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# TOTAL POSITIVITY OF TOEPLITZ MATRICES OF HYPERLUCAS SEQUENCE

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**Abstract**

In this paper, we study various positivity results for several classes of Toeplitz matrices associated with the sequence of hyperlucas numbers $L_n^{[r]}$ of the $r$-th generation. In particular, we show that such matrices with even-indexed hyperlucas numbers on the main diagonal are totally positive for large enough values of index $n$.

## 1. Introduction

The total positivity of Toeplitz and Hankel matrices are important properties of recurrence sequences, and the study of these properties appears in various areas such as orthogonal polynomials, combinatorics, algebraic geometry, stochastic processes, game theory, differential equations, representation theory, Brownian motion, electrical networks, and chemistry; see [6, 7, 16, 17, 19, 20]. A Toeplitz matrix $T = [t_{i,j}]$ is a (finite or infinite) matrix whose entries satisfy $t_{i,j} = t_{i+1,j+1}$. In the finite case,

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n} \\ t_1 & t_0 & \cdots & t_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_n & t_{n-1} & \cdots & t_0 \end{pmatrix}.$$

A Hankel matrix $H = [h_{i,j}]$ is a (finite or infinite) matrix whose entries satisfy $t_{i,j} = t_{i+1,j-1}$. In the finite case,

$$H = \begin{pmatrix} h_0 & h_1 & \cdots & h_n \\ h_1 & h_2 & \cdots & h_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n} \end{pmatrix}.$$

An infinite (real) matrix $A = [a_{i,j}]$ is said to be totally positive (or, TP, for short) if every minor of $A$ has positive determinant. It is called totally positive of order $r$ (or, $\mathrm{TP}_r$, for short), if its minors of all orders at most $r$ are nonnegative. There are many triangular matrices appearing in combinatorics that are indeed TP [6]. Recently, Ahmia and Belbachir [1] proved that the generalized Pascal triangle is totally positive of order two (or, $\mathrm{TP}_2$, for short). The total positivity of the Catalan triangle has also been established by C. Wang and Y. Wang [22]. Further general results on triangular matrices and Riordan arrays have been obtained by Chen, Liang and Wang [8, 9].

The total positivity of order two of Toeplitz and Hankel matrices is closely related to log-concavity and log-convexity, respectively, of the associated sequences. Recall that a sequence $(x_k)_k$ of positive numbers is log-concave (resp. log-convex) if $x_{i+1}^2 \geq x_i x_{i+2}$ for all $i \geq 0$ (resp. $x_{i+1}^2 \leq x_i x_{i+2}$ for all $i \geq 0$). These properties have been extensively investigated in many areas and the literature on them is vast. Besides already mentioned classical papers by Stanley [21] and Brenti [4, 5, 6] we refer the reader also to [11, 12, 18, 22] for some recently developed techniques. In particular Zheng and Liu [23] established the log-concavity and the log-convexity properties of hyperfibonacci $F_n^{[r]}$ and hyperlucas $L_n^{[r]}$ numbers by using recurrence relations. With the same technic, Ahmia et al. [2, 3] proved that the hyperpell $P_n^{[r]}$, the hyperpell-lucas $Q_n^{[r]}$, the hyperjacobsthal $J_n^{[r]}$ and the hyerjacobsthal-lucas $j_n^{[r]}$ numbers are log-concave.

An important notion when testing a matrix on total positivity is initial minor. We let $I$, $J$ denote column set and row set, respectively. A minor $a_{I,J}$ where both $I$ and $J$ consist of several consecutive indices and where $I \cup J$ contain 1, is called initial. Thus, each matrix entry is the lower-right corner of exactly one initial minor. A good way to obtain the total positivity matrices is by using the result of Gasca and Peña [15].

**Theorem 1.** *A square matrix is totally positive if and only if all its initial minors are positive.*

According to Theorem 1, Došlič et al. [13] proved the total positivity of a class of Toeplitz matrices associated with the sequence of hyperfibonacci numbers $F_n^{[r]}$. These results imply the log-concavity of $F_n^{[r]}$ [23].

The paper of Došlić et al. [13] allows us to exploit other relevant results. More precisely, we will present a new class of totally positive Toeplitz matrices whose entries are hyperlucas numbers of the $r$-th generation. The sequence of these numbers was introduced recently by Dil and Mező [14]. In Section 2, we discuss the properties of hyperlucas numbers $L_n^{[r]}$. In Sections 3 and 4, we establish the positivity of some classes of Toeplitz determinants, and we give an explicit formula for the determinant of the Hankel matrix of hyperlucas numbers of the $r$-th generation. We will find them useful in establishing our main result given in Section 5, the total positivity of the Toeplitz matrix of the same sequence with even-indexed hyperlucas number in the upper left corner.

## 2. Some Basic Properties

**Definition 1.** For a positive integer $r$, the *hyperlucas* numbers of the $r$-th generation $L_n^{[r]}$ are defined as follows:

$$L_n^{[r]} = \sum_{k=0}^{n} L_k^{[r-1]}, \quad L_n^{[0]} = L_n, \quad L_0^{[r]} = 2,$$

where $L_n$ is the $n$-th Lucas number, $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$ and $L_1 = 1$.

