ON THE DIOPHANTINE EQUATION $\frac{AX^{N+2L}+C}{ABT^2X^N+D} = BY^2$

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Received: 3/23/18, Revised: 9/16/19, Accepted: 2/6/20, Published: 2/14/20

Abstract

We obtain all solutions of the equation $\frac{ax^{n+2l}+c}{abt^2x^n+d} = by^2$ with $c,\, d \in \{\pm 1,\, \pm 2,\, \pm 4\}$.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ denote the sets of all positive integers and all integers, respectively. In this paper, we will prove the following theorem.

Theorem 1. *All solutions $(a,\, b,\, x,\, y,\, n,\, t,\, l,\, c,\, d)$ of the equation*

$$\frac{ax^{n+2l}+c}{abt^2x^n+d} = by^2, \quad a,\, x,\, y,\, n,\, t,\, l \in \mathbb{N}, \quad b \in \mathbb{Z}, \tag{1}$$

with $c,\, d \in \{\pm 1,\, \pm 2,\, \pm 4\}$ and $x > 1$, are given in Table 1, where u denotes an arbitrary positive integer.

Special cases of this general theorem are already known. The case $(a,\, b,\, n,\, t,\, c,\, d)$ $= (1,\, 1,\, 1,\, 1,\, -1,\, -1)$ was considered by W. Ljunggren [5]. This was extended later by Q. Sun and P. Yuan [7] to the case $(a,\, b,\, n,\, t) = (a,\, 1,\, 1,\, 1)$ and $c = d = \pm 1$. For further generalizations concerning the cases $(a,\, b,\, n,\, t) = (a,\, b,\, 1,\, 1)$, $c = d = \pm 1$ and $(a,\, b,\, n,\, t) = (a,\, 1,\, 1,\, 1)$, $c = d = \pm 2$ or ± 4, see the work of Z. Cao [1] and J. Luo [6], respectively. M. Filaseta, F. Luca, P. Stânicâ and R. Uuderwood [3] dealt with the case $(a,\, b,\, n,\, t) = (a,\, 1,\, n,\, 1)$, $c = d = -1$. J. Luo and P. Yuan [11] extended this result, they proved the following theorem.

