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SOME CONGRUENCES FOR $(2,5)$-REGULAR CUBIC PARTITION PAIRS

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Abstract

Let $b_{2,5}(n)$ denote the number of $(2,5)$-regular cubic partition pairs of a positive integer n. In this paper, we establish many infinite families of congruences modulo powers of 2 for $b_{2,5}(n)$. For example, for all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$,

$$b_{2,5}\left(32 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + n_1 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) \equiv 0 \pmod{16},$$

where $n_1 \in \{28, 92, 124, 156\}$.

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. An ℓ-regular partition is a partition in which none of the parts are divisible by ℓ. Let $c_\ell(n)$ denote the number of such partitions of n with $c_\ell(0) = 1$. The generating function for $c_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} c_\ell(n) q^n = \frac{f_\ell}{f_1},$$

where

$$f_\ell := (q^\ell; q^\ell)_\infty = \prod_{n=1}^{\infty} (1 - q^{n\ell}). \qquad (1)$$

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ [1] is defined by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}.$$

By using Jacobi's Triple Product identity [2, Entry 19, p. 35], the function $f(a,b)$ can be written as

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2}$$

The most important special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1 \tag{5}$$

and

$$\chi(q) = (-q;q^2)_\infty. \tag{6}$$

Three of Ramanujan's most well-known congruences of $p(n)$ are as follows [18], [19, pp.210-213]:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Chan [3] defined the cubic partition function $a(n)$ and the generating function for $a(n)$ by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{1}{f_1 f_2}.$$

He obtained the identity

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1^4 f_2^4},$$

which implies the Ramanujan-type congruence

$$a(3n+2) \equiv 0 \pmod{3}.$$

Chan also obtained an infinite family of congruences modulo large powers of 3 for $a(n)$, which are analogues of the Ramanujan-type congruences [4]. For example, for each $n, k \geq 1$,

$$a(3^k n + c_k) \equiv 0 \pmod{3^{k+\delta(k)}}, \tag{7}$$

where c_k is the reciprocal modulo 3^k of δ and

$$\delta(k) = \begin{cases} 1 & \text{if k is even,} \\ 0 & \text{if k is odd.} \end{cases}$$

Chan and Toh [5] proved higher power analogues of the Ramanujan's congruences and (7) for $a(n)$ involving the prime 5, for $n \geq 0$,

$$a(5^j n + d_j) \equiv 0 \pmod{5^{[j/2]}},$$

where $d_j = 1/8 \pmod{5^j}$.

Zhao and Zhong [21] defined the cubic partition pairs function $b(n)$ and the generating function for $b(n)$ by

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} = \frac{1}{f_1^2 f_2^2}. \tag{8}$$

They proved many Ramanujan-type congruences for $b(n)$. For example, for each $n \geq 0$,

$$\begin{aligned} b(5n+4) &\equiv 0 \pmod 5, \\ b(7n+i) &\equiv 0 \pmod 7, \\ b(9n+7) &\equiv 0 \pmod 9, \end{aligned}$$

where $i = 2, 3, 4, 6$.

B. Kim [10] obtained two partition statistics for the cubic partitions to explain the congruences, for all $n \geq 0$,

$$b(5n+4) \equiv 0 \pmod 5$$

and

$$b(7n+a) \equiv 0 \pmod 7, \text{ if } a = 2, 3, 4 \text{ or } 6.$$

Kim [11] also studied congruence properties of $c(n)$, the number of cubic partition pairs weighted by the parity of the crank. For example,

$$c(5n+4) \equiv 0 \pmod 5 \text{ and } c(7n+2) \equiv 0 \pmod 7,$$

for all nonnegative integers n. For more details, one can see [8, 12, 14].