Initial values of $(L_n)$, $(L_n^{[1]})$ and $(L_n^{[2]})$ are as follows:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_n$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 |
| $L_n^{[1]}$ | 2 | 3 | 6 | 10 | 17 | 28 | 46 | 75 | 122 | 198 | 321 |
| $L_n^{[2]}$ | 2 | 5 | 11 | 21 | 38 | 66 | 112 | 187 | 309 | 507 | 828 |

There are many properties and identities that are given for Lucas and hyperlucas numbers in the literature. For example,

$$L_n = F_{n+1} + F_{n-1}, \quad n \geq 2, \tag{1}$$
$$L_{n+2}^2 - L_{n+1}L_{n+3} = 5(-1)^n, \tag{2}$$
$$L_n^{[r]} = L_{n-1}^{[r]} + L_n^{[r-1]}, \tag{3}$$
$$L_n^{[r]} = 2F_{n+1}^{[r]} - F_n^{[r]}, \tag{4}$$
$$\left(L_{n+2}^{[1]}\right)^2 - L_{n+1}^{[1]}L_{n+3}^{[1]} = 5(-1)^n + L_{n-1}. \tag{5}$$

We start with the following lemma [10].

**Lemma 1.** *For a positive integer $r$,*

$$F_n^{[r]} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+r+k}{r-1-k}. \tag{6}$$

This lemma gives the next theorem.

**Theorem 2.** *For a positive integer $r$, we have*

$$L_{n-r}^{[r]} = L_{n+r} - S_r, \tag{7}$$

*where* $S_r = \sum_{k=0}^{r-1} \left( \binom{n+k+1}{r-1-k} + \binom{n+k}{r-2-k} \right)$.

*Proof.* Using, respectively, relations (4) and (6), we get

$$\begin{aligned}
L_n^{[r]} &= 2F_{n+1}^{[r]} - F_n^{[r]} \\
&= 2F_{n+2r+1} - F_{n+2r} - 2\sum_{k=0}^{r-1}\binom{n+r+k+1}{r-1-k} + \sum_{k=0}^{r-1}\binom{n+r+k}{r-1-k} \\
&= (F_{n+2r+1} + F_{n+2r-1}) - \sum_{k=0}^{r-1}\left(\binom{n+r+k+1}{r-1-k} + \binom{n+r+k}{r-2-k}\right).
\end{aligned}$$

From relation (1), we have

$$L_n^{[r]} = L_{n+2r} - \sum_{k=0}^{r-1}\left(\binom{n+r+k+1}{r-1-k} + \binom{n+r+k}{r-2-k}\right)$$

and furthermore

$$L_{n-r}^{[r]} = L_{n+r} - S_r,$$

where $S_r = \sum_{k=0}^{r-1}\left(\binom{n+k+1}{r-1-k} + \binom{n+k}{r-2-k}\right)$. □

## 3. Positivity of Toeplitz determinants

We let $L_{m,n}^{[r]} = [l_{i,j}]$ denote a Toeplitz matrix of order $m$ consisting of hyperlucas numbers of the $r$-the generation,

$$L_{m,n}^{[r]} = \begin{pmatrix} L_n^{[r]} & L_{n-1}^{[r]} & \cdots & L_{n-m+1}^{[r]} \\ L_{n+1}^{[r]} & L_n^{[r]} & \cdots & L_{n-m+2}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+m-1}^{[r]} & L_{n+m-2}^{[r]} & \cdots & L_n^{[r]} \end{pmatrix}$$

with the constraint $r \geq m-1$. In what follows, we will show that there exist $q(r) \in \mathbb{N}$ such that the determinant of the Toeplitz matrix $L_{m,n}^{[r]}$ is positive for $n \geq q(r)$.

**Theorem 3.** *Let $m \in \mathbb{N}$. Then there is $n_m \in \mathbb{N}$ such that* $\det\left(L_{m,n}^{[m-1]}\right) > 0$ *for all $n \geq n_m$.*

*Proof.* According to relation (3) and employing some elementary transformation on matrices we get

$$\det\left(L_{m,n}^{[m-1]}\right) = \begin{vmatrix} L_n^{[m-1]} & L_{n-1}^{[m-1]} & L_{n-2}^{[m-1]} & \cdots & L_{n-m+1}^{[m-1]} \\ L_{n+1}^{[m-1]} & L_n^{[m-1]} & L_{n-1}^{[m-1]} & \cdots & L_{n-m+2}^{[m-1]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n+m-2}^{[m-1]} & L_{n+m-3}^{[m-1]} & L_{n+m-4}^{[m-1]} & & L_{n-1}^{[m-1]} \\ L_{n+m-1}^{[m-1]} & L_{n+m-2}^{[m-1]} & L_{n+m-3}^{[m-1]} & \cdots & L_n^{[m-1]} \end{vmatrix}$$

$$= \begin{vmatrix} L_n & L_{n-1}^{[1]} & L_{n-2}^{[2]} & \cdots & L_{n-m+1}^{[m-1]} \\ L_{n-1} & L_n & L_{n-1}^{[1]} & \cdots & L_{n-m+2}^{[m-2]} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n-m+2} & L_{n-m+3} & L_{n-m+4} & \cdots & L_{n-1}^{[1]} \\ L_{n-m+1} & L_{n-m+2} & L_{n-m+3} & \cdots & L_n \end{vmatrix}.$$

And using relation (7) of Theorem 2, we obtain

$$\det\left(L_{m,n}^{[m-1]}\right) = \begin{vmatrix} L_n & L_{n+1}-S_1 & L_{n+2}-S_2 & \cdots & L_{n+m-1}-S_{m-1} \\ L_{n-1} & L_n & L_{n+1}-S_1 & \cdots & L_{n+m-2}-S_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n-m+1} & L_{n-m+2} & L_{n-m+3} & \cdots & L_n \end{vmatrix}, \tag{8}$$

where $S_1 = 1$, $S_2 = n+3$, $S_3 = \frac{n(n+1)}{2} + 2(n+2), S_4 = \frac{n(n-1)(n+1)}{6} + n^2 + 3n + 7$, etc.