$(c,\, d)$	$(a,\, b,\, x,\, y,\, n,\, t,\, l)$	Remarks
$(d,\, d)$	$(a,\, 1,\, u,\, 1,\, n,\, u^l,\, l)$	$d \in \{\pm 1,\, \pm 2,\, \pm 4\}$
$(d,\, -d)$	$(a,\, -1,\, u,\, 1,\, n,\, u^l,\, l)$	$d \in \{\pm 1,\, \pm 2,\, \pm 4\}$
$(-2d,\, d)$	$(a,\, -2,\, 2u,\, 1,\, n,\, 2^{l-1}u^l,\, l)$	$d \in \{\pm 1,\, \pm 2\}$
$(4d,\, d)$	$(a,\, 4,\, 2u,\, 1,\, n,\, 2^{l-2}u^l,\, l)$	$d \in \{-1,\, 1\}\,,\, l \geq 2$
$(4d,\, d)$	$(a,\, 1,\, 2u,\, 2,\, n,\, 2^{l-1}u^l,\, l)$	$d \in \{-1,\, 1\}$
$(-4d,\, d)$	$(a,\, -4,\, 2u,\, 1,\, n,\, 2^{l-2}u^l,\, l)$	$d \in \{-1,\, 1\}\,,\, l \geq 2$
$(-4d,\, d)$	$(a,\, -1,\, 2u,\, 2,\, n,\, 2^{l-1}u^l,\, l)$	$d \in \{-1,\, 1\}$
$(-1,\, -1)$	$((3^{2u-1}+1)/4,\, 1,\, 3,\, 3^{2u}+2,\, 1,\, 1,\, 2u)$	
$(1,\, 1)$	$((3^{2u-2}-1)/4,\, 1,\, 3,\, 3^{2u-1}-2,\, 1,\, 1,\, 2u-1)$	
$(1,\, -1)$	$((3^{2u-2}-1)/4,\, -1,\, 3,\, 3^{2u-1}-2,\, 1,\, 1,\, 2u-1)$	
$(-1,\, 1)$	$((3^{2u-1}+1)/4,\, -1,\, 3,\, 3^{2u}+2,\, 1,\, 1,\, 2u)$	
$(-2,\, -2)$	$((3^{2u-2}+1)/2,\, 1,\, 3,\, 3^{2u-1}+2,\, 1,\, 1,\, 2u-1)$	
$(2,\, 2)$	$((3^{2u-2}-1)/2,\, 1,\, 3,\, 3^{2u}-2,\, 1,\, 1,\, 2u)$	
$(-2,\, 2)$	$((3^{u-1}+1)/2,\, -1,\, 3,\, 3^{u}+2,\, 1,\, 1,\, u)$	
$(2,\, -2)$	$((3^{u-1}-1)/2,\, -1,\, 3,\, 3^{u}-2,\, 1,\, 1,\, u)$	
$(-4,\, 4)$	$(3^{u-1}+1,\, -1,\, 3,\, 3^{u}+2,\, 1,\, 1,\, u)$	
$(4,\, -4)$	$(3^{u-1}-1,\, -1,\, 3,\, 3^{u}-2,\, 1,\, 1,\, u)$	
$(-2,\, \pm 4)$	$(1,\, b,\, 2,\, t,\, 1,\, t,\, l)$	$2^l = bt^2 \pm 1$
$(1,\, 4)$	$(2 \cdot 3^{u-1}-1,\, 1,\, 3,\, 3^{u}-1,\, 1,\, 1,\, u)$	
$(-1,\, 4)$	$(2 \cdot 3^{u-1}+1,\, -1,\, 3,\, 3^{u}+1,\, 1,\, 1,\, u)$	
$(1,\, -4)$	$(2 \cdot 3^{u-1}-1,\, -1,\, 3,\, 3^{u}-1,\, 1,\, 1,\, u)$	
$(1,\, 4)$	$(1,\, 1,\, 3,\, 2,\, 1,\, 1,\, 1)$	
$(-1,\, 4)$	$(1,\, -1,\, 3,\, 4,\, 2,\, 1,\, 1)$	
$(1,\, -4)$	$(1,\, -1,\, 3,\, 2,\, 1,\, 1,\, 1)$	
$(-1,\, -4)$	$(1,\, 1,\, 3,\, 4,\, 2,\, 1,\, 1)$, $(3,\, 1,\, 3,\, 4,\, 1,\, 1,\, 1)$	
$(-1,\, 4)$	$(3,\, -1,\, 3,\, 4,\, 1,\, 1,\, 1)$	
$(4,\, \pm 1)$	$(1,\, -d,\, 2,\, 6,\, 1,\, 1,\, 2)$	
$(2,\, \mp 4)$	$(3,\, \pm 1,\, 2,\, 7,\, 1,\, 1,\, 2)$	
$(-4,\, \pm 2)$	$(1,\, \pm 1,\, 2,\, 1,\, 1,\, 1,\, 1)$	
$(4,\, \pm 2)$	$(3,\, \mp 1,\, 2,\, 5,\, 1,\, 1,\, 2)$, $(1,\, \mp 3,\, 2,\, 1,\, 1,\, 1,\, 1)$ $(1,\, \pm 1,\, 2,\, 3,\, 1,\, 1,\, 2)$, $(1,\, \mp 2,\, 2,\, 3,\, 1,\, 1,\, 2)$	
$(-4,\, -4)$	$(1,\, 1,\, 5,\, 11,\, 1,\, 1,\, 1)$	
$(1,\, 1)$	$(1,\, -1,\, 2,\, 3,\, 1,\, 1,\, 1)$	
$(2,\, 2)$	$(1,\, -1,\, 2,\, 3,\, 2,\, 1,\, 1)$	
$(4,\, 4)$	$(1,\, -1,\, 2,\, 3,\, 3,\, 1,\, 1)$, $(1,\, -1,\, 2,\, 3,\, 1,\, 2,\, 2)$	
$(2,\, -2)$	$(1,\, 1,\, 2,\, 3,\, 2,\, 1,\, 1)$, $(2,\, 1,\, 2,\, 3,\, 1,\, 1,\, 1)$	
$(4,\, -4)$	$(1,\, 1,\, 2,\, 3,\, 3,\, 1,\, 1)$, $(2,\, 1,\, 2,\, 3,\, 2,\, 1,\, 1)$ $(4,\, 1,\, 2,\, 3,\, 1,\, 1,\, 1)$, $(1,\, 1,\, 2,\, 3,\, 1,\, 2,\, 2)$	
$(-4,\, 4)$	$(1,\, -1,\, 5,\, 11,\, 1,\, 1,\, 1)$	

Table 1: Equations and their solutions

Theorem 2. ([11]) *The equation*

$$\frac{ax^{n+2l}+c}{abt^2x^n+c}=by^2, \quad c\in\{\pm1,\,\pm2,\,\pm4\}\,,(a,\,c)=1,$$

holds for some integers a, b, x, n, t and l with $x>1$, $n>0$, $t>0$, $a>0$ and $l>0$ if and only if $(a,\,b,\,x,\,y,\,n,\,t,\,l,\,c)$ is one of the following:

$$(a,\,1,\,u,\,\pm1,\,n,\,u^l,\,l,\,c),$$
$$(\frac{3^{2u-1}+1}{4},\,1,\,3,\,\pm(3^{2u}+2),\,1,\,1,\,2u,\,-1),$$
$$(\frac{3^{2u-1}-1}{4},\,1,\,3,\,\pm(3^{2u}-2),\,1,\,1,\,2u-1,\,1),$$
$$(\frac{3^{2u-2}+1}{2},\,1,\,3,\,\pm(3^{2u-1}+2),\,1,\,1,\,2u-1,\,-2),$$
$$(\frac{3^{2u-1}-1}{2},\,1,\,3,\,\pm(3^{2u}-2),\,1,\,1,\,2u,\,2),$$
$$(1,\,-1,\,2,\,\pm3,\,1,\,1,\,1,\,1),(1,\,-1,\,2,\,\pm3,\,2,\,1,\,1,\,2),$$
$$(1,\,-1,\,2,\,\pm3,\,3,\,1,\,1,\,4),(1,\,-1,\,2,\,\pm3,\,1,\,2,\,2,\,4),$$
$$(1,\,1,\,5,\,\pm11,\,1,\,1,\,1,\,-4),$$

where u denotes an arbitrary positive integer.

