GENERALIZED FIBONACCI NUMBERS AND THEIR 2-ADIC ORDER

Martin Bunder
School of Mathematics and Applied Statistics, University of Wollongong, Australia
martin.bunder@uow.edu.au

Joseph Tonien
School of Computing and Information Technology, University of Wollongong, Australia
joseph.tonien@uow.edu.au

Received: 2/24/20, Accepted: 12/9/20, Published: 12/16/20

Abstract

Lengyel and Marques have determined $v_2(F_n(k))$, the 2-adic order of the generalized Fibonacci number $F_n(k)$, for $k = 3$ and 4 and partially for $k = 5$. This paper gives new recurrence relations and a means of obtaining explicit formulas for $F_n(k)$. It also evaluates $v_2(F_n(k))$ for all k, except for a few cases where $n = t(k + 1)$. In the $k = 5$ case, the paper confirms part of a conjecture of Lengyel and Marques and shows that the rest of the conjecture fails.

1. Introduction

For a fixed integer $k \geq 2$, the generalized Fibonacci numbers are defined by

$$F_0(k) = 0, F_1(k) = \cdots = F_{k-1}(k) = 1,$$

$$F_n(k) = F_{n-1}(k) + \cdots + F_{n-k}(k), \text{ for all } n \geq k.$$

The p-adic order of an integer n is given by $v_p(n)$, which is the highest power of a prime p that divides n.

The p-adic order of a Fibonacci number, $v_p(F_n(2))$, was completely characterized; see [6, 8, 1, 2, 3]. For instance, from Lengyel [2] we have

$$v_2(F_n(2)) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 6 \pmod{12}, \\ v_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Lengyel and Marques, in [5] and [4], obtain a general formula for $v_2(F_n(3))$ and $v_2(F_n(4))$ and a partial one for $v_2(F_n(5))$:

$$v_2(F_n(3)) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 4, \\ 1, & \text{if } n \equiv 3,11 \pmod{16}, \\ 2, & \text{if } n \equiv 4,8 \pmod{16}, \\ 3, & \text{if } n \equiv 7 \pmod{16}, \\ v_2(n)-1, & \text{if } n \equiv 0 \pmod{16}, \\ v_2(n+4)-1, & \text{if } n \equiv 12 \pmod{16}, \\ v_2(n+1)+v_2(n+17)-3, & \text{if } n \equiv 15 \pmod{16}; \end{cases}$$

$$v_2(F_n(4)) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 5, \\ 1, & \text{if } n \equiv 5 \pmod{10}, \\ v_2(n)+2, & \text{if } n \equiv 0 \pmod{10}; \end{cases}$$

$$v_2(F_n(5)) = \begin{cases} 0, & \text{if } n \not\equiv 0,5 \pmod 6, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ v_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ v_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } v_2(n-6) \neq 3. \end{cases}$$

Lengyel and Marques [4] made the following conjectured values of $v_2(F_n(5))$.

Conjecture 1 (Lengyel-Marques)**.** For any positive integer n,

$$v_2(F_n(5)) = \begin{cases} v_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } v_2(n+2) < 8; \\ v_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } v_2(n+2) \geq 8. \end{cases}$$

For a general value of k, Lengyel and Marques [4] also made the following conjectured values of $v_2(F_n(k))$ when $n \equiv 0 \pmod{2k+2}$.

Conjecture 2 (Lengyel-Marques)**.** If $n \equiv 0 \pmod{2k+2}$, then

$$v_2(F_n(k)) = v_2(\frac{n}{k+1}) + c(k),$$

where

$$c(k) = \begin{cases} 2, & \text{if } k = 2; \\ 2, & \text{if } k \equiv 0 \pmod 4, \\ 1, & \text{if } k \equiv 1 \pmod 4, \\ v_2(k-2)+1, & \text{if } k \equiv 2 \pmod 8, \\ 1, & \text{if } k \equiv 3 \pmod 8, \\ 3, & \text{if } k \equiv 6 \pmod 8, \\ 1, & \text{if } k \equiv 7 \pmod 8. \end{cases}$$

Conjecture 2 is correct for $k = 2$ and 3. Lengyel and Marques [4] proved it for $k = 4$ and 5.

Sobolewski [7] showed that for all even $k \geq 4$,

$$v_2(F_n(k)) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ v_2(n) + v_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2}, \end{cases}$$

and thus proved Conjecture 2 for all even $k \geq 4$.

Young [9] showed that for all odd $k \geq 5$,

$$v_2(F_n(k)) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ v_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ v_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ v_2(n) - v_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}, \\ v_2(n-k-1), & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \quad v_2(n-k-1) < v_2(k^2-1), \\ v_2(n-2) + 1, & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \quad v_2(n-k-1) > v_2(k^2-1), \end{cases}$$

and thus confirmed part of Conjecture 2. Young [9] also showed that Conjecture 2 failed for $n = 3102462$ and $n = 6248190$.

This paper obtains, for any positive integer k, new recurrence relations for $F_n(k)$, using which, $F_n(k)$ can be expressed as a simple formula for each value of n and the 2-adic order $v_2(F_n(k))$ can be derived for all values of n except for some where $n = t(k+1)$.

Our work allows us to prove Conjecture 2 for all $k \geq 3$ in the following compact form (see Theorem 9).

Theorem 1. *For any $k \geq 3$, if $n \equiv 0 \pmod{2k+2}$, then*

$$v_2(F_n(k)) = v_2(\frac{n}{k+1}) + v_2(k-2) + 1.$$

When k is even, we have the following theorem (see Theorems 6, 7, 9, 10 and 11).

Theorem 2. *For k even,*

$$v_2(F_n(k)) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ v_2(n) + v_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2} \text{ and } k > 2, \\ v_2(n) + 2, & \text{if } n \equiv 0 \pmod{2k+2} \text{ and } k = 2. \end{cases}$$

When k is odd, we have the following theorem (see Theorems 6, 7, 8, 9 and 12).

