrac{\ell(\ell-1)}{2} + ((\ell-1)+1), \\
&= Q(\ell-1) + ((\ell-1)+1),
\end{aligned}$$

and since we are assuming $n > c + 1$, the equation

$$Q(x) + (x+1) = n$$

has two distinct non-zero roots of opposite signs if $\lambda \neq 0$, and one root if $\lambda = 0$. Let x_0 be its positive root. It is easy to check that $Q(\lfloor x_0 \rfloor) + (\lfloor x_0 \rfloor + 1) \leq n$ and $Q(\lfloor x_0 \rfloor + 1) + (\lfloor x_0 \rfloor + 1) + 1 > n$. Take $\ell_0 = \lfloor x_0 \rfloor + 1$ to complete the proof. □

Example 8. Let us assume $n = 25$, $c = 3$ and $\lambda = 2$. The condition stated in Lemma 8 is equivalent to

$$x^2 + 5x - 21 = 0,$$

which gives $\lfloor x_0 \rfloor = 2$, that is, the set $\mathbb{M}(25, 3, 2)$ has exactly three generations, and they are given in Examples 3, 5 and 6. In particular (see Lemmas 4 and 5)

$$|\mathbb{M}(25,3,2)| \;=\; |\mathbb{B}(25,3,2)| + |\mathbb{S}_2(0)| + |\mathbb{S}_2(1)| + |\mathbb{S}_3(0)| \;=\; 4 + 8 + 5 + 8 \;=\; 25.$$

Figure 1 below presents the generational tree of the set $\mathbb{M}(25, 3, 2)$.

6. The Cardinality of the Set $\mathbb{S}_\ell(t_1)$

Lemma 6 gives a general condition for the existence of an element in $\mathbb{S}_\ell(t_1)$. Now we want to establish conditions for the values of the variables $t_1, \ldots, t_\ell$ that guarantees the existence of the matrix $F(t_1, \ldots, t_\ell)$. It follows from Lemma 7 that the condition

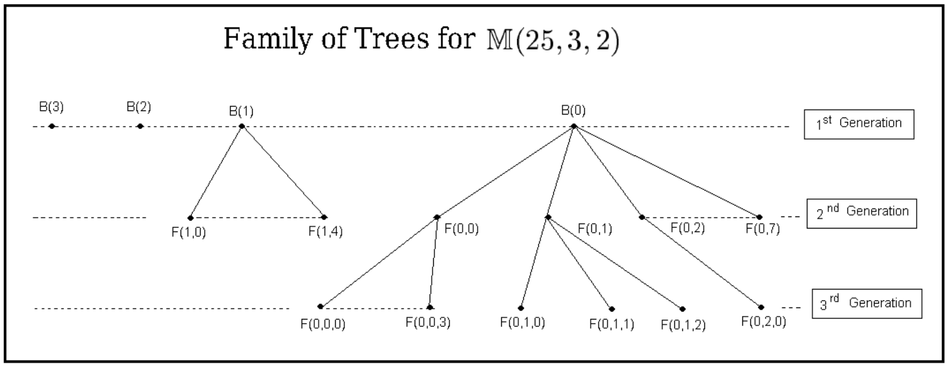

Figure 1: Generational Tree

$\mathbb{S}_\ell(t_1) \neq \emptyset$ is equivalent to $n \geq \mathcal{L}_\ell(t_1, 0, \ldots, 0)$ (see (18),(19) and (20)), thus we obtain x_ℓ as the non-negative root of the equation

$$\mathcal{L}_\ell(x, 0, \ldots, 0) = n.$$

This gives an upper bound for t_1 that guarantees that the block $B(t_1)$ has descendants at the ℓ-th generation, whenever $t_1 \leq r_\ell = \lfloor x_\ell \rfloor$.