All known results above are restricted to the condition of $c=d$. The main purpose of the paper is to consider the case $c\neq d$.

The arrangement is as follows. In Section 2, we give some necessary lemmas with respect to Diophantine equations

$$kx^2-ly^2=C, \quad C=1,\,2,\,4, \tag{2}$$

where k, l are coprime positive integers and kl is not a square. Recall that the minimal positive solution of (2) is one of the positive integer solutions $(x,\,y)$ of (2) such that $x\sqrt{k}+y\sqrt{l}$ is the smallest, which is equivalent to determining a positive integer solution $(x,\,y)$ such that x and y are the smallest. If $k=C=1$ or $l=C=1$, then such a solution is also called the fundamental solution of (2). We present some known results that will be used in the proof of our main theorem. In Section 3 we prove some necessary facts to make our proofs more concise. Finally, we prove Theorem 1 in Section 4. Because the case $c=d$ has been done as we can see in Theorem 2, we only consider the case $c\neq d$.

2. Lemmas

To prove our main Theorem 1, we need the Störmer theorem on the Pell equation $x^2 - Dy^2 = 1$ and the related results for general quadratic equations $kx^2 - ly^2 = C, C \in \{1, 2, 4\}$. We have the following.

Lemma 1. (Störmer theorem [2]) *Let D be a positive nonsquare integer and (x_1, y_1) be a positive integer solution of the Pell equation*

$$x^2 - Dy^2 = 1 \tag{3}$$

or

$$x^2 - Dy^2 = -1. \tag{4}$$

If every prime divisor of y_1 divides D, then $x_1 + y_1\sqrt{D}$ is the fundamental solution of (3) or (4).

Let k, l be coprime positive integers such that $k > 1$ and kl is not a square. For the Diophantine equation

$$kx^2 - ly^2 = 1, \tag{5}$$

D. Walker [8] obtained the following analogue of the Störmer theorem. See also Q. Sun and P. Yuan [7].

Lemma 2. ([7, 8, 11]) *Let (x, y) be a positive integer solution of Equation (5).*
(i) If every prime divisor of x divides k or x_1, then either

$$x\sqrt{k} + y\sqrt{l} = \varepsilon$$

or

$$x\sqrt{k} + y\sqrt{l} = \varepsilon^3, \ \textit{and} \quad x = 3^s x_1, \ 3 \nmid x_1, \ 3^s + 3 = 4kx_1^2, \ s \in \mathbb{N},$$

where $\varepsilon = x_1\sqrt{k} + y_1\sqrt{l}$ is the minimal positive solution of Equation (5).
(ii) If every prime divisor of y divides l or y_1, then

$$x\sqrt{k} + y\sqrt{l} = \varepsilon$$

or

$$x\sqrt{k} + y\sqrt{l} = \varepsilon^3, \ \textit{and} \quad y = 3^s y_1, \ 3 \nmid y_1, \ 3^s - 3 = 4ly_1^2, \ s \in \mathbb{N}.$$

Using the same techniques as in the work of [7], J. Luo proved the following results.

Lemma 3. ([6, 11]) *Let k, l be odd integers and (x, y) be a positive integer solution of the Diophantine equation*

$$kx^2 - ly^2 = 2. \tag{6}$$

(i) If every prime divisor of x *divides* k *or* x_1*, then*

$$x\sqrt{k}+y\sqrt{l}=\varepsilon$$

or

$$\frac{x\sqrt{k}+y\sqrt{l}}{\sqrt{2}}=\left(\frac{\varepsilon}{\sqrt{2}}\right)^3, \text{ and } \quad x=3^s x_1,\ 3\nmid x_1,\ 3^s+3=2kx_1^2,\ s\in\mathbb{N},$$

where $\varepsilon = x_1\sqrt{k}+y_1\sqrt{l}$ *is the minimal positive solution of Equation (6).*

(ii) If every prime divisor of y *divides* l *or* y_1*, then*

$$x\sqrt{k}+y\sqrt{l}=\varepsilon$$

or

$$\frac{x\sqrt{k}+y\sqrt{l}}{\sqrt{2}}=\left(\frac{\varepsilon}{\sqrt{2}}\right)^3, \text{ and } \quad y=3^s y_1,\ 3\nmid y_1,\ 3^s-3=2ly_1^2,\ s\in\mathbb{N}.$$

Lemma 4. ([6]) *Let* k*,* l *be odd integers and* (x, y) *be a positive integer solution of the equation*

$$kx^2-ly^2=4, \quad k>1. \tag{7}$$

(i) If every prime divisor of x *divides* k*, then* $x\sqrt{k}+y\sqrt{l}=\varepsilon$ *is the minimal positive solution of Equation (7) except for* $(k, l, x, y) = (5, 1, 5, 11)$*.*

(ii) If every prime divisor of y *divides* l*, then* $x\sqrt{k}+y\sqrt{l}=\varepsilon$ *is the minimal positive solution of Equation (7).*

Lemma 5. ([10]) *If Equation (7) has solutions, so does Equation (5). Let* ε_1*,* ε_2 *and* ε_3 *be the minimal positive solutions of equations (3), (5) and (7), respectively. Then* $\varepsilon_1=(\varepsilon_2)^2$ *and* $\varepsilon_2=(\varepsilon_3/2)^3$*.*