Theorem 3. *For k odd,*

$$v_2(F_n(k)) = \begin{cases} 0, & \text{if } n \not\equiv -1, 0 \pmod{k+1}, \\ v_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ v_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2} \text{ and } k > 3, \\ v_2(n+1) + v_2(n+17) - 3, & \text{if } n \equiv -1 \pmod{2k+2} \text{ and } k = 3, \\ v_2(n) - v_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}, \\ \\ v_2((k-2)n + k + 1) & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ \quad -v_2(k+1) + 1, & v_2((k-2)n + k + 1) < k + v_2(k+1). \end{cases}$$

We note that there are far fewer cases not dealt with in Young [9] (where $n \equiv k+1 \pmod{2k+2}$) than not dealt with in Theorem 3. For example, when $k = 5$, the cases not dealt with by Young have $n \equiv 30 \pmod{48}$, while the cases not dealt with in Theorem 3 have $n \equiv 126 \pmod{192}$. However, Theorem 14 evaluates $v_2(F_n(5))$ for all values of n other than $n \equiv 5886 \pmod{24576}$, thus verifying Conjecture 1 in all these cases. We notice that Conjecture 1 is not correct when $n = 3102462$; in this particular case, the conjectured value is $v_2(n + 43266) = 20$ but the correct value is $v_2(F_n(5)) = 22$. This fact was also previously observed by Young [9].

In our calculation of $v_2(F_n(k))$, we will make use of the binary digit sum function $s_2(n)$ which is defined as follows.

Definition 1. Define $s_2(0) = 0$. If $i \geq 1$ and $n_1 > n_2 > \cdots > n_i \geq 0$, then

$$s_2(2^{n_1} + 2^{n_2} + \cdots + 2^{n_i}) = i.$$

Lemma 1. *We have*

(i) $v_2(n!) = n - s_2(n)$.

(ii) $v_2(\binom{m}{n}) = s_2(n) + s_2(m-n) - s_2(m)$.

(iii) If $1 \leq n < 2^{v_2(m)}$, *then* $s_2(m-n) = s_2(m) + v_2(m) - s_2(n) - v_2(n)$.

(iv) For $n > 0$, $s_2(n-1) = s_2(n) + v_2(n) - 1$.

(v) For $n > m$, $s_2(n) - s_2(m) = n - m - \sum_{i=m+1}^{n} v_2(i)$.

Proof. (i) The result is derived from a well known formula

$$v_p(n!) = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ is the digit sum of n in base p.

(ii) Using (i), we have $v_2(\binom{m}{n}) = m - s_2(m) - (n - s_2(n)) - (m - n - s_2(m-n)) = s_2(n) + s_2(m-n) - s_2(m)$.

(iii) As $1 \leq n < 2^{v_2(m)}$ we have $m = 2^{m_1} + \cdots + 2^{m_i}$ and $n = 2^{n_1} + \cdots + 2^{n_j}$ where $i = s_2(m)$, $m_i = v_2(m)$, $j = s_2(n)$, $n_j = v_2(n)$ and $m_1 > \cdots > m_i > n_1 > \cdots > n_j$. Therefore,

$$m - n = 2^{m_1} + \cdots + 2^{m_{i-1}} + 2^{m_i - 1} + 2^{m_i - 2} + \cdots + 2^{n_j} + 2^{n_j} - (2^{n_1} + \cdots + 2^{n_j}),$$

so $s_2(m-n) = (i-1) + (m_i - n_j) + 1 - j = i + m_i - n_j - j = s_2(m) + v_2(m) - v_2(n) - s_2(n)$.

(iv) If n is odd then it is obvious. If n is even, then let $v_2(n) = v$ and write n in binary form $n = x_1 \ldots x_i 100 \ldots 0_2$ with v trailing zeros. We have $n - 1 = x_1 \ldots x_i 011 \ldots 1_2$ so $s_2(n-1) = s_2(n) - 1 + v$.

(v) The result follows from (iv). □

2. Recurrence Relations and Formulas for $F_n(k)$

Lemma 2. *For any $k \geq 2$, we have*

(i) $F_k(k) = k - 1$.

(ii) If $n \geq k + 1$,

$$F_n(k) = 2F_{n-1}(k) - F_{n-k-1}(k).$$

(iii) If $n \geq k + 1$,

$$F_n(k) = 2^{n-k}(k-1) - \sum_{i=1}^{n-k-1} 2^{n-i-k-1} F_i(k).$$

(iv) If $n \geq 2k + 1$,

$$F_n(k) = 2^{n-k-1}(2k-3) - 2^{n-2k-1}(k-3) - \sum_{i=k+1}^{n-k-1} 2^{n-k-1-i} F_i(k).$$

Proof. (i) Obvious.

(ii) If $n \geq k + 1$,

$$\begin{aligned} F_n(k) &= F_{n-1}(k) + F_{n-2}(k) + \cdots + F_{n-k}(k) \\ F_{n-1}(k) &= F_{n-2}(k) + \cdots + F_{n-k}(k) + F_{n-k-1}(k), \end{aligned}$$

so $F_n(k) = 2F_{n-1}(k) - F_{n-k-1}(k)$.

(iii) From (ii), we have

$$2^{-(j-1)} F_{j-1}(k) - 2^{-j} F_j(k) = 2^{-j} F_{j-k-1}(k), \forall j \geq k + 1.$$

If $n \geq k+1$, then

$$\sum_{j=k+1}^{n} (2^{-(j-1)} F_{j-1}(k) - 2^{-j} F_j(k)) = \sum_{j=k+1}^{n} 2^{-j} F_{j-k-1}(k),$$

$$2^{-k} F_k(k) - 2^{-n} F_n(k) = \sum_{i=0}^{n-k-1} 2^{-i-k-1} F_i(k).$$

Therefore,

$$F_n(k) = 2^{n-k} F_k(k) - \sum_{i=0}^{n-k-1} 2^{n-i-k-1} F_i(k) = 2^{n-k}(k-1) - \sum_{i=1}^{n-k-1} 2^{n-i-k-1} F_i(k).$$