Now, we let $b_{i,j}$ denote the elements of the first $m-2$ rows of the last matrix and $c_{i,j}$ the elements of the last two columns in this matrix. Then by performing elementary transformations on matrix columns and rows of (8) respectively, we obtain

$$\det\left(L_{m,n}^{[m-1]}\right) = \begin{vmatrix} S_2-S_1 & \cdots & L_{n+m-2}-S_{m-2} & L_{n+m-1}-S_{m-1} \\ S_1 & \cdots & L_{n+m-3}-S_{m-3} & L_{n+m-2}-S_{m-2} \\ 0 & \cdots & L_{n+m-4}-S_{m-4} & L_{n+m-3}-S_{m-3} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & L_n & L_{n+1}-S_1 \\ 0 & \cdots & L_{n-1} & L_n \end{vmatrix}$$

$$= \begin{vmatrix} S_2-2S_1 & \cdots & -S_{m-2}+S_{m-3}+S_{m-4} & -S_{m-1}+S_{m-2}+S_{m-3} & \\ S_1 & \cdots & -S_{m-3}+S_{m-4}+S_{m-5} & -S_{m-2}+S_{m-3}+S_{m-4} & \\ 0 & \cdots & -S_{m-4}+S_{m-5}+S_{m-6} & -S_{m-3}+S_{m-4}+S_{m-5} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -S_1 & S_1-S_2 & \\ 0 & \cdots & L_n & L_{n+1}-S_1 & \\ 0 & \cdots & L_{n-1} & L_n & \end{vmatrix},$$

where

$$b_{i,j} = b_{i+1,j+1}, \quad i = 1, \ldots, m-1, j = 1, \ldots, m-3,$$
$$c_{i,m-1} = c_{i+1,m}, \quad i = 1, \ldots, m-3.$$

By performing column transformations on the last two columns of the last determinant we obtain

$$\det\left(L_{m,n}^{[m-1]}\right) = \begin{vmatrix} S_2 - 2S_1 & \cdots & -S_{m-4} + S_{m-6} + \cdots + S_2 & -S_{m-3} + S_{m-5} + \cdots + S_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -S_2 & -S_3 \\ 0 & \cdots & -S_1 & -S_2 \\ 0 & \cdots & 0 & -S_1 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & L_n & L_{n+1} \\ 0 & \cdots & L_{n-1} & L_n \end{vmatrix}.$$

Furthermore, we perform row transformations

$$\begin{cases} c'_{i,m-1} = c_{i,m-1} + \sum_{j=3}^{m-3}(L_j - a_j) b_{i,m-j-2}, \\ c'_{i,m} = c_{i,m} + \sum_{j=3}^{m-2}(L_j - a_j) b_{i,m-j-1}, \end{cases}$$

where $a_n = a_{n-1} + a_{n-2} - 1$ with the initial conditions $a_3 = 3$ and $a_4 = 5$, we get

$$\det\left(L_{m,n}^{[m-1]}\right) = \begin{vmatrix} S_2 - 2S_1 & \cdots & b_{1,m-2} & -L_{m-2} - a_{m-2} & -L_{m-1} - a_{m-1} \\ S_1 & \cdots & b_{2,m-2} & 0 & 0 \\ 0 & \cdots & b_{3,m-2} & 0 & 0 \\ 0 & \cdots & b_{4,m-2} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & b_{m-2,m-2} & 0 & 0 \\ 0 & \cdots & S_1 & L_n & L_{n+1} \\ 0 & \cdots & 0 & L_{n-1} & L_n \end{vmatrix},$$

where $b_{1,m-2} = S_{m-1} - 2S_{m-2} - S_{m-3} + 2S_{m-4} + S_{m-5}$, $b_{2,m-2} = S_{m-2} - 2S_{m-3} - S_{m-4} + 2S_{m-5} + S_{m-6}$, etc.

By using the Lucas recurrence relation and performing columns transformations we obtain

$$\det\left(L_{m,n}^{[m-1]}\right) = \frac{1}{5(-1)^{n-3}} \begin{vmatrix} \cdots & b_{1,m-2} & -L_{m-2}+a_{m-2}-1 & -L_{m-1}+a_{m-1}-1 \\ \cdots & b_{2,m-2} & 0 & 0 \\ \cdots & b_{3,m-2} & 0 & 0 \\ \cdots & b_{4,m-2} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \cdots & b_{m-2,m-2} & 0 & 0 \\ \cdots & L_{n-1} & 0 & 5(-1)^{n-3} \\ \cdots & -L_{n-2} & 5(-1)^{n-3} & 5(-1)^{n-1} \end{vmatrix}.$$

We shall now separately treat the determinant $\det\left(L_{m,n}^{[m-1]}\right)$, for even and odd $n$. When $n$ is odd we have

$$\det\left(L_{m,n}^{[m-1]}\right) = -\frac{1}{5} \begin{vmatrix} \cdots & b'_{1,m-2} & -L_{m-3}+a_{m-3}-1 & -L_{m-2}+a_{m-2}-1 \\ \cdots & b_{2,m-2} & 0 & 0 \\ \cdots & b_{3,m-2} & 0 & 0 \\ \cdots & b_{4,m-2} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \cdots & b_{m-2,m-2} & 0 & 0 \\ \cdots & 0 & 5 & 0 \\ \cdots & 0 & 0 & 5 \end{vmatrix}, \tag{9}$$

where $b'_{1,m-2} = b_{1,m-2} + \frac{1}{5}L_{n-1}(L_{m-3} - a_{m-3} + 1) - \frac{1}{5}(L_{m-2} - a_{m-2} + 1)L_{n-2}$. It is well known that there is $q \in \mathbb{N}$ such that $L_q > P(q)$, where $P(n)$ is a polynomial of any degree. Since we can write the determinant (9) as the sum of the upper triangular determinants, and the only element containing Lucas numbers is $b'_{1,m-2}$, then the fact that the term $L_{m-3}L_{n-1}$ has a positive contribution in the determinant completes the proof.