Lemma 6. ([9, 11]) *Let* D *be a given positive nonsquare integer.*

(i) If $8|D$*, then at most one of the Diophantine equations* $kx^2-ly^2=1$ *has integer solutions, where* (k, l) *ranges over all pairs* (k, l) *such that* $k>1$*,* $kl=D$*.*

(ii) If $2|D$ *and* $8\nmid D$*, then only one of the Diophantine equations* $kx^2-ly^2=1$ *has integer solutions, where* (k, l) *ranges over all pairs* (k, l) *such that* $k>1$*,* $kl=D$*.*

(iii) If $2\nmid D$*, then only one of the Diophantine equations* $kx^2-ly^2=1$*,* $kx^2-ly^2=2$ *has integer solutions, where* (k, l) *of the former equation ranges over all pairs* (k, l) *such that* $k>1$*,* $kl=D$*, while the latter ranges over all pairs* (k, l) *such that* $k>0$*,* $kl=D$*.*

(iv) If $2\nmid D$ *and the Diophantine equation* $x^2-Dy^2=4$ *has solutions in odd integers* x *and* y*, then only one of the Diophantine equations* $kx^2-ly^2=4$ *has integer solutions, where* (k, l) *ranges over all pairs* (k, l) *such that* $k>1$*,* $kl=D$*.*

Lemma 7. ([11]) *(i) Let D be a positive nonsquare integer with $2 \nmid D$. Let (x, y) be a positive integer solution of the Pell equation (3) with $y = 2^n y'$, $n \in \mathbb{N}$. If every prime divisor of y' divides D, then $x + y\sqrt{D} = \varepsilon$ or ε^2 or ε^3, where $\varepsilon = x_1 + y_1\sqrt{D}$ is the fundamental solution of Equation (3).*

(ii) If $(2^n, y)$ is a positive integer solution of Equation (3), then $(2^n, y)$ is the fundamental solution of it.

(iii) Let (x, y) be a positive integer solution of Equation (5) with $x = 2^n x'$, $n \in \mathbb{N}$. If every prime divisor of x' divides k, then $x\sqrt{k} + y\sqrt{l} = \varepsilon$ or ε^3, where $\varepsilon = x_1\sqrt{k} + y_1\sqrt{l}$ is the minimal positive solution of Equation (5).

(iv) Let (x, y) be a positive integer solution of Equation (5) with $y = 2^n y'$, $n \in \mathbb{N}$. If every prime divisor of y' divides l, then $x\sqrt{k} + y\sqrt{l} = \varepsilon$ or ε^3, where $\varepsilon = x_1\sqrt{k} + y_1\sqrt{l}$ is the minimal positive solution of Equation (5).

Remark. In Lemma 7 (iii), if $x\sqrt{k} + y\sqrt{l} = \varepsilon^3$, then we have $x = 3^s x_1$, $3 \nmid x_1$, $3^s + 3 = 4kx_1^2$, $s \in \mathbb{N}$. And if $x\sqrt{k} + y\sqrt{l} = \varepsilon^3$ in (iv), then we have $y = 3^s y_1$, $3 \nmid y_1$, $3^s - 3 = 4ly_1^2$, $s \in \mathbb{N}$.

3. Some Preparations

To make the proof of Theorem 1 more concise, we present some simple results.

Theorem 3. *(i) The Diophantine equation*

$$2x^l = at^3x^n + 3t, \quad x > 1,\ a,\ n,\ t,\ l \in \mathbb{N},\ 2 \nmid xt \tag{8}$$

has only the integer solutions $(a, n, t, l, x) = (1, 1, 1, 1, 3)$ and $(a, n, t, l, x) = (2 \times 3^{u-1} - 1, 1, 1, u, 3)$, $u \in \mathbb{N}$.

(ii) The Diophantine equation

$$2x^l = at^3x^n - 3t, \quad x > 1,\ a,\ n,\ t,\ l \in \mathbb{N},\ 2 \nmid xt \tag{9}$$

has only the integer solutions $(a, n, t, l, x) = (3, 1, 1, 1, 3)$, $(a, n, t, l, x) = (1, 2, 1, 1, 3)$ and $(a, n, t, l, x) = (2 \times 3^{u-1} + 1, 1, 1, u, 3)$, $u \in \mathbb{N}$.

Proof. (i) For the case $l = 1$, Equation (8) becomes

$$2x = at^3x^n + 3t, \quad x > 1,\ a,\ n,\ t \in \mathbb{N},\ 2 \nmid xt,$$

so $a = t = n = 1$ and $x = 3$. Hence Equation (8) has only the integer solution $(a, n, t, l, x) = (1, 1, 1, 1, 3)$ with $l = 1$. Next we consider the case where $l > 1$. It is easy to see that $x|3t$ and $t|x$ from (8). Thus $x = t$ or $x = 3t$. If $x = t$, by (8), we get

$$2t^{l-1} = at^{n+2} + 3,\ l > 1, \tag{10}$$

so $t = 3$ and $l = 2$, which is impossible since $6 < a3^{n+2} + 3$. If $x = 3t$, then, by (8), we obtain

$$2(3t)^{l-1} = at^3x^{n-1} + 1, \qquad (11)$$

so $t = 1$, $x = 3$ and $n = 1$. It follows that $a = 2 \times 3^{l-1} - 1$. Therefore Equation (8) has only the integer solutions $(a, n, t, l, x) = (2 \times 3^{u-1} - 1, 1, 1, u, 3)$, $u \in \mathbb{N}$ with $l > 1$.