Lin [13] proved some Ramanujan-type congruences modulo 27 for $b(n)$. For example, for each $n \geq 0$,

$$b(27n+16) \equiv 0 \pmod{27},$$
$$b(27n+25) \equiv 0 \pmod{27},$$
$$b(81n+61) \equiv 0 \pmod{27}.$$

Mahadeva Naika and Gireesh [15] proved the general family of congruences modulo large powers of 3 for $b(n)$ by finding (α, β, ℓ) such that

$$b(3^{\alpha}n+\ell) \equiv 0 \pmod{3^{\beta}}.$$

Recently, in [17], the authors obtained many congruences for $b_{\ell}(n)$, the number of ℓ-regular cubic partition pairs of n. The generating function for $b_{\ell}(n)$ is given by

$$\sum_{n=0}^{\infty} b_{\ell}(n)q^n = \frac{(q^{\ell};q^{\ell})_{\infty}^2 (q^{2\ell};q^{2\ell})_{\infty}^2}{(q;q)_{\infty}^2 (q^2;q^2)_{\infty}^2} = \frac{f_{\ell}^2 f_{2\ell}^2}{f_1^2 f_2^2}. \tag{9}$$

By the motivation of the above work, we define $b_{\ell,m}(n)$, the number of (ℓ, m)-regular cubic partition pairs of n and the generating function for $b_{\ell,m}(n)$ by

$$\sum_{n=0}^{\infty} b_{\ell,m}(n)q^n = \frac{f_{\ell}^2 f_{2\ell}^2 f_m^2 f_{2m}^2}{f_1^2 f_2^2 f_{\ell m}^2 f_{2\ell m}^2}. \tag{10}$$

We prove many congruences modulo powers of 2 for $b_{2,5}(n)$ of the following form, for all $n \geq 0$ and α, $\beta \geq 0$,

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_2^3 f_5^3 \pmod{16}.$$

2. Preliminary Results

In this section, we record several identities which are useful in proving our main results.

Lemma 1. *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{11}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{12}$$

For proofs, see [2, p.40].

Lemma 2. *The following 3-dissection holds:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \tag{13}$$

Proof. From [2, p.345, Entry 1(iv)], we have

$$3 + \frac{f^3(-q^{1/3})}{q^{1/3} f^3(-q^3)} = \frac{1}{v} + 4v^2, \tag{14}$$

where

$$v = \frac{q^{1/3}\chi(-q)}{\chi(-q^3)}. \tag{15}$$

Substituting (15) in (14) and replacing q by q^3, we obtain

$$\frac{f^3(-q)}{f^3(-q^9)} = \frac{\chi^3(-q^9)}{\chi(-q^3)} - 3q + 4q^3 \frac{\chi^2(-q^3)}{\chi^6(-q^9)}. \tag{16}$$

By using (2), (5) and (6) in (16), we get (13). □

Lemma 3. *The following 2-dissections hold:*

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2} \tag{17}$$

and

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{18}$$

Equation (17) was proved by Hirschhorn and Sellers [9]; see also [19]. Replacing q by $-q$ in (17) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$, we obtain (18).

Lemma 4. *We have*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q \frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q \frac{f_4^3 f_{20}}{f_2^8} + 2q^2 \frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}, \tag{19}$$

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3 + 2q^2 \frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2} \tag{20}$$

and

$$f_1 f_5^3 = f_2^3 f_{10} - q \frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}. \tag{21}$$

For proofs, see [16].

Lemma 5. *Ramanujan recorded the following identity*

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{22}$$

where

$$R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

Equation (22) is the same as (8.1.1) in [7]. For more details, one can see [6], [20].

Lemma 6. *We have*

$$\begin{aligned} f_1 =& f_{49}\left(\left(\begin{matrix} q^{14}, & q^{35} \\ q^{7}, & q^{42} \end{matrix}; q^{49}\right)_\infty - q\left(\begin{matrix} q^{21}, & q^{28} \\ q^{14}, & q^{35} \end{matrix}; q^{49}\right)_\infty \right. \\ & \left. -q^2 + q^5\left(\begin{matrix} q^{7}, & q^{42} \\ q^{21}, & q^{28} \end{matrix}; q^{49}\right)_\infty\right). \end{aligned} \tag{23}$$

Lemma 6 is an exercise in [7]. Also, we can see [2, p.303, Entry 17(v)].