When $n$ is even we have

$$\det\left(L_{m,n}^{[m-1]}\right) = -\frac{1}{5} \begin{vmatrix} \cdots & b_{1,m-2} & -L_{m-2}+a_{m-2}-1 & -L_{m-1}+a_{m-1}-1 \\ \cdots & b_{2,m-2} & 0 & 0 \\ \cdots & b_{3,m-2} & 0 & 0 \\ \cdots & b_{4,m-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & b_{m-2,m-2} & 0 & 0 \\ \cdots & L_{n-1} & 0 & -5 \\ \cdots & -L_{n-2} & -5 & -5 \end{vmatrix}.$$

The proof is completed by using the same arguments as in the case of odd $n$. □

For further explanation of this theorem, we give the following example.

**Example 1.** When $m = 4$ and $n$ is odd we have

$$\begin{aligned}\det\left(L_{4,n}^{[3]}\right) &= \frac{1}{5}\begin{vmatrix} S_2-2S_1 & S_3-2S_2-S_1 & -1_1 & -2 \\ S_1 & S_2-2S_1 & 0 & 0 \\ 0 & L_{n-1} & 0 & 5 \\ 0 & -L_{n-2} & 5 & 5 \end{vmatrix} \\ &= \frac{1}{5}\begin{vmatrix} S_2-2S_1 & S_3-2S_2-S_1 & -1 & -1 \\ S_1 & S_2-2S_1 & 0 & 0 \\ 0 & L_{n-1} & 0 & 5 \\ 0 & -L_{n-2} & 5 & 0 \end{vmatrix} \\ &= \frac{1}{5}\begin{vmatrix} S_2-2S_1 & S_3-2S_2-S_1+\frac{1}{5}L_{n-3} & -1 & -1 \\ S_1 & S_2-2S_1 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{vmatrix} \\ &= -\frac{1}{5}\begin{vmatrix} n+1 & \frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{5}L_{n-3}-3 & -1 & -1 \\ 1 & n+1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix} \\ &= L_{n-3}-\frac{5}{2}n^2-\frac{15}{2}n-20.\end{aligned}$$

The determinant is positive if

$$L_{n-3} > \frac{5}{2}n^2+\frac{15}{2}n+20.$$

Hence, this inequality is true for $n \geq 18$ and consequently $\det\left(L_{4,n}^{[3]}\right) > 0$ for $n \geq 18$.

Similarly, when $n$ is even

$$\begin{aligned}\det\left(L_{4,n}^{[3]}\right) &= \frac{1}{5}\begin{vmatrix} n+1 & \frac{1}{2}n^2+\frac{1}{2}n-\frac{1}{5}L_{n-3}-3 & -1 & -1 \\ 1 & n+1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix} \\ &= L_{n-3}+\frac{5}{2}n^2+\frac{15}{2}+20 > 0.\end{aligned}$$

Thus, it follows from these two cases that $\det\left(L_{4,n}^{[3]}\right) > 0$ for $n \geq 18$.

**Theorem 4.** *Let $m, n, r \in \mathbb{N}$ and $r \geq m-1$. Then there is $q \in \mathbb{N}$ such that for all $n \geq q$,*

$$\det\left(L_{m,n}^{[r]}\right) > 0.$$

*Proof.* It suffices to proceed by induction over $r$. The case $r = m-1$ is provided by Theorem 3. Suppose that this hypothesis is true for $m-1 \le p \le r-1$. We show that it remains true for $p = r$. So our task is to show that the determinant

$$\det\left(L_{m,n}^{[r]}\right) = \begin{vmatrix} L_n^{[r]} & L_{n-1}^{[r]} & \cdots & L_{n-m+1}^{[r]} \\ L_{n+1}^{[r]} & L_n^{[r]} & \cdots & L_{n-m+2}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+m-1}^{[r]} & L_{n+m-2}^{[r]} & \cdots & L_n^{[r]} \end{vmatrix}$$

is also positive.