(ii) We also divide the proof into two cases according to $l = 1$ and $l > 1$. For $l = 1$, Equation (8) becomes

$$2x = at^3x^n - 3t, \quad x > 1,\ a,\ n,\ t \in \mathbb{N},\ 2 \nmid xt,$$

so $t = 1$ and $x = 3$. It follows that $2 = a \times 3^n - 1$, so $(a, n) = (3, 1)$ and $(1, 2)$. Hence Equation (11) has only the integer solutions $(a, n, t, l, x) = (3, 1, 1, 1, 3)$ and $(1, 2, 1, 1, 3)$ with $l = 1$. The case $l > 1$ can be solved by a similar argument as in (i). This completes the proof. □

Theorem 4. *(i) Each of the Diophantine equations*

$$X^2 - b(bt^2+1)Y^2 = 1, \quad X^2 - \frac{b}{2}(2bt^2+1)Y^2 = 1,\ 2|b, \quad X^2 - 2b(2bt^2+1)Y^2 = 1,$$

$$X^2 - \frac{b}{2}(2bt^2-1)Y^2 = 1,\ 2|b, \quad X^2 - \frac{b}{4}(4bt^2+1)Y^2 = 1,\ 4 \mid b,$$

$$X^2 - b(4bt^2+1)Y^2 = 1 \ \ \textit{and}\ \ X^2 - \frac{b}{4}(4bt^2-1)Y^2 = 1,\ 4 \mid b$$

has no integer solutions (X, Y) with $X = 2^\alpha$, $\alpha \in \mathbb{N}$.

(ii) The Diophantine equation $2X^2 - Y^2 = 1$ has no integer solutions (X, Y) with $Y = 2^\alpha$, $\alpha \in \mathbb{N}$.

(iii) If $b > 0$, then $(X, Y) = (bt^2 + 1, t)$ and $(X, Y) = (bt^2 - 1, t)$ are the fundamental solutions of the equations $X^2 - b(bt^2+2)Y^2 = 1$ and $X^2 - b(bt^2-2)Y^2 = 1$, respectively. If $b < 0$, then $(X, Y) = (-bt^2 - 1, t)$ and $(X, Y) = (-bt^2 + 1, t)$ are the fundamental solutions of $X^2 - b(bt^2+2)Y^2 = 1$ and $X^2 - b(bt^2-2)Y^2 = 1$, respectively .

Proof. For $b > 0$, let $(X, Y) = (u, v)$ be the fundamental solution of the Pell equation $X^2 - b(bt^2+1)Y^2 = 1$. It is easy to see that $(X, Y) = (1, t)$ is the minimal solution of the equation $(bt^2+1)X^2 - bY^2 = 1$. By Lemma 5, we obtain $u + v\sqrt{b(bt^2+1)} = (\sqrt{bt^2+1} + t\sqrt{b})^2$, from which $u = 2bt^2 + 1$ and $v = 2t$ if $b > 0$. Applying Lemma 7 (ii), we get $2bt^2 + 1 = 2^\alpha$, which is impossible.

As in the proof above, by Lemmas 5 and 7, the key step is to find the minimal solutions of the related equations. Since the proofs are similar for other equations, we just list their minimal solutions in the following table. □

Equation	Condition	Minimal solution
$X^2 - b(bt^2+1)Y^2 = 1$	$b > 0$	$(2bt^2+1,\, 2t)$
	$b < 0$	$(-2bt^2-1,\, 2t)$
$X^2 - \frac{b}{2}(2bt^2+1)Y^2 = 1\,(2\|b)$	$b > 0$	$(4bt^2+1,\, 4t)$
	$b < 0$	$(-4bt^2-1,\, 4t)$
$X^2 - 2b(2bt^2+1)Y^2 = 1$	$b > 0$	$(4bt^2+1,\, 2t)$
	$b < 0$	$(-4bt^2-1,\, 2t)$
$X^2 - \frac{b}{2}(2bt^2-1)Y^2 = 1\,(2\|b)$	$b > 0$	$(4bt^2-1,\, 4t)$
	$b < 0$	$(-4bt^2+1,\, 4t)$
$X^2 - \frac{b}{4}(4bt^2+1)Y^2 = 1\,(4 \mid b)$	$b > 0$	$(8bt^2+1,\, 8t)$
	$b < 0$	$(-8bt^2-1,\, 8t)$
$X^2 - b(4bt^2+1)Y^2 = 1$	$b > 0$	$(8bt^2+1,\, 4t)$
	$b < 0$	$(-8bt^2-1,\, 4t)$
$X^2 - \frac{b}{4}(4bt^2-1)Y^2 = 1\,(4 \mid b)$	$b > 0$	$(8bt^2-1,\, 4t)$
	$b < 0$	$(-8bt^2+1,\, 4t)$
$2X^2 - Y^2 = 1$		$(1,\, 1)$
$X^2 - b(bt^2+2)Y^2 = 1$	$b > 0$	$(bt^2+1,\, t)$
	$b < 0$	$(-bt^2-1,\, t)$
$X^2 - b(bt^2-2)Y^2 = 1$	$b > 0$	$(bt^2-1,\, t)$
	$b < 0$	$(-bt^2+1,\, t)$

Table 2: Some equations and their minimal solutions

Theorem 5. *(1) The simultaneous Diophantine equations*

$$x^l = 3^s t \quad \textit{and} \quad 3^s - 3 = 4ax^n t^2, \quad a,\, n,\, t,\, l,\, x \in \mathbb{N},$$

have only the integer solutions $(a,\, n,\, t,\, l,\, x) = (\frac{3^{s-1}-1}{4},\, 1,\, 1,\, s,\, 3)$, $2 \nmid s$, *with* $3 \nmid t$.

