AN IMPROVED INEQUALITY OF ROSSER AND SCHOENFELD AND ITS APPLICATION

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Abstract

In this note, using Robin's inequality and Pierre Dusart's inequality, we refine Rosser-Schoenfeld's inequality. As a consequence, we show that every even integer greater than 30 can be represented as the sum of a composite number c and a prime p not dividing c.

1. Introduction

For any positive integer n, let $\omega(n)$ be the number of distinct prime factors of n and $\pi(n)$ be the number of prime numbers which are less than or equal to n. In the 1960s, Rosser and Schoenfeld [6] proved that $2\pi(n) > \pi(2n)$ for $n > 2$. In 1988, Ehrhart [3] again proved Rosser-Schoenfeld's inequality by using a simple method. However, Rosser-Schoenfeld's inequality has not been improved for a long time. The main reason is that better explicit upper and lower bounds on $\pi(n)$ have not been obtained. We assume that $\frac{n}{\ln n - A} < \pi(n) < \frac{n}{\ln n - B}$. In order to prove that $2\pi(n) > \pi(2n) + 1$, the constants A and B must satisfy $A < B < \ln 2 + A$. This requires very little difference between A and B. It is fortunate that Pierre Dusart showed recently that for $n \geq 5393$, $\pi(n) > \frac{n}{\ln n - 1}$ and for $n \geq 60184$, $\pi(n) < \frac{n}{\ln n - 1.1}$[2]. For some better bounds on $\pi(n)$, see [1]. Therefore, Pierre Dusart's results are expected to improve Rosser-Schoenfeld's inequality. In this note, combining with Robin's inequality [5] on $\omega(n)$, we will prove the following theorems.

Theorem 1. *For $n \geq 59$, we have $2\pi(n) > \pi(2n) + \omega(2n)$.*

Theorem 2. *Every even integer greater than 30 is the sum of a composite number c and a prime p not dividing c.*

2. Proof of Theorems

Lemma 1 ([2]). *For $n \geq 5393$, we have $\pi(n) > \frac{n}{\ln n - 1}$.*

Lemma 2 ([2]). *For $n \geq 60184$, we have $\pi(n) < \frac{n}{\ln n - 1.1}$.*

Lemma 3 ([5]). *For $n \geq 3$, we have $\omega(n) \leq c\frac{\ln n}{\ln \ln n}$, where $c = 1.38402...$.*

Proof of Theorem 1. We estimate roughly that for $n \geq 30092$,

$$\frac{2n}{\ln n - 1} - 1.385\frac{\ln 2n}{\ln \ln 2n} - \frac{2n}{\ln 2n - 1.1} > 0.$$

By Lemmas 1, 2, and 3, we have that

$$2\pi(n) - \omega(2n) > \frac{2n}{\ln n - 1} - 1.385\frac{\ln 2n}{\ln \ln 2n} > \frac{2n}{\ln 2n - 1.1} > \pi(2n).$$

The cases $58 < n < 30092$ are verified by computer. Thus the proof of Theorem 1 is completed. □

Remark 1. 59 is the best bound because $2\pi(58) - \omega(2 \times 58) = \pi(2 \times 58)$.

Corollary 1. *For $n \geq 17$, we have $2\pi(n) > \pi(2n)+1$.*

Proof. By Theorem 1, we have that $2\pi(n) > \pi(2n)+1$ for $n \geq 59$. Do the cases $16 < n < 59$ separately. Thus the proof of Corollary 1 is completed. □

Proof of Theorem 2. Let $2n$ be an even positive integer greater than 120, and k be the number of primes which are coprime to $2n$ and less than n. Note that if n is an odd prime, then $k = \pi(n) - 2$. Otherwise $k = \pi(n) - \omega(2n)$. Denote these prime numbers by $q_1, ..., q_k$. Consider the following k distinct numbers: $2n - q_i, 1 \leq i \leq k$. If $k > \pi(2n) - \pi(n)$, then there is a composite number c among $2n - q_i, 1 \leq i \leq k$, and $2n$ is the sum of a composite number c and a prime p not dividing c.