By performing row transformations and according to relation (3), we obtain

$$\begin{aligned} \det\left(L_{m,n}^{[r]}\right) &= \begin{vmatrix} L_n^{[r]} & L_{n-1}^{[r]} & \cdots & L_{n-m+1}^{[r]} \\ L_{n+1}^{[r-1]} & L_n^{[r-1]} & \cdots & L_{n-m+2}^{[r-1]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+m-1}^{[r-1]} & L_{n+m-2}^{[r-1]} & \cdots & L_n^{[r-1]} \end{vmatrix} \\ &= \sum_{j=1}^{m} (-1)^{j-1} L_{n-j+1}^{[r]} M_j \\ &= \sum_{j=1}^{m} (-1)^{j-1} \frac{L_{n-j+1}^{[r]}}{L_{n-j+1}^{[r-1]}} L_{n-j+1}^{[r-1]} M_j, \end{aligned}$$

where $M_j$ denotes the determinant obtained from $\det\left(L_{m,n}^{[r]}\right)$ by omitting the first row and $j$-th column for $1 \le j \le m$. Let us denote $x_j = \frac{L_{n-j+1}^{[r]}}{L_{n-j+1}^{[r-1]}}$ and define a function $f : \mathbb{R}^m \to \mathbb{R}$ by

$$f(x_1, \ldots, x_m) = \sum_{j=1}^{m} (-1)^{j-1} x_j L_{n-j+1}^{[r-1]} M_j.$$

Obviously, $f(1, \ldots, 1) = \det\left(L_{m,n}^{[r-1]}\right) > 0$, and hence $f(c, \ldots, c) = c \det\left(L_{m,n}^{[r-1]}\right) > 0$, for any positive constant $c$. In particular, $f(\phi^2, \ldots, \phi^2) > 0$, where $\phi = \frac{1+\sqrt{5}}{2}$.

Since $f$ is continuous, there must exist a neighborhood

$$I = \left(\phi^2 - \delta_1, \phi^2 + \delta_1\right) \times \cdots \left(\phi^2 - \delta_m, \phi^2 - \delta_m\right)$$

such that $f$ is positive on $I$. Now we use the explicit expression

$$L_n^{[r]} = L_{n+2r} - \sum_{k=0}^{r-1} \left( \binom{n+r+k+1}{r-1-k} + \binom{n+r+k}{r-2-k} \right),$$

from Theorem 2. By dividing it through by the analogous expression for $L_n^{[r-1]}$ and passing to the limit when $n \to \infty$, one readily obtains

$$\lim_{n\to\infty} \frac{L_n^{[r]}}{L_n^{[r-1]}} = \phi^2.$$

That further implies that, for large enough $n$, the coefficient $x_j = \frac{L_{n-j+1}^{[r]}}{L_{n-j+1}^{[r-1]}}$ falls into $\left(\phi^2 - \delta_j, \phi^2 + \delta_j\right)$ for all $j$, and hence

$$f\left(\frac{L_n^{[r]}}{L_n^{[r-1]}}, \ldots, \frac{L_{n-m+1}^{[r]}}{L_{n-m+1}^{[r-1]}}\right) = \det\left(L_{m,n}^{[r]}\right) > 0.$$

That completes the proof. □

## 4. Hankel Determinants

In this section, we give an explicit formula for the determinant of the Hankel matrix of hyperlucas numbers. We let

$$H_{r,n} = \begin{pmatrix} L_n^{[r]} & L_{n+1}^{[r]} & \cdots & L_{n+r+1}^{[r]} \\ L_{n+1}^{[r]} & L_{n+2}^{[r]} & \cdots & L_{n+r+2}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+r+1}^{[r]} & L_{n+r+2}^{[r]} & \cdots & L_{n+2r+2}^{[r]} \end{pmatrix}$$

denote the Hankel matrix consisting of hyperlucas numbers of the $r$-th generation.

To prove Theorem 5, we need the following lemma.

**Lemma 2.** *For $m, n \in \mathbb{N}$ and $m \geq 2$, we have*

$$\det\left(L_{m,n}^{[m-2]}\right) = 5(-1)^{n-4}.$$

*Proof.* By the same technique used to prove Theorem 3, we get

$$\det\left(L_{m,n}^{[m-2]}\right) = \begin{vmatrix} L_n^{[m-2]} & L_{n-1}^{[m-2]} & L_{n-2}^{[m-2]} & \cdots & L_{n-m+1}^{[m-2]} \\ L_{n+1}^{[m-2]} & L_n^{[m-2]} & L_{n-1}^{[m-2]} & \cdots & L_{n-m+2}^{[m-2]} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{n+m-2}^{[m-2]} & L_{n+m-3}^{[m-2]} & L_{n+m-2}^{[m-2]} & \cdots & L_{n-1}^{[m-2]} \\ L_{n+m-1}^{[m-2]} & L_{n+m-2}^{[m-2]} & L_{n+m-3}^{[m-2]} & \cdots & L_n^{[m-2]} \end{vmatrix}$$

$$= \begin{vmatrix} L_n & L_{n-1} & \cdots & L_{n+m-3}-S_{m-2} & L_{n+m-2}-S_{m-1} \\ L_{n-1} & L_{n-2} & \cdots & L_{n+m-4}-S_{m-3} & L_{n+m-3}-S_{m-2} \\ \vdots & \vdots & & \vdots & \\ L_{n-m+2} & L_{n-m+3} & \cdots & L_{n-2} & L_{n-1} \\ L_{n-m+1} & L_{n-m+2} & \cdots & L_{n-1} & L_n \end{vmatrix}$$

$$= \begin{vmatrix} S_2 & S_2-S_1 & \cdots & -S_{m-2}+S_{m-3}+S_{m-4} & -S_{m-1}+S_{m-2}+S_{m-3} \\ S_1 & S_1 & \cdots & -S_{m-3}+S_{m-4}+S_{m-5} & -S_{m-2}+S_{m-3}+S_{m-4} \\ 0 & 0 & \cdots & -S_{m-4}+S_{m-5}+S_{m-6} & -S_{m-3}+S_{m-4}+S_{m-5} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & L_{n-2} & L_{n-1} \\ 0 & 0 & \cdots & L_{n-1} & L_n \end{vmatrix}$$