We prove the following theorems.

Theorem 1. *Let $n_1 \in \{62, 78\}$, $n_2 \in \{62, 158\}$, $n_3 \in \{166, 214\}$, $n_4 \in \{142, 238\}$, $n_5 \in \{86, 134\}$, $n_6 \in \{10, 26\}$, $n_7 \in \{28, 92, 124, 156\}$ and $n_8 \in \{124, 156\}$. Then for all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have, modulo 16,*

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 5^{2\beta} n + 6 \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_1^9 + 8 f_4 f_5, \tag{24}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 5^{2\beta+1} n + 14 \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8 f_1 f_{20} + 8 q f_5^9, \tag{25}$$

$$b_{2,5}\left(16 \cdot 5^{2\beta+2} n + n_1 \cdot 5^{2\beta+1} + 1\right) \equiv 0, \tag{26}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_2^3 f_5^3, \tag{27}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} n + 6 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8 q f_1^3 f_{10}^3, \tag{28}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_2 f_5, \tag{29}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} n + 22 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8 f_1 f_{10}, \tag{30}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} n + 2 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8 q^2 f_6^3 f_{15}^3, \tag{31}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+46\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)q^n \equiv 8f_5f_6^3+8qf_2f_{15}^3, \quad (32)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+38\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8f_3^3f_{10}+8q^3f_1f_{30}^3, \quad (33)$$

$$b_{2,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+n_2\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)\equiv 0, \quad (34)$$

$$b_{2,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+n_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)\equiv 0, \quad (35)$$

$$b_{2,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+n_4\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)\equiv 0, \quad (36)$$

$$b_{2,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+n_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)\equiv 0, \quad (37)$$

$$b_{2,5}\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}n+n_6\cdot 3^{2\alpha+1}\cdot 5^{2\beta}+1\right)\equiv 0, \quad (38)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n \equiv 8f_1^9, \quad (39)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8q^2f_7^9, \quad (40)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8qf_5^9, \quad (41)$$

$$b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+n_7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \quad (42)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+44\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n \equiv 8f_2f_3^3, \quad (43)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+76\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n \equiv 8f_1f_6^3, \quad (44)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8q^2f_{10}f_{15}^3, \quad (45)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+92\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n \equiv 8q^3f_5f_{30}^3, \quad (46)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 5^{2\beta}n+28\cdot 5^{2\beta}+1\right)q^n \equiv 8f_1f_{20}+8f_2^3f_5^3, \quad (47)$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 5^{2\beta+1}n+12\cdot 5^{2\beta+1}+1\right)q^n \equiv 8f_4f_5+8qf_1^3f_{10}^3, \quad (48)$$

$$b_{2,5}\left(32\cdot 5^{2\beta+1}n+n_8\cdot 5^{2\beta}+1\right)\equiv 0. \quad (49)$$

Theorem 2. *Let* $n_9 \in \{22, 38\}$*,* $n_{10} \in \{34, 66\}$*,* $n_{11} \in \{26, 42, 58, 74\}$*,* $n_{12} \in \{44, 76\}$*,* $n_{13} \in \{68, 132\}$ *and* $n_{14} \in \{52, 84, 116, 148\}$*. Then for all* $n \geq 0$ *and* $\alpha, \beta, \gamma \geq 0$*, we have, modulo 4,*

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1\right) q^n \equiv 2f_1^3, \tag{50}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1\right) q^n \equiv 2f_7^3, \tag{51}$$

$$\begin{aligned} & b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1\right) \\ \equiv & \begin{cases} 2 & \text{if } \ n = k(3k+1)/2 \ \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{52}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1\right) q^n \equiv 2f_3^3, \tag{53}$$

$$b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 34 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1\right) \equiv 0, \tag{54}$$

$$b_{2,5}\left(16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + n_9 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1\right) \equiv 0, \tag{55}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} + 1\right) q^n \equiv 2f_5^3, \tag{56}$$