Lemma 9. *Supppose $\mathbb{S}_\ell(t_1) \neq \emptyset$ and, for $\jmath \in \{2, \ldots, \ell\}$, consider*

$$\begin{aligned} \mathcal{P}_1(x) &= n - Q(x) = -\tfrac{\lambda}{2}x^2 - x(c + \tfrac{\lambda}{2}) - c + n, \quad \textit{and} \\ \mathcal{P}_j(x) &= n - \mathcal{L}_j(x, 0, \ldots, 0) \\ &= -\tfrac{\lambda}{2}x^2 - x(c + \tfrac{\lambda}{2}(2j-1)) - (j(c+1) + \tfrac{\lambda}{2}j(j-1) - n). \end{aligned} \tag{23}$$

If x_i is the non-negative root of $\mathcal{P}_i(x) = 0$, then $x_1 > x_2 > \cdots > x_\ell$. In particular,

$$r_1 \geq r_2 \geq \cdots \geq r_\ell.$$

Proof. If $\lambda = 0$ these equations are all linear and the result is easy to obtain. So let us consider $\lambda \neq 0$. For the quadratic equation $\mathcal{P}_j(x) = 0$, let us denote by Δ_j its discriminant, thus

$$\Delta_2 = (c + \frac{3\lambda}{2})^2 - 2\lambda(2(c+1) + \lambda - n) = \Delta_1 - 4\lambda,$$

and for $\jmath \in \{3, \ldots, \ell\}$ we have

$$\begin{aligned} \Delta_j &= (c + \tfrac{\lambda}{2}(2j-1))^2 - 2\lambda(j(c+1) + \tfrac{\lambda}{2}j(j-1) - n) \\ &= \Delta_{j-1} + 2\lambda(c + \tfrac{\lambda}{2}(2j-3)) + \lambda^2 - 2\lambda(c + 1 + \lambda(j-1)) \\ &= \Delta_{j-1} - 2\lambda. \end{aligned}$$

Hence $x_1 < x_2$ and

$$x_j = -\frac{1}{\lambda}(c + \frac{\lambda}{2}(2j-1) + \sqrt{\Delta_j}\ ) \ < \ -\frac{1}{\lambda}(c + \frac{\lambda}{2}(2j-3) + \sqrt{\Delta_{j-1}}\ ) \ = \ x_{j-1},$$

completing the proof. □

Corollary 1. *Assume $\lambda \neq 0$ and let Δ_j be the discriminant of the quadratic equation $\mathcal{P}_j = 0$. Then*

$$\begin{array}{ll} (a) & \Delta_2 = \Delta_1 - 4\lambda, \quad \textit{and} \quad \Delta_i = \Delta_{i-1} - 2\lambda, \quad \textit{for} \ \ i \geq 2; \\ (b) & \Delta_j = \Delta_1 - 2j\lambda. \end{array}$$

Proof. This is a direct consequence of the proof of Lemma 9. □

Before going any further, let us collect all the notation used so far in the form of a lemma.

Lemma 10. *Let $n, \ell \in \mathbb{N}$ and $n, \ell \geq 2$. The value of $m_\ell(t_1, \ldots, t_s, 0, \ldots, 0)$ can be obtained by the formula:*

$$m_\ell(t_1, \ldots, t_s, 0, \ldots, 0) \ = \ n - \mathcal{L}_\ell(t_1, \ldots, t_s, 0, \ldots, 0) \ = \ \mathcal{P}_\ell(t_1) \ - \sum_{i=\ell-s+2}^{\ell} i \cdot t_{\ell-i+2}.$$

Proof. It follows directly from (19), (20), (21) and (23). □

Let us assume that $F(t_1, \ldots, t_{\ell-1}) \in \mathbb{S}_{\ell-1}(t_1)$ and recall that the condition for this matrix to have a descendant is

$$m_\ell(t_1, \ldots, t_{\ell-1}, 0) \ \geq \ 0, \tag{24}$$

and in this case, the number of descendants of $F(t_1, \ldots, t_{\ell-1})$ (see (15)) is equal to

$$\left\lfloor \frac{m_\ell(t_1, \ldots, t_{\ell-1}, 0)}{2} \right\rfloor + 1. \tag{25}$$

Let us assume that for a fixed t_1 we have $\mathbb{S}_\ell(t_1) \neq \emptyset$, thus $t_1 \leq r_\ell$, according to Lemma 9. A condition for the existence of a descendant $F_\ell(t_1, t_2, 0, \ldots, 0)$ is that (see (24) and Lemma 10)

$$m_\ell(t_1, t_2, 0, \ldots, 0) \ = \ \mathcal{P}_\ell(t_1) - \ell t_2 \ \geq \ 0.$$