(iv) If $n \geq 2k+1$, using (iii), we have

$$\begin{aligned}
F_n(k) &= 2^{n-k}(k-1) - 2^{n-2k-1} F_k(k) - \sum_{i=1}^{k-1} 2^{n-i-k-1} F_i(k) - \sum_{i=k+1}^{n-k-1} 2^{n-i-k-1} F_i(k) \\
&= 2^{n-k}(k-1) - 2^{n-2k-1}(k-1) - \sum_{i=1}^{k-1} 2^{n-i-k-1} - \sum_{i=k+1}^{n-k-1} 2^{n-i-k-1} F_i(k) \\
&= 2^{n-k}(k-1) - 2^{n-2k-1}(k-1) - (2^{n-k-1} - 2^{n-2k}) - \sum_{i=k+1}^{n-k-1} 2^{n-i-k-1} F_i(k) \\
&= 2^{n-k-1}(2k-3) - 2^{n-2k-1}(k-3) - \sum_{i=k+1}^{n-k-1} 2^{n-k-1-i} F_i(k).
\end{aligned}$$

□

The following theorem gives an explicit formula for $F_n(k)$.

Theorem 4. *Let $n \geq k+1$ and $t = \lfloor \frac{n+1}{k+1} \rfloor$. Then*

$$F_n(k) = 1 + 2^{n-(k+1)} A_1(n,k) + 2^{n-2(k+1)} A_2(n,k) + \cdots + 2^{n-t(k+1)} A_t(n,k), \quad (1)$$

where $A_1(n,k), A_2(n,k), A_3(n,k), \ldots, A_t(n,k)$ are integers defined as

$$\begin{aligned}
A_s(n,k) &= \begin{cases} 2k-3, & \text{if } s = 1, \\ (-1)^{s-1}\left((2k-3)\binom{n-sk}{s-1} - \binom{n-sk-1}{s-2}\right), & \text{if } s > 1 \end{cases} \\
&= (-1)^{s-1} \frac{(n-sk-1)!\ ((2k-3)n - s(2k^2-3k+1)+1)}{(s-1)!(n-sk-s+1)!}. \quad (2)
\end{aligned}$$

Proof. When $k+1 \le n < 2(k+1)-1$, using Lemma 2(iii),

$$\begin{aligned} F_n(k) &= 2^{n-k}(k-1) - \sum_{i=1}^{n-k-1} 2^{n-i-k-1} F_i(k) \\ &= 2^{n-k}(k-1) - \sum_{i=1}^{n-k-1} 2^{n-i-k-1} = 1 + 2^{n-(k+1)}(2k-3). \end{aligned}$$

When $n \ge 2(k+1)-1$, the result can be proved by induction on n by using Lemma 2(ii) and by considering the two separate cases: $n = t(k+1)-1$ and $t(k+1)-1 < n < (t+1)(k+1)-1$. □

Using Theorem 4, we can calculate all the values of $F_n(k)$. Below are some examples.

Lemma 3. *We have*

(i) If $k+1 \le n \le 2k$,

$$F_n(k) = 2^{n-(k+1)}(2k-3) + 1.$$

(ii) If $2k+1 \le n \le 3k+1$,

$$F_n(k) = 2^{n-(k+1)}(2k-3) - 2^{n-2(k+1)}((2k-3)n - 4k^2 + 6k - 1) + 1.$$

(iii) If $3k+2 \le n \le 4k+2$,

$$\begin{aligned} F_n(k) =& 2^{n-(k+1)}(2k-3) - 2^{n-2(k+1)}((2k-3)n - 4k^2 + 6k - 1) \\ &+ 2^{n-3(k+1)}\frac{1}{2}(n-3k-1)((2k-3)n - 6k^2 + 9k - 2) + 1. \end{aligned}$$

(iv) If $4k+3 \le n \le 5k+3$,

$$\begin{aligned} F_n(k) = {}& 2^{n-(k+1)}(2k-3) - 2^{n-2(k+1)}((2k-3)n - 4k^2 + 6k - 1) \\ &+ 2^{n-3(k+1)}\frac{1}{2}(n-3k-1)((2k-3)n - 6k^2 + 9k - 2) \\ &- 2^{n-4(k+1)}\frac{1}{6}(n-4k-1)(n-4k-2)((2k-3)n - 8k^2 + 12k - 3) + 1. \end{aligned}$$

3. The 2-adic Order of $F_n(k)$

Our method of finding $v_2(F_n(k))$, for $n \ge k+1$, is mainly based on formula (1) of Theorem 4. Throughout this paper, we will write $t = \lfloor \frac{n+1}{k+1} \rfloor$ and $n = t(k+1) + r$ where $-1 \le r \le k-1$. In formula (1), all indices in the power of 2 are greater than or equal to -1. The lowest index $n-t(k+1)$ is negative only in exactly one case, $n=t(k+1)-1$. In this case,

$$2^{n-t(k+1)}A_t(n,k) = (-1)^{t-1}(k-2)$$

is an integer and we have the following theorem.

Theorem 5. *Let $n \geq k+1$ and $t = \lfloor \frac{n+1}{k+1} \rfloor$ and*

$$\begin{aligned} F_n(k) &= 2^{n-(k+1)}A_1(n,k) + 2^{n-2(k+1)}A_2(n,k) + \cdots + (2^{n-t(k+1)}A_t(n,k)+1), \\ &= f_{t-1}(n,k) + f_{t-2}(n,k) + \cdots + f_0(n,k), \end{aligned} \tag{3}$$

where $f_0(n,k) = 2^{n-t(k+1)}A_t(n,k)+1$ and $f_i(n,k) = 2^{n-(t-i)(k+1)}A_{t-i}(n,k)$. Then
(i) all the values $f_i(n,k)$ are integers;
(ii) for any $j \geq 0$, if $t \geq j+1$ and for all $j+1 \leq i \leq t-1$, we have

$$v_2(f_i(n,k)) > v_2(f_j(n,k) + f_{j-1}(n,k) + \cdots + f_0(n,k)), \tag{4}$$

then

$$v_2(F_n(k)) = v_2(f_j(n,k) + f_{j-1}(n,k) + \cdots + f_0(n,k)).$$

Proof. (i) In formula (3), all indices in the power of 2 are greater than or equal to -1. The lowest index $n-t(k+1)$ is negative only in exactly one case, $n = t(k+1)-1$. In this case,

$$2^{n-t(k+1)}A_t(n,k) = (-1)^{t-1}(k-2)$$

is an integer, and therefore, all $f_i(n,k)$ are integers.
(ii) The result follows from (i). □

As $n = t(k+1)+r$, $v_2(f_i(n,k)) = i(k+1) + r + v_2(A_{t-i}(n,k)) \geq i(k+1)+r$, the following result is a direct corollary of Theorem 5(ii).