$$b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + n_{10} \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} + 1\right) \equiv 0, \tag{57}$$

$$b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + n_{11} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) \equiv 0, \tag{58}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) q^n \equiv 2f_1^3, \tag{59}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} + 1\right) q^n \equiv 2f_7^3, \tag{60}$$

$$\begin{aligned} & b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) \\ \equiv & \begin{cases} 2 & \text{if } \ n = k(3k+1)/2 \ \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{61}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) q^n \equiv 2f_3^3, \tag{62}$$

$$b_{2,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 34 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) \equiv 0, \tag{63}$$

$$b_{2,5}\left(16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + n_9 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} + 1\right) \equiv 0, \tag{64}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{65}$$

$$b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+n_{10}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{66}$$

$$b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+n_{11}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{67}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_1^3, \tag{68}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n\equiv 2f_7^3, \tag{69}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right) \equiv \begin{cases} 2 & \textit{if } \ n=k(3k+1)/2 \ \textit{ for some } k\in\mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \tag{70}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_3^3, \tag{71}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{72}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+n_{12}\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{73}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{74}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+n_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{75}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+n_{14}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{76}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_1^3, \tag{77}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}+1\right)q^n\equiv 2f_7^3, \tag{78}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right) \equiv \begin{cases} 2 & \textit{if } \ n=k(3k+1)/2 \ \textit{ for some } k\in\mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \tag{79}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_3^3, \tag{80}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{81}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+n_{12}\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{82}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{83}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+n_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{84}$$

$$b_{2,5}\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+n_{14}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)\equiv 0. \tag{85}$$

3. Proof of Theorem 1

From Equation (10) with $\ell=2$ and $m=5$, we see that

$$\sum_{n=0}^{\infty} b_{2,5}(n)\,q^n=\frac{f_4^2 f_5^2}{f_1^2 f_{20}^2}. \tag{86}$$

Using (18) in (86) and then comparing the coefficients of q^{2n+1} on both sides of the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} b_{2,5}(2n+1)\,q^n=2\frac{f_2^5 f_5}{f_1^5 f_{10}}. \tag{87}$$

From the binomial theorem and Equation (1), for any positive integers k and m, we can easily prove the following identities:

$$f_k^{2m}\equiv f_{2k}^{m} \pmod 2, \tag{88}$$

$$f_k^{4m}\equiv f_{2k}^{2m} \pmod 4, \tag{89}$$

$$f_k^{8m}\equiv f_{2k}^{4m} \pmod 8. \tag{90}$$

Employing (12) and (18) along with (88) and (90) in (87), we get, modulo 16,

$$\sum_{n=0}^{\infty} b_{2,5}(4n+1)\,q^n\equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_1^3 f_5 f_{20}}+8qf_2^7 f_{10} \tag{91}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}(4n+3)\,q^n\equiv 2\frac{f_2^5 f_{20}}{f_1^4 f_4 f_{10}}+8\frac{f_4^5}{f_1^3 f_5}. \tag{92}$$

Utilizing (12) and (19) in (92), we arrive at

$$\sum_{n=0}^{\infty} b_{2,5}(8n+3)\,q^n\equiv 2\frac{f_2 f_{10}}{f_1 f_5}+8\frac{f_2^7}{f_1^3 f_5} \tag{93}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(8n+7\right)q^n \equiv 8f_8f_{10} + 8f_1^3f_2^5f_5. \tag{94}$$

Using (20) in (94), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(16n+7\right)q^n \equiv 8f_1^9 + 8f_4f_5 \tag{95}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(16n+15\right)q^n \equiv 8f_2^3f_5^3. \tag{96}$$

Congruence (95) is the $\beta = 0$ case of (24). Suppose that Congruence (24) is true for some integer $\beta \geq 0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 5^{2\beta}n + 6\cdot 5^{2\beta} + 1\right)q^n &\equiv 8f_1^9 + 8f_4f_5 \\ &\equiv 8f_{25}^9\left(R(q^5)^{-1} - q - q^2R(q^5)\right)^9 \\ &\quad + 8f_5f_{100}(R(q^{20})^{-1} - q^4 - q^8R(q^{20})). \end{aligned}\tag{97}$$