(2) The simultaneous Diophantine equations

$$x^l = 3^s t \quad \textit{and} \quad 3^s + 3 = 4ax^n t^2, \quad a,\, n,\, t,\, l,\, x \in \mathbb{N},$$

have only the integer solutions $(a,\, n,\, t,\, l,\, x) = (\frac{3^{s-1}+1}{4},\, 1,\, 1,\, s,\, 3)$, $2|s$, *with* $3 \nmid t$.

(3) The simultaneous Diophantine equations

$$x^l = 3^s t \quad \textit{and} \quad 3^s + 3 = 2ax^n t^2, \quad a,\, n,\, t,\, l,\, x \in \mathbb{N},$$

have only the integer solutions $(a,\, n,\, t,\, l,\, x) = (\frac{3^{s-1}+1}{2},\, 1,\, 1,\, s,\, 3)$, $s \in \mathbb{N}$, *with* $3 \nmid t$.

(4) The simultaneous Diophantine equations

$$x^l = 3^s t \quad \textit{and} \quad 3^s - 3 = 2ax^n t^2, \quad a,\, n,\, t,\, l,\, x \in \mathbb{N},$$

have only the integer solutions $(a,\, n,\, t,\, l,\, x) = (\frac{3^{s-1}-1}{2},\, 1,\, 1,\, s,\, 3)$, $s > 1$, *with* $3 \nmid t$.

(5) The simultaneous Diophantine equations

$$x^l = 3^s t \quad \textit{and} \quad 3^s + 3 = ax^n t^2, \quad a,\, n,\, t,\, l,\, x \in \mathbb{N},$$

have only the integer solutions $(a, n, t, l, x) = (3^{s-1}+1, 1, 1, s, 3)$*,* $s \in \mathbb{N}$*, with* $3 \nmid t$*.*

(6) The simultaneous Diophantine equations

$$x^l = 3^s t \quad \text{and} \quad 3^s - 3 = ax^n t^2, \quad a, n, t, l, x \in \mathbb{N},$$

have only the integer solutions $(a, n, t, l, x) = (3^{s-1}-1, 1, 1, s, 3)$*,* $s > 1$*, with* $3 \nmid t$*.*

(7) The simultaneous Diophantine equations

$$x^l = 2 \cdot 3^s t \quad \text{and} \quad 3^s \pm 3 = 8ax^n t^2, \quad a, n, t, l, x \in \mathbb{N},$$

have no integer solutions (a, n, t, l, x)*, with* $3 \nmid t$ *and* $2|ax$*.*

(8) The simultaneous Diophantine equations

$$x^l = 2 \cdot 3^s t \quad \text{and} \quad 3^s \pm 3 = 4ax^n t^2, \quad a, n, t, l, x \in \mathbb{N},$$

have no integer solutions (a, n, t, l, x)*, with* $3 \nmid t$ *and* $2|x$*.*

(9) The simultaneous Diophantine equations

$$x^l = 4 \cdot 3^s t \quad \text{and} \quad 3^s \pm 3 = 16ax^n t^2, \quad a, n, t, l, x \in \mathbb{N},$$

have no integer solutions (a, n, t, l, x)*, with* $3 \nmid t$ *and* $4|ax^n$*.*

(10) The simultaneous Diophantine equations

$$x^l = 2 \cdot 3^s t \quad \text{and} \quad 3^s \pm 3 = 4ax^n t^2, \quad a, n, t, l, x \in \mathbb{N},$$

have no integer solutions (a, n, t, l, x)*, with* $3 \nmid t$ *and* $4|ax^n$*.*

(11) The simultaneous Diophantine equations

$$x^l = 2 \cdot 3^s t \quad \text{and} \quad 3^s \pm 3 = 16ax^n t^2, \quad a, n, t, l, x \in \mathbb{N},$$

have no integer solutions (a, n, t, l, x)*, with* $3 \nmid t$ *and* $4|ax^n$*.*

Proof. (1) Since $x^l = 3^s t$ and $3 \nmid t$, we obtain $3|x$. As $3^s - 3 = 4ax^n t^2$, we have $3||4ax^n t^2$ and hence $3||x$. Thus $l = s$ and $t = {t_1}^s$, where t_1 is a positive integer with $3\|t_1$. Therefore $x = 3t_1$, which implies $3^s - 3 = 4a \cdot 3^n t_1^{2s+1}$, and $n = 1$ as $3||3^s - 3$. If $t_1 > 1$, then $t_1^{2s+1} \geq 2^{2s+1} > 3^{s-1} - 1$, which is impossible. So we get that $t_1 = 1$, $x = 3$ and $a = \frac{3^{s-1}-1}{4}$, $2 \nmid s$.