Corollary 1. *Let $n \geq k+1$, $t = \lfloor \frac{n+1}{k+1} \rfloor$ and $n = t(k+1)+r$. For any $j \geq 0$, if $t \geq j+1$ and $(j+1)(k+1)+r > v_2(f_j(n,k) + f_{j-1}(n,k) + \cdots + f_0(n,k))$, then*

$$v_2(F_n(k)) = v_2(f_j(n,k) + f_{j-1}(n,k) + \cdots + f_0(n,k)).$$

Using formula (2) for $A_s(n,k)$, and the formulas for $v_2(n!)$ and $s_2(n) - s_2(m)$ from Lemma 1, we can calculate $f_0(n,k)$ and $v_2(f_i(n,k))$.

Lemma 4. *Let $n \geq k+1$, $t = \lfloor \frac{n+1}{k+1} \rfloor$ and $n = t(k+1)+r$. Then*

$$f_0(n,k) = \begin{cases} (-1)^{t-1}\ (k-2)+1, & \text{if } r=-1, \\ (-1)^{t-1}\ (t(2k-4)+1)+1, & \text{if } r=0, \\ 2^r A_t(n,k)+1, & \text{if } 1 \leq r \leq k-1, \end{cases}$$

and for $1 \leq i \leq t-1$ *and* $-1 \leq r \leq k-1$,

$$v_2(f_i(n,k)) = v_2\Big(2t(k-2) + i(2k^2-3k+1) + 1 + r(2k-3)\Big) + s_2(ik+i+1+r) - 1 + \sum_{\ell=t-i}^{t+ik+r-1} v_2(\ell).$$

We will use Theorem 5 to determine $v_2(F_n(k))$ for $n \geq k+1$. With $n = t(k+1)+r$ where $-1 \leq r \leq k-1$, there are different formulas for $v_2(F_n(k))$ depending on the value of r. When $r \neq 0$, that is, $n \not\equiv 0 \pmod{k+1}$, it is quite easy to determine $v_2(F_n(k))$. The case $n \equiv 0 \pmod{k+1}$ requires more work. Theorem 6 deals with the case $1 \leq r \leq k-1$, that is, $n \not\equiv -1, 0 \pmod{k+1}$. Theorem 7 deals with the case $n \equiv -1 \pmod{k+1}$. Finally, Theorems 9 and 11 deal with the remaining case $n \equiv 0 \pmod{k+1}$.

3.1. The Case $n \not\equiv 0 \pmod{k+1}$

Theorem 6. *If* $n \not\equiv -1, 0 \pmod{k+1}$, *then* $v_2(F_n(k)) = 0$.

Proof. It is obvious for the case $n \leq k$. When $n \geq k+1$ and $n \not\equiv -1, 0 \pmod{k+1}$, we have $n = t(k+1)+r$ where $1 \leq r \leq k-1$. In this case, by Lemma 4, $f_0(n,k)$ is an odd integer and $v_2(f_0(n,k)) = 0$. By Corollary 1 with $j = 0$ we have $v_2(F_n(k)) = v_2(f_0(n,k)) = 0$. $\square$

Theorem 7. *If* $n \equiv -1 \pmod{k+1}$, *then*

$$v_2(F_n(k)) = \begin{cases} v_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2} \text{ and } k \neq 3, \\ v_2(k-1), & \text{if } n \equiv k \pmod{2k+2}. \end{cases}$$

Proof. It is obvious for the case $n = k$ as $F_k(k) = k-1$. If $n \geq k+1$ and $n \equiv -1 \pmod{k+1}$, we have $n = t(k+1)-1$ with $t \geq 2$. The case t is even corresponds to $n \equiv -1 \pmod{2k+2}$ and t is odd corresponds to $n \equiv k \pmod{2k+2}$. By Lemma 4,

$$f_0(n,k) = \begin{cases} -(k-3), & \text{if } t \text{ is even}, \\ k-1, & \text{if } t \text{ is odd}. \end{cases}$$

When t is odd, $v_2(k-1) < k$, so by Corollary 1 with $j = 0$ we have $v_2(F_n(k)) = v_2(f_0(n,k)) = v_2(k-1)$.

When t is even and $k \neq 3$, $v_2(k-3) < k$, so again by Corollary 1 with $j = 0$ we have $v_2(F_n(k)) = v_2(f_0(n,k)) = v_2(k-3)$. $\square$

Theorem 7 does not cover the case $k = 3$ and $n \equiv -1 \pmod 8$ so we will look at this exceptional case now.

Theorem 8. *If* $n \equiv -1 \pmod 8$, *then* $v_2(F_n(3)) = v_2(n+1) + v_2(n+17) - 3$.

Proof. Write $n = 4t - 1$ with t even. By Lemma 4, $f_0(n,3) = 0$, $v_2(f_1(n,3)) = 1 + v_2(t) + v_2(t+4)$, and for any $i \geq 2$,

$$\begin{aligned} v_2(f_i(n,3)) &= v_2(2t+10i-2) + s_2(4i) - 1 + \sum_{\ell=t-i}^{t+3i-2} v_2(\ell) \\ &\geq 1 + \sum_{\ell=t-i}^{t+3i-2} v_2(\ell) > 1 + v_2(t) + v_2(t+4). \end{aligned}$$

By Theorem 5(ii) with $j = 1$ we obtain $v_2(F_n(3)) = v_2(f_1(n,3)) = 1 + v_2(t) + v(t+4) = 1 + v_2(\frac{n+1}{4}) + v(\frac{n+1}{4} + 4) = v_2(n+1) + v_2(n+17) - 3$. $\square$

3.2. The Case $n = t(k+1)$

In this section we consider the case $n \geq k+1$ and $n \equiv 0 \pmod{k+1}$. So $n = t(k+1)$. We consider two separate cases: t even and t odd. Theorem 9 deals with the case t is even where we prove Conjecture 2 of Lengyel and Marques.