Comparing the coefficients of q^{5n+4} on both sides of the above equation, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 5^{2\beta+1}n + 14\cdot 5^{2\beta+1} + 1\right)q^n &\equiv 8f_1f_{20} + 8qf_5^9 \\ &\equiv 8f_{20}f_{25}\left(R(q^5)^{-1} - q - q^2R(q^5)\right) + 8qf_5^9. \end{aligned}\tag{98}$$

Collecting the coefficients of q^{5n+1} from both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 5^{2\beta+2}n + 6\cdot 5^{2\beta+2} + 1\right)q^n \equiv 8f_1^9 + 8f_4f_5, \tag{99}$$

which implies that Congruence (24) is true for $\beta+1$. By mathematical induction, Congruence (24) is true for all integers $\beta \geq 0$.

Using (22) in (24) and then comparing the coefficients of q^{5n+4} on both sides, we arrive at (25).

Collecting the coefficients of q^{5n+i} for $i = 3, 4$ from (98), we obtain (26).

Congruence (96) is the $\alpha = \beta = 0$ case of (27). Suppose that Congruence (27) holds for $\alpha \geq 0$ with $\beta = 0$. Using (13) in (27) with $\beta = 0$, we have

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}n + 14\cdot 3^{2\alpha} + 1\right)q^n \equiv 8f_6f_{15} + 8q^2f_{15}f_{18}^3 + 8q^5f_6f_{45}^3 + 8q^7f_{18}^3f_{45}^3, \tag{100}$$

which implies

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+1}n+14\cdot 3^{2\alpha}+1\right)q^n \equiv 8f_2f_5, \tag{101}$$

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+1}n+10\cdot 3^{2\alpha+1}+1\right)q^n \equiv 8q^2f_6^3f_{15}^3 \tag{102}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+1}n+46\cdot 3^{2\alpha}+1\right)q^n \equiv 8f_5f_6^3+8qf_2f_{15}^3. \tag{103}$$

Comparing the coefficients of q^{3n+2} on both sides of Equation (102), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+2}n+14\cdot 3^{2\alpha+2}+1\right)q^n \equiv 8f_2^3f_5^3, \tag{104}$$

which implies that Congruence (27) is true for $\alpha+1$ with $\beta=0$. Hence, by induction, Congruence (27) is true for any integer $\alpha\geq 0$ with $\beta=0$. Suppose that Congruence (27) holds for some integers $\alpha,\beta\geq 0$. Employing (22) in (27) and then comparing the coefficients of q^{5n+1}, we find that

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+6\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8qf_1^3f_{10}^3. \tag{105}$$

Employing (22) in (105) and then extracting the coefficients of q^{5n+4}, we arrive at

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+1\right)q^n \equiv 8f_2^3f_5^3, \tag{106}$$

which implies that Congruence (27) is true for $\beta+1$. Hence, by induction, Congruence (27) is true for all integers $\alpha,\beta\geq 0$.

Using (22) in (27) and then collecting the coefficients of q^{5n+1} from the resultant equation, we get (28).

Employing (13) in (27) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (29), (31) and (32) respectively.

Employing (22) in (29) and (32), we get (30) and (33) respectively.

Using (29) and (30) along with (22), we obtain (34) and (35) respectively.

Using (32) and (33) along with (22), we obtain (36) and (37) respectively.

Collecting the coefficients of q^{3n} and q^{3n+1} from (31), we get (38).