Using arguments similar to those above, we get the results for the related simultaneous Diophantine equations in (2)-(6).

(7) Since $x^l = 2 \cdot 3^s t$, we obtain $3|x$. It follows from $3^s + 3 = 8ax^n t^2$ that $s > 1$ and $3||8ax^n t^2$. Hence $3||x$ and $n = 1$. Therefore $l = s$ and $t = 2^{s-1}{t_1}^s$, where t_1 is a positive integer with $3 \nmid t_1$. Hence $x = 6t_1$, so $3^s + 3 = 8a \cdot 6t_1 \cdot 2^{2s-2} t_1^{2s}$, which implies $t_1 = 1$ and $a = \frac{3^{s-1}+1}{2^{2s+2}}$, which is impossible as $2^{2s+2} > 3^{s-1} + 1$.

Using arguments similar to those in (7) show that the related simultaneous Diophantine equations in (8)-(11) have no integer solutions. This completes the proof. $\square$

4. Proof of Theorem 1

We will divide the proof into 30 cases according to the different values of c and d. For simplicity, we only give a complete proof of the case $c = -4$ and $d = -1$.
`The case` $c = -4$, $d = -1$. From (1) we have

$$ax^{n+2l} - b(abt^2x^n - 1)y^2 = 4. \tag{12}$$

We divide the proof into four subcases according to $ax^n = 4$, $4 \mid ax^n$ and $ax^n > 4$, $2 || ax^n$, and $2 \nmid ax^n$.

`Subcase 1.` $ax^n = 4$. Then $2 \nmid abt^2x^n - 1$, so $4 \mid by^2$. As $x > 1$, we get $(a, x, n) = (1, 2, 2)$ or $(1, 4, 1)$ or $(2, 2, 1)$.

If $4 \mid b$, by (12) we get $2^{2l} - \frac{b}{4}(4bt^2 - 1)y^2 = 4$ or $4^{2l} - \frac{b}{4}(4bt^2 - 1)y^2 = 4$, a contradiction to Theorem 4 (i). Similarly, if $4 \nmid b$, then $2 \mid y$, from (12) we have $2^{2l} - b(4bt^2 - 1)(\frac{y}{2})^2 = 1$ or $4^{2l} - b(4bt^2 - 1)(\frac{y}{2})^2 = 1$, a contradiction to Theorem 4 (i).

`Subcase 2.` If $4 \mid ax^n$ and $ax^n > 4$. Then $2 \nmid abt^2x^n + 1$, so $4 \mid by^2$.

(i) If $4 \mid b$, from (12) we have $\frac{ax^n}{4}(x^l)^2 - \frac{b(abt^2x^n-1)}{4}y^2 = 1$, which implies $(X, Y) = (x^l, y)$ is a solution of

$$\frac{ax^n}{4}X^2 - \frac{b(abt^2x^n - 1)}{4}Y^2 = 1.$$

If $b > 0$, then $(X, Y) = (4t, 1)$ is the minimal positive solution of

$$\frac{abx^n}{16}X^2 - (abt^2x^n - 1)Y^2 = 1.$$

Since $\frac{ax^n}{4} = 1 (b > 4)$, by Lemma 6 we must get $\frac{abx^n}{16} = 1$ or $\frac{ax^n}{4} = \frac{abx^n}{16}$. The former gives $(a, b, x, n) = (1, 4, 2, 2)$ or $(1, 4, 4, 1)$ or $(2, 4, 2, 1)$. Thus $X^2 - (4t^2 - 1)Y^2 = 1$ has a solution $(X, Y) = (2^l, y)$ or $(4^l, y)$. Moreover $(X, Y) = (2t, 1)$ is the fundamental solution of $X^2 - (4t^2 - 1)Y^2 = 1$. Thus we have $2^l = 2t$, $y = 1$ or $4^l = 2t$, $y = 1$ by Lemma 7 (ii). The latter yields $b = 4$. Again by Lemma 2 (i) we obtain $x^l = 4t$, $y = 1$ or $x^l = 3^s 4t$, $3 \nmid 4t$, $3^s + 3 = 4\frac{ax^n}{4}(4t)^2$. The latter is impossible by Theorem 5 (9).

If $b < 0$, then $(X, Y) = (1, 4t)$ is the minimal positive solution of

$$(-abt^2x^n + 1)X^2 - \frac{-abx^n}{16}Y^2 = 1.$$

Since $\frac{ax^n}{4} > 1$, $-abt^2x^n + 1 > 1$ and $\frac{ax^n}{4} \neq -abt^2x^n + 1$, by Lemma 6 (12) has no solutions.