3.2.1. t is Even

The case t is even corresponds to $n \equiv 0 \pmod{2k+2}$. We will prove Conjecture 2 of Lengyel and Marques for all $k \geq 3$ in the following simpler form.

Theorem 9. *For any $k \geq 3$, if $n = t(k+1)$ and t is even, then*

$$v_2(F_n(k)) = v_2(2t(k-2)) = v_2(n) - v_2(k+1) + v_2(k-2) + 1.$$

Proof. When t is even, by Lemma 4, we have

$$f_0(n,k) = -2t(k-2),$$

and for all $1 \leq i \leq t-1$,

$$v_2(f_i(n,k)) = v_2\big(2t(k-2)+i(2k^2-3k+1)+1\big)+s_2(ik+i+1)-1+\sum_{\ell=t-i}^{t+ik-1} v_2(\ell). \quad (5)$$

When k is odd, $v_2(f_i(n,k)) \geq 1 + \sum_{\ell=t-i}^{t+ik-1} v_2(\ell) > 1 + v_2(t) = v_2(f_0(n,k))$.

When k is even, we consider two cases: i odd and i even.

If i is even, then $v_2(f_i(n,k)) \geq 1 + \sum_{\ell=t-i}^{t+ik-1} v_2(\ell)$. There are $\frac{i+ik}{2}$ even numbers ℓ in the interval $t - i \leq \ell \leq t + ik - 1$ and one of those is t. So $v_2(f_i(n,k)) \geq 1 + v_2(t) + \frac{i+ik}{2} - 1 \geq 1 + v_2(t) + k > 1 + v_2(t) + v_2(k-2) = v_2(f_0(n,k))$.

If i is odd, then $v_2(f_i(n,k)) \geq 1 + \sum_{\ell=t-i}^{t+ik-1} v_2(\ell)$. There are $\frac{i+ik-1}{2}$ even numbers ℓ in the interval $t - i \leq \ell \leq t + ik - 1$ and one of those is t. So $v_2(f_i(n,k)) \geq$

$1 + v_2(t) + \frac{i+ik-1}{2} - 1 > 1 + v_2(t) + v_2(k-2) = v_2(f_0(n,k))$ for all $i \geq 3$, or $i = 1$ and $k > 6$. The remaining case is $i = 1$ and $k = 4$ or 6.

If $i = 1$ and $k = 4$ equation (5) gives $v_2(f_i(n,k)) = v_2(4t + 22) + s_2(6) - 1 + v_2(t) + v_2(t+2) > v_2(t) + 2 = v_2(f_0(n,k))$.

If $i = 1$ and $k = 6$ equation (5) gives $v_2(f_i(n,k)) = v_2\big(8t + 56\big) + s_2(8) - 1 + v_2(t) + v_2(t+2) + v_2(t+4) > v_2(t) + 3 = v_2(f_0(n,k))$.

In every case, we have shown that $v_2(f_i(n,k)) > v_2(f_0(n,k))$ for all $1 \leq i \leq t-1$, and by Theorem 5(ii) with $j = 0$, we have $v_2(F_n(k)) = v_2(f_0(n,k)) = v_2(2t(k-2))$. $\square$

Theorem 9 does not cover the case $k = 2$ so we will look at this exceptional case now.

Theorem 10. *If $n \equiv 0 \pmod 6$, then $v_2(F_n(2)) = v_2(n) + 2$.*

Proof. Write $n = 3t$ with t even. By Lemma 4, $f_0(n,2) = 0$, $v_2(f_1(n,2)) = 2 + v_2(t)$, and for any $i \geq 2$,

$$v_2(f_i(n,2)) = v_2(3i+1) + s_2(3i+1) - 1 + \sum_{\ell=t-i}^{t+2i-1} v_2(\ell) > 2 + v_2(t).$$

By Theorem 5(ii) with $j = 1$ we obtain $v_2(F_n(2)) = v_2(f_1(n,2)) = 2 + v_2(t) = 2 + v_2(n)$. $\square$

3.2.2. t is Odd

In this section we consider the case $n = t(k+1)$ and t is odd, that is, $n \equiv k+1 \pmod{2k+2}$. By Lemma 4,

$$f_0(n,k) = 2(t(k-2)+1).$$

Theorem 11. *If k is even and $n \equiv k+1 \pmod{2k+2}$, then $v_2(F_n(k)) = 1$.*

Proof. Write $n = t(k+1)$ where t is odd. By Lemma 4, $f_0(n,k) = 2(t(k-2)+1)$ and as k is even, we have $v_2(f_0(n,k)) = 1$. By Corollary 1 with $j = 0$ we have $v_2(F_n(k)) = v_2(f_0(n,k)) = 1$. $\square$

From here on, we consider the case k is odd.