This gives the following condition on t_2 for the existence of $F_\ell(t_1, t_2, 0, \ldots, 0)$:

$$t_2 \ \leq \ \left\lfloor \frac{\mathcal{P}_\ell(t_1)}{\ell} \right\rfloor \ = \ T_2^{(\ell)}(t_1). \tag{26}$$

Similarly, given t_1, t_2, the condition on t_3 for the existence of $F_\ell(t_1, t_2, t_3, 0, \dots, 0)$ is given by

$$m_\ell(t_1, t_2, t_3, 0, \dots, 0) \; = \mathcal{P}_\ell(t_1) - \ell t_2 - (\ell - 1)t_3 \; \geq \; 0,$$

which gives

$$t_3 \; \leq \; \left\lfloor \frac{\mathcal{P}_\ell(t_1) - \ell t_2}{\ell - 1} \right\rfloor \; = \; T_3^{(\ell)}(t_1, t_2). \tag{27}$$

In general, given $t_1, t_2, \dots, t_{j-1}$, the condition on t_j for the existence of

$$F_\ell(t_1, t_2, t_3, \dots, t_j, 0 \dots, 0)$$

is given by

$$t_j \; \leq \; \left\lfloor \frac{\mathcal{P}_\ell(t_1) - \sum_{i=\ell-j+3}^{\ell} i \cdot t_{\ell-i+2}}{\ell - (j-2)} \right\rfloor \; = \; T_j^{(\ell)}(t_1, t_2, \dots, t_{j-1}), \tag{28}$$

for any $\jmath \leq \ell - 1$. These arguments together with (25) proves the next theorem

Theorem 9. *For any $\ell \geq 2$ such that $r_\ell \geq 0$ (see Lemma 9), the number of elements of $\mathbb{M}(n, c, \lambda)$ that are descendants of any block at the ℓ-th generation is equal to*

$$\mathbb{D}_\ell \; = \; \sum_{t_1=0}^{r_\ell} \sum_{t_2=0}^{T_2^{(\ell)}} \cdots \sum_{t_{\ell-1}=0}^{T_{\ell-1}^{(\ell)}} \left(\left\lfloor \frac{\mathcal{P}_\ell(t_1) - \displaystyle\sum_{i=3}^{\ell} i \cdot t_{\ell-i+2}}{2} \right\rfloor + 1 \right).$$

Therefore, for ℓ_0 given in Lemma 8 we have

$$|\mathbb{M}(n, c, \lambda)| \; = \; (r_1 + 1) + \sum_{\ell=2}^{\ell_0} \mathbb{D}_\ell.$$

The calculation of $|\mathbb{M}(n, c, \lambda)|$ can be done in a very effective way. We will illustrate this procedure in the next example.

Example 10. We want to calculate $|\mathbb{M}(31, 1, 2)|$. Starting with Lemma 8 we have that the maximum number of generations is given by $\ell_0 = \lfloor x_0 \rfloor + 1$ where x_0 is the root of $\mathcal{P}_0(x) = -x^2 - 3x + 29$. Hence

$$\lfloor x_0 \rfloor = 4 \quad \text{and} \quad \ell_0 = 5.$$

From Lemma 9 we have

$$\begin{aligned}
\mathcal{P}_1(x) &= -x^2 - 2x + 30, \\
\mathcal{P}_2(x) &= -x^2 - 4x + 25, \\
\mathcal{P}_3(x) &= -x^2 - 6x + 19, \\
\mathcal{P}_4(x) &= -x^2 - 8x + 11, \\
\mathcal{P}_5(x) &= -x^2 - 10x + 1.
\end{aligned}$$

Denoting by Δ_j the discriminant of the equation $\mathcal{P}_j(x) = 0$ we have (see Corollary 1)

$$\Delta_1 = 124, \qquad \Delta_2 = \Delta_1 - 8 = 116, \quad \Delta_3 = \Delta_2 - 4 = 112,$$

$$\Delta_4 = \Delta_3 - 4 = 108, \quad \Delta_5 = \Delta_4 - 4 = 104.$$

Hence

$$r_1 = 4, \ \ r_2 = 3, \ \ r_3 = 2, \ \ r_4 = 1, \ \ r_5 = 0.$$

Therefore

$$\mathbb{D}_2 = \sum_{t_1=0}^{r_2} (\lfloor \frac{\mathcal{P}_2(t_1)}{2} \rfloor + 1) = 13 + 11 + 7 + 3 = 34.$$