If n is a composite number, then by Theorem 1, we have that $k = \pi(n) - \omega(2n) > \pi(2n) - \pi(n)$. If n is an odd prime, then $2\pi(n) - 2 = 2\pi(n-1)$ and $n-1$ is a composite number. By Theorem 2 again, we have that $2\pi(n-1) > \omega(2(n-1)) + \pi(2(n-1)) \geq \pi(2n)$. Thus, $k = \pi(n) - 2 > \pi(2n) - \pi(n)$. Therefore, each even positive integer greater than 120 can be expressed as the sum of a composite number c and a prime p not dividing c.

Finally, do the cases $30 < 2n \leq 120$ separately: $32 = 7 + 25, 34 = 7 + 27, 36 = 11 + 25, 38 = 13 + 25, 40 = 7 + 33, 42 = 17 + 25, 44 = 17 + 27, 46 = 7 + 39, 48 = 13 + 35, 50 = 11 + 39, 52 = 7 + 45, 54 = 5 + 49, 56 = 11 + 45, 58 = 13 + 45, 60 = 11 + 49, 62 = 5+57, 64 = 7+57, 66 = 41+25, 68 = 41+27, 70 = 43+27, 72 = 37+35, 74 = 29 + 45, 76 = 31 + 45, 78 = 29 + 49, 80 = 29 + 51, 82 = 31 + 51, 84 = 29 + 55, 86 =$

$31+55, 88 = 31+57, 90 = 13+77, 92 = 11+81, 94 = 13+81, 96 = 71+25, 98 = 73+25, 100 = 73+27, 102 = 17+85, 104 = 17+87, 106 = 19+87, 108 = 83+25, 110 = 83+27, 112 = 31+81, 114 = 89+25, 116 = 19+87, 118 = 9+109, 120 = 71+49.$ Thus, the proof of Theorem 2 is completed. □

Corollary 2. *Every odd integer greater than 23 can be represented as $p+q+c$, where p and q are prime, and c is a composite number satisfying that p, q and c are pairwise coprime.*

Proof of Corollary 2. Let n be an odd integer greater than 73. A result of Nagura [4] states that there is a prime in the interval $[x, 1.2x]$ for $x \geq 25$. This implies that there is an odd prime q in the interval $[\frac{n}{2}, n-30)$ for $n \geq 75$. But $n-q$ is even and greater than 30. By Theorem 2, $n-q$ can be expressed as the sum of a composite number c and a prime p not dividing c. Since $q > \frac{n}{2}$, it follows that p, q and c are pairwise coprime. Finally, do the cases $23 < n < 75$ separately: $73 = 41+5+27, 71 = 43+3+25, 69 = 41+3+25, 67 = 31+11+25, 65 = 29+11+25, 63 = 29+7+27, 61 = 29+5+27, 59 = 19+7+33, 57 = 17+7+33, 55 = 17+13+25, 53 = 17+11+25, 51 = 17+7+27, 49 = 17+7+25, 47 = 13+7+27, 45 = 13+7+25, 43 = 11+7+25, 41 = 19+13+9, 39 = 17+13+9, 37 = 17+11+9, 35 = 19+7+9, 33 = 17+7+9, 31 = 17+5+9, 29 = 13+7+9, 27 = 11+7+9, 25 = 11+5+9.$ But, 23 cannot be represented as $p+q+c$, where p and q are prime, and c is a composite number satisfying that p, q and c are pairwise coprime. Thus, the proof is completed. □

Remark 2. Based on the method of proof of Theorem 2 and Corollary 1, one will see that every even integer greater than 10 can be represented as the sum of a composite number and a prime. Note that $12k = 3+(12k-3), 12k+2 = 5+(12k-3), 12k+4 = 7+(12k-3), 12k+6 = 3+(12k+3), 12k+8 = 11+(12k-3), 12k+10 = 7+(12k+3)$. But, this method fails as a proof of Theorem 2. So, a refinement of Rosser-Schoenfeld's inequality is of some reference value.

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