Lemma 5. *If k is odd, $n \equiv k+1 \pmod{2k+2}$, and*

$$n \not\equiv (k+1)\left(\frac{(k-1)^{\lfloor \frac{k-1}{v_2(k-1)} \rfloor} - 1}{k-2} + 2^{\lfloor \frac{k-1}{v_2(k-1)} \rfloor v_2(k-1)}\right) \pmod{(k+1)2^{\lfloor \frac{k-1}{v_2(k-1)} \rfloor v_2(k-1)+1}}, \tag{6}$$

then
$$v_2(F_n(k)) = v_2((k-2)n+k+1) - v_2(k+1) + 1.$$

Proof. Let $m = \left\lfloor \frac{k-1}{v_2(k-1)} \right\rfloor$. Then $m\, v_2(k-1) < k$ and by Lemma 4,

$$f_0(n,k) = 2(t(k-2)+1) = 2\left(\left(t - \frac{(k-1)^m - 1}{k-2}\right)(k-2) + (k-1)^m\right).$$

If $t - \frac{(k-1)^m-1}{k-2} = 0$ or, if $t - \frac{(k-1)^m-1}{k-2} \neq 0$ and $v_2\left(t - \frac{(k-1)^m-1}{k-2}\right) \neq m\, v_2(k-1)$, then $v_2(f_0(n,k)) \leq 1 + m\, v_2(k-1) < k+1$. In this case,

$$v_2(F_n(k)) = v_2(f_0(n,k)) = v_2(2(t(k-2)+1)) = v_2((k-2)n+k+1) - v_2(k+1) + 1.$$

The remaining case is when $v_2\left(t - \frac{(k-1)^m-1}{k-2}\right) = mv_2(k-1)$. Write

$$t - \frac{(k-1)^m - 1}{k-2} = 2^{mv_2(k-1)}(2Z+1).$$

Then

$$n = (k+1)\left(\frac{(k-1)^m - 1}{k-2} + 2^{mv_2(k-1)}\right) + (k+1)2^{mv_2(k-1)+1}Z,$$

which contradicts the assumption and the lemma is proved. $\square$

Lemma 6. *If k is odd, $k > 3$, $n \equiv k+1 \pmod{2k+2}$, and*

$$n \not\equiv (k+1)\left(\frac{(k-3)^{2\lfloor \frac{k-1}{2v_2(k-3)} \rfloor} - 1}{k-2} + 2^{2\lfloor \frac{k-1}{2v_2(k-3)} \rfloor v_2(k-3)}\right) \pmod{(k+1)2^{2\lfloor \frac{k-1}{2v_2(k-3)} \rfloor v_2(k-3)+1}}, \tag{7}$$

then
$$v_2(F_n(k)) = v_2((k-2)n+k+1) - v_2(k+1) + 1.$$

Proof. Let $m = 2\left\lfloor \frac{k-1}{2v_2(k-3)} \right\rfloor$. Then m is even and $mv_2(k-3) < k$, and by Lemma 4,

$$f_0(n,k) = 2(t(k-2)+1) = 2\left(\left(t - \frac{(k-3)^m - 1}{k-2}\right)(k-2) + (k-3)^m\right).$$

If $t - \frac{(k-3)^m-1}{k-2} = 0$ or, if $t - \frac{(k-3)^m-1}{k-2} \neq 0$ and $v_2\left(t - \frac{(k-3)^m-1}{k-2}\right) \neq mv_2(k-3)$, then $v_2(f_0(n,k)) \leq 1 + mv_2(k-3) < k+1$. In this case,

$$v_2(F_n(k)) = v_2(f_0(n,k)) = v_2(2(t(k-2)+1)) = v_2((k-2)n+k+1) - v_2(k+1) + 1.$$

The remaining case is when $v_2\Big(t - \frac{(k-3)^m-1}{k-2}\Big) = mv_2(k-3)$. Write

$$t - \frac{(k-3)^m - 1}{k-2} = 2^{mv_2(k-3)}(2Z+1).$$

Then

$$n = (k+1)\left(\frac{(k-3)^m-1}{k-2} + 2^{mv_2(k-3)}\right) + (k+1)2^{mv_2(k-3)+1}Z,$$

which contradicts the assumption and the lemma is proved. $\square$

Theorem 12. *If k is odd, $n \equiv k+1 \pmod{2k+2}$, and*

$$v_2((k-2)n + k + 1) < k + v_2(k+1),$$

then

$$v_2(F_n(k)) = v_2((k-2)n + k + 1) - v_2(k+1) + 1.$$

Proof. When $k \equiv 1 \pmod 4$ and $n \equiv k+1 \pmod{2k+2}$, let $n = (k+1)(2T+1)$ and $k = 1 + 4K$. Then the condition (7) of Lemma 6 is equivalent to

$$n \not\equiv (k+1)\left(\frac{(k-3)^{k-1}-1}{k-2} + 2^{k-1}\right) \pmod{(k+1)2^k}$$

$$\Leftrightarrow \quad 2T+1 \not\equiv \frac{(k-3)^{k-1}-1}{k-2} + 2^{k-1} \pmod{2^k}$$
$$\Leftrightarrow \quad (2T+1)(k-2) \not\equiv (k-3)^{k-1} - 1 + 2^{k-1}(k-2) \pmod{2^k}$$
$$\Leftrightarrow \quad (2T+1)(k-2) \not\equiv 2^{k-1}(2K-1)^{4K} - 1 + 2^{k-1}(4K-1) \pmod{2^k}$$
$$\Leftrightarrow \quad (2T+1)(k-2) \not\equiv -1 \pmod{2^k}$$
$$\Leftrightarrow \quad n(k-2) + k + 1 \not\equiv 0 \pmod{(k+1)2^k}$$
$$\Leftrightarrow \quad v_2(n(k-2)+k+1) < k + v_2(k+1).$$

Therefore, the case $k \equiv 1 \pmod 4$ follows from Lemma 6.

When $k \equiv 3 \pmod 4$ and $n \equiv k+1 \pmod{2k+2}$, let $n = (k+1)(2T+1)$ and $k = 3 + 4K$. Then the condition (6) of Lemma 5 is equivalent to

$$n \not\equiv (k+1)\left(\frac{(k-1)^{k-1}-1}{k-2} + 2^{k-1}\right) \pmod{(k+1)2^k}$$

$$\Leftrightarrow \quad 2T+1 \not\equiv \frac{(k-1)^{k-1}-1}{k-2} + 2^{k-1} \pmod{2^k}$$
$$\Leftrightarrow \quad (2T+1)(k-2) \not\equiv (k-1)^{k-1} - 1 + 2^{k-1}(k-2) \pmod{2^k}$$

$$\Leftrightarrow \quad (2T+1)(k-2) \not\equiv 2^{k-1}(2K+1)^{4K+2} - 1 + 2^{k-1}(4K+1) \pmod{2^k}$$
$$\Leftrightarrow \quad (2T+1)(k-2) \not\equiv -1 \pmod{2^k}$$
$$\Leftrightarrow \quad n(k-2)+k+1 \not\equiv 0 \pmod{(k+1)2^k}$$
$$\Leftrightarrow \quad v_2(n(k-2)+k+1) < k + v_2(k+1).$$

Therefore, the case $k \equiv 3 \pmod 4$ follows from Lemma 5. □

Using Theorem 12, we can now prove Young [9] result.