Now, $r_3 = 2$, and since (see (28))

$$T_2^{(3)}(t_1) = \lfloor \frac{\mathcal{P}_3(t_1)}{3} \rfloor,$$

we have

$$T_2^{(3)}(0) = 6, \quad T_2^{(3)}(1) = 4, \quad T_2^{(3)}(2) = 1.$$

Hence

$$\begin{aligned}
\mathbb{D}_3 &= \sum_{t_1=0}^{2} \sum_{t_2=0}^{T_2^{(3)}(t_1)} (\lfloor \frac{\mathcal{P}_3(t_1) - 3t_2}{2} \rfloor + 1) \\
&= \sum_{t_2=0}^{6} (\lfloor \frac{19 - 3t_2}{2} \rfloor + 1) + \sum_{t_2=0}^{4} (\lfloor \frac{12 - 3t_2}{2} \rfloor + 1) + \sum_{t_2=0}^{1} (\lfloor \frac{3 - 3t_2}{2} \rfloor + 1) \\
&= 62.
\end{aligned}$$

Now $r_4 = 1$, and since (see (28))

$$T_2^{(4)}(t_1) = \lfloor \frac{\mathcal{P}_4(t_1)}{4} \rfloor \quad \text{and} \quad T_3^{(4)}(t_1, t_2) = \lfloor \frac{\mathcal{P}_4(t_1) - 4t_2}{3} \rfloor$$

we have

$$\begin{aligned}
&T_2^{(4)}(0) = 2, \qquad T_2^{(4)}(1) = 0, \\
&T_3^{(4)}(0,0) = 3, \quad T_3^{(4)}(0,1) = 2, \\
&T_3^{(4)}(0,2) = 1, \quad T_3^{(4)}(1,0) = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{D}_4 &= \sum_{t_1=0}^{r_4} \sum_{t_2=0}^{T_2^{(4)}(t_1)} \sum_{t_3=0}^{T_3^{(4)}(t_1,t_2)} \left(\left\lfloor \tfrac{\mathcal{P}_4(t_1)-4t_2-3t_3}{2} \right\rfloor + 1 \right) \\
&= \sum_{t_3=0}^{3} \left(\left\lfloor \frac{11-3t_3}{2} \right\rfloor + 1 \right) + \sum_{t_3=0}^{2} \left(\left\lfloor \frac{7-3t_3}{2} \right\rfloor + 1 \right) \\
&\quad + \sum_{t_3=0}^{1} \left(\left\lfloor \frac{3-3t_3}{2} \right\rfloor + 1 \right) + \left(\left\lfloor \tfrac{2}{2} \right\rfloor + 1 \right) \\
&= 29.
\end{aligned}$$

Finally, $r_5 = 0$ and (see (28))

$$T_2^{(5)}(0) = \lfloor \frac{\mathcal{P}_5(0)}{5} \rfloor = 0, \ \ T_3^{(5)}(0,0) = \lfloor \frac{\mathcal{P}_5(0)}{4} \rfloor = 0 \ \text{ and } \ T_4^{(5)}(0,0,0) = \lfloor \frac{\mathcal{P}_5(0)}{3} \rfloor = 0,$$

hence

$$\mathbb{D}_5 = \lfloor \frac{\mathcal{P}_5(0)}{2} \rfloor + 1 = 1.$$

Therefore

$$\mathbb{M}(31,1,2) = (r_1+1) + \mathbb{D}_2 + \mathbb{D}_3 + \mathbb{D}_4 + \mathbb{D}_5 = 5 + 34 + 62 + 29 + 1 = 131.$$

In particular, we have also proved that the number of partitions of 31 with the smallest part greater than or equal to 1 and distance between parts at least 2 is equal to 131. The partitions of n with the restrictions given by $c = 1$ and $\lambda = 2$ are the ones arising from the Rogers-Ramanujan Identities. For this reason 131 is also the number of partititons of 31 into parts that are congruent to ± 1 modulo 5.