Using (12) and (17) in (91), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(8n+1\right)q^n \equiv 2\frac{f_2^{14}f_{10}^2}{f_1^{11}f_4^3f_5f_{20}}+8q\frac{f_2^5f_4^3f_{20}}{f_1^8f_{10}} \tag{107}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(8n+5\right)q^n \equiv 8\frac{f_4^5}{f_1^3 f_5} - 2\frac{f_2^5 f_{20}}{f_1^4 f_4 f_{10}} + 8f_1^3 f_2^2 f_5. \tag{108}$$

Utilizing (88) and (90) in (107), we see that

$$b_{2,5}(8n+1) \equiv b_{2,5}(4n+1).$$

By using the above relation and by induction on α, we get

$$b_{2,5}(2^{\alpha+2}n+1) \equiv b_{2,5}(4n+1).$$

Using (12), (19) and (20) in (108), we obtain

$$\sum_{n=0}^{\infty} b_{2,5}\left(16n+5\right)q^n \equiv 8\frac{f_2^6}{f_1 f_5} + 8f_2^3 - 2\frac{f_2 f_{10}}{f_1 f_5} \tag{109}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(16n+13\right)q^n \equiv 8f_8 f_{10} + 8f_1 f_2 f_5^3 + 8f_1^3 f_2^5 f_5. \tag{110}$$

Employing (20) and (21) in (110), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(32n+13\right)q^n \equiv 8f_1^9 \tag{111}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(32n+29\right)q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3. \tag{112}$$

Congruence (111) is the $\alpha = \beta = \gamma = 0$ case of (39). Suppose that Congruence (39) is true for $\alpha \geq 0$ with $\beta = \gamma = 0$. Equation (39) with $\beta = \gamma = 0$ becomes

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}n + 12\cdot 3^{4\alpha} + 1\right)q^n \equiv 8f_1^9. \tag{113}$$

Using (13) in (113) and then comparing the coefficients of q^{3n}, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+1}n + 12\cdot 3^{4\alpha} + 1\right)q^n &\equiv 8f_1^3 + 8qf_3^9 \\ &\equiv 8f_3 + 8qf_3^9 + 8qf_9^3,\end{aligned} \tag{114}$$

which implies

$$\begin{aligned}\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+2}n + 12\cdot 3^{4\alpha+2} + 1\right)q^n &\equiv 8f_1^9 + 8f_3^3 \\ &\equiv 8qf_6 f_9^3 + 8q^2 f_3 f_9^6 + 8q^3 f_9^9,\end{aligned} \tag{115}$$

which yields

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+3}n+12\cdot 3^{4\alpha+2}+1\right)q^n \equiv 8qf_3^9, \tag{116}$$

which implies

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha+4}n+12\cdot 3^{4\alpha+4}+1\right)q^n \equiv 8f_1^9, \tag{117}$$

which implies that Congruence (39) is true for $\alpha+1$. By mathematical induction, Congruence (39) is true for all integers $\alpha \geq 0$ with $\beta=\gamma=0$. Suppose that Congruence (39) is true for $\alpha,\beta \geq 0$ with $\gamma=0$. From Equation (39) with $\gamma=0$ and employing (22), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+1}n+28\cdot 3^{4\alpha}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8qf_5^9, \tag{118}$$

which implies

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta+2}+1\right)q^n \equiv 8f_1^9, \tag{119}$$

which implies that Congruence (39) with $\gamma=0$ is true for $\beta+1$. So, by induction, Congruence (39) with $\gamma=0$ is true for all integers $\alpha,\beta \geq 0$. Suppose that Congruence (39) is true for $\alpha,\beta,\gamma \geq 0$ and utilizing (23) in (39), we arrive at

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8q^2f_7^9. \tag{120}$$

Collecting the coefficients of q^{7n+2} from both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty} b_{2,5}\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n \equiv 8f_1^9, \tag{121}$$

which implies that Congruence (39) is true for $\gamma+1$. By induction, Congruence (39) is true for all integers $\alpha,\beta,\gamma \geq 0$.

Using (23) in (39) and then collecting the coefficients of q^{7n+4} from the resultant equation, we get (40).

Using (22) in (39) and then comparing the coefficients of q^{5n+4} on both sides of the resultant equation, we obtain (41).

Congruence (41) implies (42).