$$= \begin{vmatrix} S_2 & S_2-S_1 & \cdots & b_{1,m-1} & b_{1,m} \\ 0 & \frac{S_1^2}{S_2} & \cdots & b_{2,m-1}-\frac{S_1}{S_2}b_{1,m-1} & b_{2,m}-\frac{S_1}{S_2}b_{1,m} \\ 0 & 0 & \cdots & b_{3,m-1} & b_{3,m} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & L_{n-2} & L_{n-1} \\ 0 & 0 & \cdots & L_{n-1} & L_n \end{vmatrix},$$

where $b_{1,m-1} = b_{2,m} = -S_{m-2} + S_{m-3} + S_{m-4}$, $b_{2,m-1} = b_{3,m} = -S_{m-3} + S_{m-4} + S_{m-5}$ and $b_{3,m-1} = -S_{m-4} + S_{m-5} + S_{m-6}$. And by performing column transformations on the last two columns of the last determinant we obtain

$$\det\left(L_{m,n}^{[m-2]}\right) =$$

$$\begin{vmatrix} S_2 & \cdots & b_{1,m-1}-\frac{L_{n-1}}{L_n}b_{1,m} & b_{1,m} \\ 0 & \cdots & b_{2,m-1}-\frac{S_1}{S_2}b_{1,m-1}-\frac{L_{n-1}}{L_n}\left(b_{2,m}-\frac{S_1}{S_2}b_{1,m}\right) & b_{2,m}-\frac{S_1}{S_2}b_{1,m} \\ 0 & \cdots & b_{3,m-1}-\frac{L_{n-1}}{L_n}b_{3,m} & b_{3,m} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & \frac{L_{n-2}L_n-L_{n-1}^2}{L_n} & L_{n-1} \\ 0 & \cdots & 0 & L_n \end{vmatrix}$$

$$= L_{n-2}L_n - L_{n-1}^2$$

$$= 5(-1)^{n-4},$$

as required. □

From relation (3), the hyperlucas sequence can be defined by the vector recurrence relation

$$\begin{pmatrix} L_{n+1}^{[r]} \\ L_{n+2}^{[r]} \\ \vdots \\ L_{n+r+2}^{[r]} \end{pmatrix} = D_{r+2} \begin{pmatrix} L_n^{[r]} \\ L_{n+1}^{[r]} \\ \vdots \\ L_{n+r+1}^{[r]} \end{pmatrix}, \tag{10}$$

where $D_{r+2}$ is the square matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ q_1 & q_2 & q_3 & \cdots & q_{r+1} & q_{r+2} \end{pmatrix}. \tag{11}$$

In order to determine elements $q_1, \ldots, q_{r+2}$, we use the fact that terms from $-r$ through 0 of the $r$-th generation of hyperlucas numbers take values 0.

In particular, when $n = -r-1$ we get

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 2 \\ L_1^{[r]} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ q_1 & q_2 & q_3 & \cdots & q_{r+1} & q_{r+2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 2 \end{pmatrix},$$

meaning that $q_{r+2} = \frac{1}{2} L_1^{[r]}$. In the same way we obtain relations for all elements of $D_{r+2}$,

$$\begin{aligned} q_{r+2} &= \frac{1}{2} L_1^{[r]} \\ q_{r+1} &= \frac{1}{2} \left[ L_2^{[r]} - L_1^{[r]} q_{r+2} \right] \\ q_r &= \frac{1}{2} \left[ L_3^{[r]} - L_2^{[r]} q_{r+2} - L_1^{[r]} q_{r+1} \right] \\ &\vdots \\ q_1 &= \frac{1}{2} \left[ L_{r+2}^{[r]} - L_{r+1}^{[r]} q_{r+2} - \cdots - L_1^{[r]} q_2 \right]. \end{aligned}$$

This reasoning gives the next lemma.

**Lemma 3.** *For the hyperlucas sequence we have*

$$H_{r,n} = (D_{r+2})^n \, H_{r,0}. \tag{12}$$

*Proof.* Relation (10) can be written as $H_{r,n} = D_{r+2} H_{r,n-1}$. Now the statement of lemma follows immediately:

$$H_{r,n} = D_{r+2} H_{r,n-1} = (D_{r+2})^2 \, H_{r,n-2} = \cdots = (D_{r+2})^n \, H_{r,0}.$$

□

It remains to find the determinant of the matrix $D_{r+2}$ as follows.

**Lemma 4.** *For $r \in \mathbb{N}$ the determinant of a matrix $D_{r+2}$ takes value $-1$,*

$$\det(D_{r+2}) = -1.$$

*Proof.* We prove this statement by means of comparing determinants of the matrices $H_{r,-r}$ and $H_{r,-r-1}$,

$$H_{r,-r} = D_{r+2} H_{r,-r-1}.$$

It follows from Lemma 2 that

$$\begin{aligned}
\det(H_{r,-r}) &= \begin{vmatrix} L_{-r}^{[r]} & L_{-r+1}^{[r]} & \cdots & L_{1}^{[r]} \\ L_{-r+1}^{[r]} & L_{-r+2}^{[r]} & \cdots & L_{2}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{1}^{[r]} & L_{2}^{[r]} & \cdots & L_{r+2}^{[r]} \end{vmatrix} \\
&= (-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor} \begin{vmatrix} L_{1}^{[r]} & \cdots & L_{-r+1}^{[r]} & L_{-r}^{[r]} \\ L_{2}^{[r]} & \cdots & L_{-r+2}^{[r]} & L_{-r+1}^{[r]} \\ \vdots & \ddots & \vdots & \vdots \\ L_{r+2}^{[r]} & \cdots & L_{2}^{[r]} & L_{1}^{[r]} \end{vmatrix} \\
&= (-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor} \det\left(L_{r+2,1}^{[r]}\right) \\
&= -5(-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor}.
\end{aligned}$$