The most important application of our procedure is the presentation of a closed combinatorial formula for the number of unrestricted partitions. The details are given below.

6.1. Unrestricted Partitions

The number of unrestricted partitions of n is equal to the cardinality of the set $\mathbb{M}(n,1,0)$. The number ℓ_0 of generations is given by the root of

$$\mathcal{P}_0(x) = n - 2(x+1), \ \text{ that is, } \ \ell_0 = \left\lfloor \frac{n-2}{2} \right\rfloor + 1,$$

which gives the generating polynomials

$$\mathcal{P}_1(x) = n - (x+1) \quad \text{and} \quad \mathcal{P}_j(x) = n - (x+2j), \quad \text{for } \jmath = 2, \ldots, \ell_0, \qquad (29)$$

and the roots of these polynomials are

$$r_1 = n-1,\ \ r_2 = n-4,\ \ldots,\ r_{\ell_0} = n-2\ell_0.$$

From (28) we can obtain the values of $T_j^{(\ell)}(t_1,\ldots,t_{j-1})$. Now we have all the necessary ingredients to obtain from Theorem 9 a formula for the number of unrestricted partitions of n. Let us illustrate this formula with the following example.

Example 11. For $n = 10$ we have $\mathcal{P}_0(x) = 10-2(x+1)$, hence $x_0 = 4$ and $\ell_0 = 5$ (see Lemma 8), that is, there are five generations. Consider the polynomials (see (29))

$$\begin{array}{lll} \mathcal{P}_1(x) = 10-(x+1), & \text{hence} & r_1 = 9, \\ \mathcal{P}_2(x) = 10-(x+4), & \text{hence} & r_2 = 6, \\ \mathcal{P}_3(x) = 10-(x+6), & \text{hence} & r_3 = 4, \\ \mathcal{P}_4(x) = 10-(x+8), & \text{hence} & r_4 = 2, \\ \mathcal{P}_5(x) = 10-(x+10), & \text{hence} & r_5 = 0. \end{array}$$

In this case $\mathbb{D}_1 = r_1+1 = 10$, that is, there are 10 blocks. The number of elements at the second generation is $\mathbb{D}_2$ which is given by (recall that $r_2 = 6$)

$$\sum_{t_1=0}^{6}\left(\left\lfloor\frac{\mathcal{P}_2(t_1)}{2}\right\rfloor+1\right) = 16.$$

Now (see (26))

$$T_2^{(3)}(t_1) = \lfloor\frac{\mathcal{P}_3(t_1)}{3}\rfloor = \lfloor\frac{4-t_1}{3}\rfloor.$$

Since $r_3 = 4$, we have

$$T_2^{(3)}(0) = 1,\quad T_2^{(3)}(1) = 1,\quad T_2^{(3)}(2) = T_2^{(3)}(3) = T_2^{(3)}(4) = 0,$$

and

$$\mathbb{D}_3 = \sum_{t_1=0}^{4}\sum_{t_2=0}^{T_2^{(3)}(t_1)}\left(\lfloor\frac{\mathcal{P}_3(t_1)-3t_2}{2}\rfloor+1\right) = 11.$$

From (26) and (27) we have

$$T_2^{(4)}(t_1) = \lfloor\frac{\mathcal{P}_4(t_1)}{4}\rfloor = \lfloor\frac{2-t_1}{4}\rfloor \quad\text{and}\quad T_3^{(4)}(t_1,t_2) = \lfloor\frac{\mathcal{P}_4(t_1)-4t_2}{3}\rfloor = \lfloor\frac{2-t_1-4t_2}{3}\rfloor,$$

hence

$$T_2^{(4)}(0) = T_2^{(4)}(1) = T_3^{(4)}(0,0) = T_3^{(4)}(1,0) = 0.$$

Therefore

$$\mathbb{D}_4 = \sum_{t_1=0}^{4}\sum_{t_2=0}^{T_2^{(4)}(t_1)}\sum_{t_3=0}^{T_3^{(4)}(t_1,t_2)}\left(\lfloor\frac{\mathcal{P}_4(t_1)-4t_2-3t_3}{2}\rfloor+1\right) = 4.$$