Corollary 2. *If k is odd and $n \equiv k+1 \pmod{2k+2}$, then*

$$v_2(F_n(k)) = \begin{cases} v_2(n-k-1), & \textit{if } v_2(n-k-1) < v_2(k^2-1), \\ v_2(n-2)+1, & \textit{if } v_2(n-k-1) > v_2(k^2-1). \end{cases}$$

Proof. Since

$$(k-2)n+k+1 = (k-2)(n-k-1)+k^2-1,$$

and k is odd, we have

$$v_2((k-2)n+k+1) = \begin{cases} v_2(n-k-1), & \text{if } v_2(n-k-1) < v_2(k^2-1), \\ v_2(k^2-1), & \text{if } v_2(n-k-1) > v_2(k^2-1). \end{cases}$$

Let $n = (k+1)(2T+1)$ and consider two separate cases.

Case 1: $v_2(k^2-1) > v_2(n-k-1)$. It follows that $v_2(k-1) > v_2(\frac{n}{k+1}-1) = v_2(2T)$ and so $k \equiv 1 \pmod 4$. We have

$$\begin{aligned} v_2((k-2)n+k+1) &= v_2(n-k-1) \\ &< v_2(k^2-1) = v_2(k-1)+v_2(k+1) \\ &< k + v_2(k+1), \end{aligned}$$

therefore, by Theorem 12,

$$\begin{aligned} v_2(F_n(k)) &= v_2((k-2)n+k+1) - v_2(k+1) + 1 \\ &= v_2(n-k-1) - 1 + 1 \\ &= v_2(n-k-1). \end{aligned}$$

Case 2: $v_2(n-k-1) > v_2(k^2-1)$. It follows that $v_2(k-1) < v_2(\frac{n}{k+1}-1) = v_2(2T) < v_2(4T)$, and

$$v_2(n-2) = v_2((k-1)(2T+1)+4T) = v_2(k-1).$$

We have

$$\begin{aligned} v_2((k-2)n+k+1) &= v_2(k^2-1) = v_2(k-1)+v_2(k+1) \\ &< k + v_2(k+1), \end{aligned}$$

therefore, by Theorem 12,

$$\begin{aligned} v_2(F_n(k)) &= v_2((k-2)n+k+1) - v_2(k+1) + 1 \\ &= v_2(k^2-1) - v_2(k+1) + 1 \\ &= v_2(k-1) + 1 \\ &= v_2(n-2) + 1. \end{aligned}$$

□

3.3. The $k = 3$ and $k = 5$ Cases

Theorem 13. *For any positive integer n, we have*

$$v_2(F_n(3)) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 4, \\ 1, & \text{if } n \equiv 3 \pmod 8, \\ v_2(n+1) + v_2(n+17) - 3, & \text{if } n \equiv 7 \pmod 8, \\ v_2(n) - 1, & \text{if } n \equiv 0 \pmod 8, \\ v_2(n+4) - 1, & \text{if } n \equiv 4 \pmod 8. \end{cases}$$

Proof. The case $n \equiv 1, 2 \pmod 4$ is by Theorem 6. The case $n \equiv 3 \pmod 8$ is by Theorem 7. The case $n \equiv 7 \pmod 8$ is by Theorem 8. The case $n \equiv 0 \pmod 8$ is by Theorem 9.

In the case $n \equiv 4 \pmod 8$, write $n = 4t$ where t is odd. By Lemma 4, $f_0(n, 3) = 2(t+1)$ and for any $i \geq 1$,

$$v_2(f_i(n,3)) = s_2(4i+1) - 1 + \sum_{\ell=t-i}^{t+3i-1} v_2(\ell).$$

Hence $v_2(f_i(n,3)) > v_2(t+1) + 1 = v_2(f_0(n,3))$, and therefore, $v_2(F_n(3)) = v_2(f_0(n,3)) = v_2(t+1) + 1 = v_2(n+4) - 1$. □

Theorem 14. *For any positive integer n, we have*

$$v_2(F_n(5)) = \begin{cases} 0, & \text{if } \not\equiv 0,5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ v_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ v_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } n \not\equiv 126 \pmod{192}, \\ 6, & \text{if } n \equiv 318 \pmod{384}, \\ 7, & \text{if } n \equiv 126 \pmod{768}, \\ 8, & \text{if } n \equiv 510 \pmod{1536}, \\ 9, & \text{if } n \equiv 1278 \pmod{3072}, \\ 10, & \text{if } n \equiv 2814 \pmod{6144}, \\ 11, & \text{if } n \equiv 12030 \pmod{12288}, \\ 12, & \text{if } n \equiv 18174 \pmod{24576}. \end{cases}$$

Proof. The case $n \not\equiv -1, 0 \pmod 6$ is by Theorem 6. The case $n \equiv -1, 5 \pmod{12}$ is by Theorem 7. The case $n \equiv 0 \pmod{12}$ is by Theorem 9. The case $n \equiv 6 \pmod{12}$, $n \not\equiv 126 \pmod{192}$, is by Theorem 12.

The remaining case is $n \equiv 126 \pmod{192}$. Write $n = 6t$ where $t \equiv 21 \pmod{32}$, that is, $t \equiv 10101_2 \pmod{2^5}$. By Lemma 4, $f_0(n,5) = 2(3t+1)$ and for any $i \geq 1$,

$$v_2(f_i(n,5)) = s_2(6i+1) - 1 + \sum_{\ell=t-i}^{t+5i-1} v_2(\ell).$$

Particularly, we have $v_2(f_1(n,5)) = 2 + v_2(t-1) + v_2(t+1) + v_2(t+3) = 2+2+1+3 = 8$ and $v_2(f_2(n,5)) = 2 + v_2(t-1) + v_2(t+1) + v_2(t+3) + v_2(t+5) + v_2(t+7) + v_2(t+9) = 2+2+1+3+1+2+1 = 12$.