Using (13) in (39) and then collecting the coefficients of q^{3n+1} and q^{3n+2} from the resultant equation, we obtain (43) and (44) respectively.

Using (43) and (44) along with (22), we obtain (45) and (46) respectively.

Congruence (112) is the $\beta = 0$ case of (47). Suppose that Congruence (47) is true for some integer $\beta \geq 0$. Using (22) in (47) and then comparing the coefficients of q^{5n+1}, we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(32 \cdot 5^{2\beta+1}n + 12 \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3. \tag{122}$$

Employing (22) in (122) and then comparing the coefficients of q^{5n+4} on both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} b_{2,5}\left(32 \cdot 5^{2\beta+2}n + 28 \cdot 5^{2\beta+2} + 1\right) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3, \tag{123}$$

which implies that Congruence (47) is true for $\beta + 1$. By mathematical induction, Congruence (47) is true for all integers $\beta \geq 0$.

Using (22) in (47) and then collecting the coefficients of q^{5n+1} from the resultant equation, we get (48).

Collecting the coefficients of q^{5n+i} for $i = 3, 4$ from (47) along with (22), we obtain (49).

4. Proof of Theorem 2

From Equation (93), we have, modulo 4,

$$\begin{aligned}\sum_{n=0}^{\infty} b_{2,5}(8n+3)\, q^n &\equiv 2\frac{f_1 f_{10}}{f_5} \\ &\equiv 2f_2^3 + 2q f_{10}^3, \end{aligned} \tag{124}$$

which implies

$$\sum_{n=0}^{\infty} b_{2,5}(16n+3)\, q^n \equiv 2f_1^3 \tag{125}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}(16n+11)\, q^n \equiv 2f_5^3. \tag{126}$$

Equation (125) is the $\alpha = \beta = \gamma = 0$ case of (50). Suppose that Congruence (50) is true for $\alpha \geq 0$ with $\beta = \gamma = 0$. From Equation (50) with $\beta = \gamma = 0$, we have

$$\sum_{n=0}^{\infty} b_{2,5}\left(16 \cdot 3^{2\alpha}n + 2 \cdot 3^{2\alpha} + 1\right) q^n \equiv 2f_1^3. \tag{127}$$

Utilizing (13) in (127), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+1}n+2\cdot 3^{2\alpha+2}+1\right)q^n \equiv 2f_3^3, \tag{128}$$

which yields

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha+2}n+2\cdot 3^{2\alpha+2}+1\right)q^n \equiv 2f_1^3, \tag{129}$$

which implies that Congruence (50) is true for $\alpha+1$ with $\beta=\gamma=0$. By mathematical induction, Congruence (50) is true for all integers $\alpha\geq 0$. Suppose that Congruence (50) holds for $\alpha,\beta\geq 0$ with $\gamma=0$ and employing (22) in (50) with $\gamma=0$, we obtain

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+1\right)q^n \equiv 2f_5^3, \tag{130}$$

which implies

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+1\right)q^n \equiv 2f_1^3, \tag{131}$$

which implies that Congruence (50) is true for $\beta+1$ with $\gamma=0$. By mathematical induction, Congruence (50) is true for all non-negative integers α,β with $\gamma=0$. Suppose that Congruence (50) holds for $\alpha,\beta,\gamma\geq 0$. Employing (23) in (50), we get

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n \equiv 2f_7^3, \tag{132}$$

which implies

$$\sum_{n=0}^{\infty} b_{2,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n \equiv 2f_1^3, \tag{133}$$

which implies that Congruence (50) is true for $\gamma+1$. By mathematical induction, Congruence (50) is true for all integers $\alpha,\beta,\gamma\geq 0$.

Employing (23) in (50) and then extracting the coefficients of q^{7n+6} from the resultant equation, we arrive at (51).

Using (13) in (50) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (52), (53) and (54), respectively.

Collecting the coefficients of q^{3n+1} and q^{3n+2} from (53), we get (55).

Employing (22) in (50) and then comparing the coefficients of q^{5n+3}, we obtain (56).