Finally, since $r_5 = 0$, we have (see (26), (27), (28))

$$T_2^{(5)}(0) = \lfloor \frac{\mathcal{P}_5(0)}{5} \rfloor = 0, \;\; T_3^{(5)}(0,0) = \lfloor \frac{\mathcal{P}_5(0)}{4} \rfloor = 0, \;\; T_4^{(5)}(0,0,0) = \lfloor \frac{\mathcal{P}_5(0)}{3} \rfloor = 0,$$

hence

$$\mathbb{D}_5 = \left\lfloor \frac{\mathcal{P}_5(0)}{2} \right\rfloor + 1 \;=\; 1.$$

Therefore, the number of unrestrited partitions of $n = 10$ is

$$(r_1 + 1) + \mathbb{D}_2 + \mathbb{D}_3 + \mathbb{D}_4 + \mathbb{D}_5 = 42.$$

As expected we can get the cardinality of $\mathbb{M}(n + k, c, \lambda)$ from the cardinality of $\mathbb{M}(n, c, \lambda)$. In our final section we will describe how this can be done using our methods.

7. The Cardinality of the Set $\mathbb{M}(n + k, c, \lambda)$

We start this section with a simple lemma on the roots of quadratic equations. These ideas will be later applied to the polynomials $\mathcal{P}_j(x)$.

Lemma 11. *Let $a, b, c, k \in \mathbb{N}$ and define*

$$f(x) = c - ax^2 - bx \quad \text{ and } \quad f_k(x) = f(x) + k.$$

Let us denote by x_0, x_1 the roots of $f(x)$ with $x_0 < 0$ and $x_1 > 0$, and $x_0^{(k)}, x_1^{(k)}$ the roots of $f_k(x)$ with $x_0^{(k)} < 0$ and $x_1^{(k)} > 0$. If we denote by Δ and Δ_k the discriminants of $f(x)$ and $f(x)_k$, respectively, then

(*a*) $\Delta_k = \Delta + 4ak$;

(*b*) $x_0^{(k)} = x_0 \; - \; \dfrac{\sqrt{\Delta + 4ak} - \sqrt{\Delta}}{2a}$ *and* $x_1^{(k)} = x_1 \; + \; \dfrac{\sqrt{\Delta + 4ak} - \sqrt{\Delta}}{2a}$.

Proof. Since part (a) follows directly from the definition, we concentrate on part (b). It is easy to check that the roots of both $f(x)$ and $f_k(x)$ are symmetric with respect to the vertical line $x = -b/2a$. Therefore, since $k \geq 1$, there is a $t \in \mathbb{R}$, $t > 0$, such that

$$x_0^{(k)} = x_0 - t \quad \text{and} \quad x_1^{(k)} = x_1 + t.$$

Then

$$\begin{aligned} f_k(x) \;&=\; -a\,(x - x_0^{(k)})\,(x - x_1^{(k)}) \;=\; -a\,(x - (x_0 - t))\,(x - (x_1 + t)) \\ &=\; -ax^2 + ax(x_0 + x_1) - a\,(x_0 x_1 + t(x_0 - x_1) - t^2) \\ &=\; -ax^2 - bx + (c + t\sqrt{\Delta} + at^2). \end{aligned}$$

Hence $c+k = c + t\sqrt{\Delta} + at^2$, and the positive root of $at^2 + \sqrt{\Delta}t - k = 0$ is

$$t = \frac{\sqrt{\Delta + 4ak} - \sqrt{\Delta}}{2a},$$

completing the proof. $\square$

Now we can establish a simple procedure to calculate the value of $|\mathbb{M}(n+k, c, \lambda)|$ once we have the value of $|\mathbb{M}(n, c, \lambda)|$. The first step is to calculate $\ell_0^{(k)}$, the maximum number of generations of $\mathbb{M}(n+k, c, \lambda)$, which is given by the positive root of the polynomial (see Lemma 8)

$$\mathcal{P}_0^{(k)} = (n+k) - Q(x) - (x+1) = -\frac{\lambda}{2}x^2 - x(c + \frac{\lambda}{2} + 1) - c - 1 + n + k.$$