If $t \equiv 110101_2 \pmod{2^6}$, that is, $n \equiv 318 \pmod{384}$, then $v_2(3t+1) = 5$ and $v_2(f_0(n,5)) = 6$. In this case, $v_2(F_n(5)) = 6$.

Now consider the case $t \equiv 010101_2 \pmod{2^6}$. If $t \equiv 0010101_2 \pmod{2^7}$, that is, $n \equiv 126 \pmod{768}$, then $v_2(3t+1) = 6$ and $v_2(f_0(n,5)) = 7$. In this case, $v_2(F_n(5)) = 7$.

Now consider the case $t \equiv 1010101_2 \pmod{2^7}$. If $t \equiv 01010101_2 \pmod{2^8}$, that is, $n \equiv 510 \pmod{1536}$, then $v_2(3t+1) \geq 8$ and $v_2(f_0(n,5)) \geq 9$. As $v_2(f_1(n,5)) = 8$, we have $v_2(f_0(n,5) + f_1(n,5)) = 8$ and $v_2(F_n(5)) = v_2(f_0(n,5) + f_1(n,5)) = 8$.

Now consider the case $t \equiv 11010101_2 \pmod{2^8}$. We have

$$f_0(n,5) + f_1(n,5) = -2^6 \frac{(t+4)(t+3)(t+2)(t+1)t(t-1)(6t+37)}{7!} + 2(3t+1).$$

Let $s_1(n,5) = 315(f_0(n,5) + f_1(n,5))$. Then $v_2(f_0(n,5) + f_1(n,5)) = v_2(s_1(n,5))$. We have

$$s_1(n,5) = -2^2(t+4)(t+3)(t+2)(t+1)t(t-1)(6t+37) + 630(3t+1).$$

Write $t = 11010101_2 + 2^8 A$. Then $s_1(n,5) \equiv 2^9(A+1) \pmod{2^{10}}$.

If A is even, that is, $t \equiv 11010101_2 \pmod{2^9}$ and $n \equiv 1278 \pmod{3072}$, then $v_2(s_1(n,5)) = 9$ and $v_2(F_n(5)) = 9$.

Consider the case A is odd, that is, $t \equiv 111010101_2 \pmod{2^9}$. Write $t = 111010101_2 + 2^9 B$. Then $s_1(n,5) \equiv 2^{10}(B+1) \pmod{2^{11}}$.

If B is even, that is, $t \equiv 111010101_2 \pmod{2^{10}}$ and $n \equiv 2814 \pmod{6144}$, then $v_2(s_1(n,5)) = 10$ and $v_2(F_n(5)) = 10$.

Consider the case B is odd, that is, $t \equiv 1111010101_2 \pmod{2^{10}}$. Write $t = 1111010101_2 + 2^{10} C$. Then $s_1(n,5) \equiv 2^{11} C \pmod{2^{12}}$.

If C is odd, that is, $t \equiv 11111010101_2 \pmod{2^{11}}$ and $n \equiv 12030 \pmod{12288}$, then $v_2(s_1(n,5)) = 11$ and $v_2(F_n(5)) = 11$.

Consider the case C is even, that is, $t \equiv 01111010101_2 \pmod{2^{11}}$. Write $t = 01111010101_2 + 2^{11} D$. Then $s_1(n,5) \equiv 2^{12}(D+1) \pmod{2^{13}}$.

If D is odd, that is, $t \equiv 101111010101_2 \pmod{2^{12}}$ and $n \equiv 18174 \pmod{24576}$, then $v_2(s_1(n,5)) \geq 13$. Since $v_2(f_2(n,5)) = 12$, we have $v_2(f_0(n,5) + f_1(n,5) + f_2(n,5)) = 12$ and $v_2(F_n(5)) = v_2(f_0(n,5) + f_1(n,5) + f_2(n,5)) = 12$. □

In the above proof of Theorem 14, when D is even, $t \equiv 001111010101_2 \pmod{2^{12}}$ and $n \equiv 5886 \pmod{24576}$. This remaining case is not covered by Theorem 14. In this case, Conjecture 1 of Lengyel and Marques does not hold. For instance, when $n = 3102462$, we have $v_2(n+2) = 8$, the conjectured value is $v_2(n + 43266) = 20$ but our calculation shows $v_2(F_n(5)) = 22$.

Acknowledgement. The authors wish to thank the anonymous reviewers for many helpful comments and suggestions that helped us to improve our paper.

References

[1] J. H. Halton, On the divisibility properties of Fibonacci numbers, *Fibonacci Quart.* **4** (1966), 217-240.

[2] T. Lengyel, The order of the Fibonacci and Lucas numbers, *Fibonacci Quart.* **33** (1995), 234-239.

[3] T. Lengyel, Divisibility properties by multisection, *Fibonacci Quart.* **41** (2003), 72-79.

[4] T. Lengyel and D. Marques, The 2-adic order of some generalized Fibonacci numbers, *Integers* **17**(2017), #A5.

[5] D. Marques and T. Lengyel, The 2-adic order of the Tribonacci numbers and the equation $T_n = m!$, *J. Integer Seq.* **17**(2014), Article 14.10.1.

[6] D. W. Robinson, The Fibonacci matrix modulo m, *Fibonacci Quart.* **1** (1963), 29-36.

[7] B. Sobolewski, The 2-adic valuation of generalized Fibonacci sequences with an application to certain Diophantine equations, *J. Number Theory* **180** (2017), 730-742.

[8] J. Vinson, The relation of the period modulo m to the rank of apparition of m in the Fibonacci sequence, *Fibonacci Quart.* **1** (1963), 37-45.

[9] P. T. Young, 2-adic valuations of generalized Fibonacci numbers of odd order, *Integers* **18** (2018), #A1.