Similarly using Lemma 2, we get

$$\det(H_{r,-r-1}) = (-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor} \det\left(L_{r+2,0}^{[r]}\right) = 5(-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor},$$

which proves that

$$\det(H_{r,-r}) = -\det(H_{r,-r-1}).$$

Now, the statement of lemma follows immediately by the Binet-Cauchy theorem. □

**Theorem 5.** *For the sequence $\left(L_k^{[r]}\right)_{k\geq 0}$, $r \in N$ and $n \in N$ a determinant of $H_{r,n}$ takes values $\pm 5$,*

$$\det(H_{r,n}) = 5(-1)^{n+1+\left\lfloor \frac{r+3}{2} \right\rfloor}. \tag{13}$$

*Proof.* Using elementary transformations on the matrix and Lemma 4 we get

$$\begin{aligned}
\det(H_{r,0}) &= \begin{vmatrix} L_0^{[r]} & L_1^{[r]} & \cdots & L_{r+1}^{[r]} \\ L_1^{[r]} & L_2^{[r]} & \cdots & L_{r+2}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{r+1}^{[r]} & L_{r+2}^{[r]} & \cdots & L_{2r+2}^{[r]} \end{vmatrix} \\
&= -\begin{vmatrix} L_{-1}^{[r]} & L_0^{[r]} & \cdots & L_r^{[r]} \\ L_0^{[r]} & L_1^{[r]} & \cdots & L_{r+1}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_r^{[r]} & L_{r+1}^{[r]} & \cdots & L_{2r+1}^{[r]} \end{vmatrix} \\
&= (-1)^{r+1}\begin{vmatrix} L_{r-1}^{[r]} & L_{-r}^{[r]} & \cdots & L_0^{[r]} \\ L_{-r}^{[r]} & L_{-r+2}^{[r]} & \cdots & L_1^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_0^{[r]} & L_1^{[r]} & \cdots & L_{r+1}^{[r]} \end{vmatrix} \\
&= (-1)^{r+1}\det(H_{r,-r-1}) \\
&= 5(-1)^{\lfloor \frac{r+3}{2} \rfloor + 1}.
\end{aligned}$$

According to Lemma 3 we obtain

$$\begin{aligned}
\det(H_{r,n}) &= \det(D_{r+2})^n \det(H_{r,0}) = (-1)^n \det(H_{r,0}) \\
&= 5(-1)^{n+1+\lfloor \frac{r+3}{2} \rfloor},
\end{aligned}$$

which completes the statement of the theorem. □

## 5. Total Positivity of Toeplitz Matrices

We let $B_{r,n}$ denote the matrix of order $r+2$ consisting of hyperlucas numbers of the $r$-th generation,

$$B_{r,n} := \begin{pmatrix} L_{2n}^{[r]} & L_{2n-1}^{[r]} & \cdots & L_{2n-r-1}^{[r]} \\ L_{2n+1}^{[r]} & L_{2n}^{[r]} & \cdots & L_{2n-r}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{2n+r+1}^{[r]} & L_{2n+r}^{[r]} & \cdots & L_{2n}^{[r]} \end{pmatrix}.$$

The following lemma due to Zheng et al. [23] plays a key role in the proof of our next results.

**Lemma 5.** *For $r \geq 1$, the sequence $\left(L_n^{[r]}\right)_{n\geq 3}$ is log-concave.*

**Lemma 6.** *For $n \geq 3$ an integer, the matrix*

$$B_{1,n} = \begin{pmatrix} L_{2n}^{[1]} & L_{2n-1}^{[1]} & L_{2n-2}^{[1]} \\ L_{2n+1}^{[1]} & L_{2n}^{[1]} & L_{2n-1}^{[1]} \\ L_{2n+2}^{[1]} & L_{2n+1}^{[1]} & L_{2n}^{[1]} \end{pmatrix}$$

*is totally positive.*

*Proof.* By Lemma 5, the log-concavity of the hyperlucas sequence $\left(L_n^{[1]}\right)_{n\geq 3}$ implies that the three initial minors of order 2 of $B_{1,n}$ are positive. It is immediately seen from Theorem 5 that the determinant of $B_{1,n}$ is positive. □

If the element $b_{1,1}$ of the matrix $B_{1,n} = [b_{i,j}]$ has odd index, then the matrix $B_{1,n}$ is not totally positive because the determinant of order three of $B_{1,n}$ is not positive for odd indices (by Theorem 5), while it keeps the positivity of minors of order two. Thus, from the proof of Lemma 6 we have the following result.