According to Lemma 11, this number can be obtained from the previous calculations regarding $\mathcal{P}_0$, that is, if Δ_0 is the discrimant of $\mathcal{P}_0$ and x_0 is its positive root we obtain

$$\ell_0^{(k)} = \lfloor x_0^{(k)} \rfloor + 1 = \left\lfloor x_0 - \frac{\sqrt{\Delta_0 + 2\lambda k} - \sqrt{\Delta_0}}{\lambda} \right\rfloor + 1. \tag{30}$$

The next step is to find the maximum number of blocks inside $\mathbb{M}(n+k, c, \lambda)$, which is given by the positive root of the polynomial (see Lemma 9)

$$\mathcal{P}_1^{(k)} = (n+k) - Q(x) = -\frac{\lambda}{2}x^2 - x(c + \frac{\lambda}{2}) - c + n + k.$$

Again by Lemma 11, assuming that Δ_1 is the discrimant of $\mathcal{P}_1$ and x_1 is its positive root, we obtain

$$r_1^{(k)} = \lfloor x_1^{(k)} \rfloor = \left\lfloor x_1 + \frac{\sqrt{\Delta_1 + 4\lambda k} - \sqrt{\Delta_1}}{\lambda} \right\rfloor, \tag{31}$$

and the number of blocks in $\mathbb{M}(n+k, c, \lambda)$ is equal to $r_1^{(k)} + 1$. For the values of $r_j^{(k)}$ we need the roots of the polynomials

$$\mathcal{P}_j^{(k)} = (n+k) - \mathcal{L}_j(x, 0, \ldots, 0) = -\frac{\lambda}{2}x^2 - x(c + \frac{\lambda}{2}(2j-1)) - (j(c+1) + \frac{\lambda}{2}j(j-1) - n - k),$$

for $j = 2, \ldots, \ell_0^{(k)}$. From Corollary 1 and Lemma 11 it follows that all these values can be obtained from a formula based on the value of Δ_1, in the following way:

$$\begin{aligned} r_j^{(k)} = \lfloor x_j^{(k)} \rfloor \quad &= \quad \left\lfloor \frac{(c + \frac{\lambda}{2}(2j-1)) + \sqrt{\Delta_j^{(k)}}}{\lambda} \right\rfloor \\ &= \quad \left\lfloor \frac{(c + \frac{\lambda}{2}(2j-1)) + \sqrt{\Delta_1^{(k)} - 2j\lambda}}{\lambda} \right\rfloor \\ &= \quad \left\lfloor \frac{(c + \frac{\lambda}{2}(2j-1)) + \sqrt{\Delta_1 - 2\lambda(j-k)}}{\lambda} \right\rfloor, \end{aligned}$$

for $j = 2, \ldots, \ell_0^{(k)}$. For a fixed $\ell \in \{2, \ldots, \ell_0^{(k)}\}$ we obtain (see (26), (27), (28)) for $2 \leq j \leq \ell - 1$

$$T_j^{(\ell)(k)}(t_1, t_2, \ldots, t_{j-1}) = \left\lfloor \frac{(\mathcal{P}_\ell(t_1) - \sum_{i=\ell-j+3}^{\ell} i \cdot t_{\ell-i+2}) \, + \, k}{\ell - (j-2)} \right\rfloor .$$

Once we have all these values we can go back to Section 5, and obtain from Theorem 9 the values of

$$\mathbb{D}_\ell^{(k)} \;=\; \sum_{t_1=0}^{r_\ell^{(k)}} \sum_{t_2=0}^{T_2^{(\ell)(k)}} \cdots \sum_{t_{\ell-1}=0}^{T_{\ell-1}^{(\ell)(k)}} \left(\left\lfloor \frac{(\; \mathcal{P}_\ell(t_1) - \displaystyle\sum_{i=3}^{\ell} i \cdot t_{\ell-i+2} \;) \, + \, k}{2} \right\rfloor + 1 \right)$$

and

$$|\mathbb{M}(n+k, c, \lambda)| \;=\; (r_1^{(k)} + 1) + \sum_{\ell=2}^{\ell_0^{(k)}} \mathbb{D}_\ell^{(k)}.$$

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