(ii) If $4 \nmid b$, then $2 \mid y$, from (12) we have $\frac{ax^n}{4}(x^l)^2 - b(abt^2x^n - 1)(\frac{y}{2})^2 = 1$, which implies $(X, Y) = (x^l, \frac{y}{2})$ is a solution of

$$\frac{ax^n}{4}X^2 - b(abt^2x^n - 1)Y^2 = 1.$$

If $b > 0$, then $(X, Y) = (2t, 1)$ is the minimal positive solution of

$$\frac{abx^n}{4}X^2 - (abt^2x^n - 1)Y^2 = 1.$$

Since $\frac{ax^n}{4} > 1$, by Lemma 6 we must get $\frac{abx^n}{4} = 1$ or $\frac{ax^n}{4} = \frac{abx^n}{4}$. The equation $\frac{abx^n}{4} = 1$ yields $(a, b, x, n) = (1, 1, 2, 2)$ or $(1, 1, 4, 1)$ or $(2, 1, 2, 1)$. Thus the equation $X^2 - (4t^2 - 1)Y^2 = 1$ has a solution $(X, Y) = (2^l, \frac{y}{2})$ or $(4^l, \frac{y}{2})$. Note that $(X, Y) = (2t, 1)$ is the fundamental solution of $X^2 - (4t^2 - 1)Y^2 = 1$. Thus by Lemma 7 (ii) we have $2^l = 2t$, $\frac{y}{2} = 1$ and $4^l = 2t$, $\frac{y}{2} = 1$.

The equation $\frac{ax^n}{4} = \frac{abx^n}{4}$ implies $b = 1$. Again by Lemma 2 we obtain that $x^l = 2t$, $\frac{y}{2} = 1$ or $x^l = 3^s 2t$, $3 \nmid 2t$, $3^s + 3 = 4\frac{ax^n}{4}(2t)^2$. The latter is impossible by Theorem 5 (10).

If $b < 0$, then $(X, Y) = (1, 2t)$ is the minimal positive solution of

$$(-abt^2x^n + 1)X^2 - \frac{-abx^n}{4}Y^2 = 1.$$

Since $\frac{ax^n}{4} > 1$, $-abt^2x^n + 1 > 1$ and $\frac{ax^n}{4} \neq -abt^2x^n + 1$, by Lemma 6 (12) has no solutions.

`Subcase 3.` $2 \parallel ax^n$. Then $2||b(abt^2x^n - 1)y^2$, so $2||b$ and $\frac{abx^n}{4}(abt^2x^n - 1)$ is odd. From (12) we have $\frac{ax^n}{2}(x^l)^2 - \frac{b}{2}(abt^2x^n - 1)y^2 = 2$, which implies $(X, Y) = (x^l, y)$ is a solution of

$$\frac{ax^n}{2}X^2 - \frac{b}{2}(abt^2x^n - 1)Y^2 = 2.$$

If $b > 0$, then $(X, Y) = (2t, 1)$ is a solution of

$$\frac{abx^n}{4}X^2 - (abt^2x^n - 1)Y^2 = 1.$$

Since $\frac{ax^n}{2} > 0$ and $\frac{abx^n}{4} > 1$, by Lemma 6 (iii) the equation (12) has no solutions.

If $b < 0$, then $(1,\ 2t)$ is a solution of

$$(-abt^2x^n + 1)X^2 - \frac{-abx^n}{4}Y^2 = 1.$$

Since $\frac{ax^n}{2} > 0$ and $-abt^2x^n + 1 > 1$, by Lemma 6 (iii) the equation (12) has no solutions.

`Subcase 4.` If ax^n is odd, then $abx^n(abt^2x^n - 1)$ is odd. From (12) $(X, Y) = (x^l, y)$ is a solution of

$$ax^nX^2 - b(abt^2x^n - 1)Y^2 = 4.$$

If $b > 0$, then $(X, Y) = (t, 1)$ is the minimal positive solution of

$$abx^nX^2 - (abt^2x^n - 1)Y^2 = 1.$$

Since $ax^n > 1$, by Lemma 5 the equation $ax^nX^2 - b(abt^2x^n - 1)Y^2 = 1$ has an integer solution (X, Y). Because $ax^n > 1$ and $abx^n > 1$, by Lemma 6 we have $ax^n = abx^n$, which implies $b = 1$. Let $(X, Y) = (u, v)$ be the minimal solution of $ax^nX^2 - b(abt^2x^n - 1)Y^2 = 4$. Thus by Lemma 5 we obtain $t\sqrt{ax^n} + \sqrt{at^2x^n - 1)} = \frac{1}{8}(u\sqrt{ax^n} + v\sqrt{at^2x^n - 1})^3$, from which $\frac{1}{8}(v^3l + 3u^2vk) = 1$, which is impossible as $k = ax^n \geq 3$ and $l = at^2x^n - 1 \geq 2$.

If $b < 0$, then $(X, Y) = (1, t)$ is the minimal positive solution of

$$(-abt^2x^n + 1)X^2 - (-abx^n)Y^2 = 1.$$

Since $ax^n > 1$, by Lemma 5 the equation $ax^nX^2 - b(abt^2x^n - 1)Y^2 = 1$ has solution. Because $ax^n > 1$, $-abt^2x^n + 1 > 1$ and $ax^n \neq -abt^2x^n + 1$, by Lemma 6 (12) has no solutions. This proves the case of $c = -4$ and $d = -1$.

The proofs of the other 29 cases are similar and perhaps we need other lemmas and theorems in Sections 2 and 3 to complete the proof of the main theorem, and we omit the details. This completes the proof of Theorem 1. □

Acknowledgement. The authors express their gratitude to the anonymous referee for carefully examining this paper and providing a number of important comments and suggestions. The second author is also supported by the National Natural Science Foundation of China (grant no. 11671153).

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