**Corollary 1.** *For $n \geq 5$ an integer, the matrix*

$$B'_{1,n} = \begin{pmatrix} L_{n}^{[1]} & L_{n-1}^{[1]} & L_{n-2}^{[1]} \\ L_{n+1}^{[1]} & L_{n}^{[1]} & L_{n-1}^{[1]} \\ L_{n+2}^{[1]} & L_{n+1}^{[1]} & L_{n}^{[1]} \end{pmatrix}$$

*is totally positive of order two.*

**Lemma 7.** *For $n \geq 6$ an integer, the matrix*

$$B_{2,n} = \begin{pmatrix} L_{2n}^{[2]} & L_{2n-1}^{[2]} & L_{2n-2}^{[2]} & L_{2n-3}^{[2]} \\ L_{2n+1}^{[2]} & L_{2n}^{[2]} & L_{2n-1}^{[2]} & L_{2n-2}^{[2]} \\ L_{2n+2}^{[2]} & L_{2n+1}^{[2]} & L_{2n}^{[2]} & L_{2n-1}^{[2]} \\ L_{2n+3}^{[2]} & L_{2n+2}^{[2]} & L_{2n+1}^{[2]} & L_{2n}^{[2]} \end{pmatrix}$$

*is totally positive.*

*Proof.* By Lemma 5, the log-concavity of the hyperlucas sequence $\left(L_n^{[2]}\right)_{n\geq 3}$ implies that the five initial minors of order two of $B_{2,n}$ are positive. However, the three initial minors of order three are not all positive when $n = 5$ by Theorem 4 and the determinant of $B_{2,n}$ is positive by Theorem 5, so the matrix $B_{2,n}$ is totally positive for $n \geq 6$. □

The positivity determinant of the matrix $L_{3,n}^{[2]}$ for $n \geq 10$ (by Theorem 3), and the log-concavity of the hyperlucas sequence $\left(L_n^{[2]}\right)_{n\geq 3}$, imply the following corollary.

**Corollary 2.** *For $n \geq 11$ an integer, the matrix*

$$B^{'}_{2,n} = \begin{pmatrix} L^{[2]}_{n} & L^{[2]}_{n-1} & L^{[2]}_{n-2} & L^{[2]}_{n-3} \\ L^{[2]}_{n+1} & L^{[2]}_{n} & L^{[2]}_{n-1} & L^{[2]}_{n-2} \\ L^{[2]}_{n+2} & L^{[2]}_{n+1} & L^{[2]}_{n} & L^{[2]}_{n-1} \\ L^{[2]}_{n+3} & L^{[2]}_{n+2} & L^{[2]}_{n+1} & L^{[2]}_{n} \end{pmatrix}$$

*is totally positive of order three.*

**Theorem 6.** *There is a $q \geq 3$ such that the matrix $B_{r,n}$ of order $r+2$*

$$B_{r,n} = \begin{pmatrix} L^{[r]}_{2n} & L^{[r]}_{2n-1} & \dots & L^{[r]}_{2n-r-1} \\ L^{[r]}_{2n+1} & L^{[r]}_{2n} & \dots & L^{[r]}_{2n-r} \\ \vdots & \vdots & \ddots & \vdots \\ L^{[r]}_{2n+r+1} & L^{[r]}_{2n+r} & \dots & L^{[r]}_{2n} \end{pmatrix}$$

*is totally positive for $n \geq q$.*

*Proof.* The log-concavity of hyperlucas numbers and Theorem 3 imply the positivity of all initial minors of order two, which are of the form $L^{[r]}_{2,n_2}$ where $n_2 \geq 2n-r+1$, of the matrix $B_{r,n}$ for $r \geq 1$ and $n \geq 3$. According to Theorem 4, there exist $q_3, \dots, q_{r+1} \in \mathbb{N}$ such that the other initial minors of order $3, \dots, r+1$, which are of the form $L^{[r]}_{3,n_3}, \dots, L^{[r]}_{r+1,n_3}$, are positive and $n_3 \geq q_3, \dots, n_{r+1} \geq q_{r+1}$.

Finally, we show that $\det(B_{r,n})$ is positive. According to Theorem 5 and by reversing the order of columns we obtain

$$\begin{aligned} \det(B_{r,n}) &= (-1)^{\lfloor \frac{r+2}{2} \rfloor} \begin{vmatrix} L^{[r]}_{2n-r-1} & \dots & L^{[r]}_{2n-1} & L^{[r]}_{2n} \\ L^{[r]}_{2n-r} & \dots & L^{[r]}_{2n} & L^{[r]}_{2n+1} \\ \vdots & \ddots & \vdots & \vdots \\ L^{[r]}_{2n} & \dots & L^{[r]}_{2n+r} & L^{[r]}_{2n+r+1} \end{vmatrix} \\ &= (-1)^{\lfloor \frac{r+2}{2} \rfloor} \det(H_{r,2n-r-1}) \\ &= 5(-1)^{2n-r+\lfloor \frac{r+2}{2} \rfloor + \lfloor \frac{r+3}{2} \rfloor} \\ &= 5, \end{aligned}$$

for all $r \geq 1$. That completes the proof. □

We conclude this section with another result that follows directly from Corollary 2.

**Corollary 3.** *There is a $q \geq 3$ such that the matrix $B'_{r,n}$ of order $r+2$*

$$B'_{r,n} = \begin{pmatrix} L_n^{[r]} & L_{n-1}^{[r]} & \dots & L_{n-r-1}^{[r]} \\ L_{n+1}^{[r]} & L_n^{[r]} & \dots & L_{n-r}^{[r]} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+r+1}^{[r]} & L_{n+r}^{[r]} & \dots & L_n^{[r]} \end{pmatrix}$$

*is totally positive of order $r+1$ for $n \geq q$.*

## 6. Conclusion

In this paper we have established various positivity results for several classes of Toeplitz matrices associated with the sequence of hyperlucas numbers of a given generation. In particular, we proved that such matrices with even-indexed hyperlucas numbers on the main diagonal are totally positive for large enough values of index $n$. When the restriction to even-valued indices is omitted, the total positivity is not preserved, but we established that those matrices are totally positive of order $r+1$ for a given generation $r$ and large enough $n$.

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