Collecting the coefficients of q^{5n+2} and q^{5n+4} from (50) along with (22), we obtain (57).

From Congruence (56), we arrive at (58).

From Equation (126), we have

$$\sum_{n=0}^{\infty} b_{2,5}\left(80n+11\right)q^{n} \equiv 2f_1^3. \tag{134}$$

Congruence (134) is the $\alpha=\beta=\gamma=0$ case of (59). The rest of the proofs of the identities (59)–(67) are similar to the proofs of the identities (50)–(58). So, we omit the details.

Employing (17) in (109), we obtain

$$\sum_{n=0}^{\infty} b_{2,5}\left(32n+5\right)q^{n} \equiv 2f_1^3 \tag{135}$$

and

$$\sum_{n=0}^{\infty} b_{2,5}\left(32n+21\right)q^{n} \equiv 2f_5^3. \tag{136}$$

The rest of the proofs of the identities (68)–(85) are similar to the proofs of the identities (50)–(58). So, we omit the details.

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References

[1] C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, Chapter 16 of Ramanujan's Second Notebook: Theta functions and q-series, *Mem. Amer. Math. Soc.* **315** (1985), 1-91.

[2] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, New York, 1991.

[3] H. C. Chan, Ramanujan's cubic continued fraction and a generalization of his "most beautiful identity", *Int. J. Number Theory* **6** (2010), 673-680.

[4] H. C. Chan, Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function, *Int. J. Number Theory* **6** (2010), 819-834.

[5] H. H. Chan and P. C. Toh, New analogues of Ramanujan's partition identities, *J. Number Theory* **130** (2010), 1898-1913.

[6] M. D. Hirschhorn, Ramanujan's "most beautiful identity", *Amer. Math. Monthly* **118** (2011), 839-845.

[7] M. D. Hirschhorn, *The Power of q*, Springer International Publishing, Switzerland, 2017.

[8] M. D. Hirschhorn, A conjecture of B. Lin on cubic partition pairs, *Ramanujan J.* **45** (2018), 781-795.

[9] M. D. Hirschhorn and J. A. Sellers, Elementary proofs of parity results for 5-regular partitions, *Bull. Aust. Math. Soc.* **81** (2010), 58-63.

[10] B. Kim, Partition statistics for cubic partition pairs, *Electron. J. Combin.* **18** (2011), P128.

[11] B. Kim, Cubic partition pairs weighted by the parity of the crank, *Korean J. Math.* **23** (2015), 637-642.

[12] B. Kim and P. C. Toh, On the crank function of cubic partition pairs, *Ann. Comb.* **22** (2018), 803-818.

[13] B. L. S. Lin, Congruences modulo 27 for cubic partition pairs, *J. Number Theory* **171** (2017), 31-42.

[14] B. L. S. Lin, L. Wang and E. X. W. Xia, Congruences for cubic partition pairs modulo powers of 3, *Ramanujan J.* **46** (2018), 563-578.

[15] M. S. Mahadeva Naika and D. S. Gireesh, General family of congruences modulo large power of 3 for cubic partition pairs, *New Zealand J. of Mathematics* **47** (2017), 43-56.

[16] M. S. Mahadeva Naika, B. Hemanthkumar and H. S. Sumanth Bharadwaj, Color partition identities arising from Ramanujan's theta functions, *Acta Math. Vietnam* **41** (4), (2016), 633-660.

[17] M. S. Mahadeva Naika and S. Shivaprasada Nayaka, Congruences for ℓ-regular cubic partition pairs, *Rend. Circ. Mat. Palermo Ser. II* **67** (3), (2018), 465-476.

[18] S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.* **19** (1919), 210-213.

[19] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, RI, 2000.

[20] G. N. Watson, Theorems stated by Ramanujan (VII): Theorems on continued fractions, *J. London Math. Soc.* **4** (1929), 39-48.

[21] H. Zhao and Z. Zhong, Ramanujan type congruences for a partition function, *Electron. J. Combin.* **18** (2011), P58.