PARTITION INEQUALITIES AND APPLICATIONS TO SUM-PRODUCT CONJECTURES OF KANADE-RUSSELL

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Abstract

We consider differences of one- and two-variable finite products and provide combinatorial proofs of the nonnegativity of certain coefficients. Since the products may be interpreted as generating functions for certain integer partitions, this amounts to showing a partition inequality. This extends results due to Berkovich-Garvan and McLaughlin. We then apply the first inequality and Andrews’ Anti-telescoping Method to give a solution to an “Ehrenpreis Problem” for recently conjectured sum-product identities of Kanade-Russell. That is, we provide significant further evidence for Kanade-Russell’s conjectures by showing nonnegativity of coefficients in differences of product-sides as Andrews-Baxter and Kadell did for the product sides of the Rogers-Ramanujan identities.

1. Introduction

A partition λ of an integer n is a multi-set of positive integers \{λ₁, \ldots, λ_ℓ\}, whose parts satisfy

\[ λ₁ \geq λ₂ \geq \cdots \geq λ_ℓ \geq 1 \quad \text{and} \quad \sum_{j=1}^{ℓ} λ_j = n. \]

We will often use frequency notation to refer to a partition, where \((r^{m_r}, \ldots, 2^{m_2}, 1^{m_1})\) represents the partition in which the part \(i\) occurs \(m_i\) times for \(1 \leq i \leq r\).

Visually, a partition \(λ\) may be represented by its Ferrer’s diagram, in which parts are displayed as rows of dots. For example, the Ferrer’s diagram of the partition \((5, 3, 2^2)\) is the array below:

\[
\begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array}
\]

For two \(q\)-series \(f(q) = \sum_{n \geq 0} a_n q^n\) and \(g(q) = \sum_{n \geq 0} b_n q^n\), we write \(f(q) \geq g(q)\) if \(a_n \geq b_n\) for all \(n\). If \(f(q) \geq 0\), we will say that \(f\) is a nonnegative series.
We will use the standard $q$-Pochhammer symbol,

\[(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad (a; q)_\infty := \lim_{n \to \infty} (a; q)_n, \quad \text{and} \quad (a_1, \ldots, a_r; q)_n := (a_1; q)_n \cdots (a_r; q)_n.\]

By convention, an empty product equals 1.

The study of the type of partition inequality we consider began at the 1987 A.M.S. Institute on Theta Functions with a question of Leon Ehrenpreis about the Rogers-Ramanujan Identities ([3], Cor. 7.6 and Cor. 7.7):

\[\mathcal{R}_1 : \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}},\]

\[\mathcal{R}_2 : \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2, q^3; q^4)_{\infty}}.\]

The identity $\mathcal{R}_1$ may be interpreted as an equality of certain partition generating functions, giving that the number of partitions of $n$ such that the gap between successive parts is at least 2 equals the number of partitions of $n$ into parts congruent to ±1 (mod 5). Similarly, $\mathcal{R}_2$ gives that the number of partitions of $n$ such that the gap between successive parts is at least 2 and 1 does not occur as a part equals the number of partitions of $n$ into parts congruent to ±2 (mod 5). Thus, both combinatorially and algebraically, it is easy to see that

\[\sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} - \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n} \geq 0.\]

Therefore, it also holds that

\[\frac{1}{(q, q^4; q^5)_{\infty}} - \frac{1}{(q^2, q^3; q^4)_{\infty}} \geq 0. \tag{1}\]

Ehrenpreis’ Problem was to provide a proof of (1) that did not reference the (heavily-handed and quite nontrivial) Rogers-Ramanujan identities.

Solutions to Ehrenpreis’ Problem have been given in various ways. In the course of proving (1), Andrews-Baxter [2] were led to a new “motivated” proof of the Rogers-Ramanujan Identities themselves. A direct combinatorial proof of (1) was provided by Kadell [8], who constructed an injection from the set of partitions of $n$ with parts congruent to ±2 (mod 5) to those with parts congruent to ±1 (mod 5). Later, Andrews developed the Anti-telescoping Method for showing positivity in differences of products like (1) [1]. This method was used by Berkovich-Grizzell in [5] to prove infinite classes of partition inequalities, such as the following.
Theorem (Theorem 1.2 of [5]). For any octuple of positive integers $(L, m, x, y, z, r, s, u)$,

$$\frac{1}{(q^x, q^y, q^z, q^{x+y+z}; q^m)_L} - \frac{1}{(q^x, q^y, q^z, q^{x+y+z}; q^m)_L} \geq 0.$$ 

In [4], Berkovich-Garvan generalized (1) to an arbitrary modulus as follows.

Theorem (Theorem 5.3 of [4]). Suppose $L \geq 1$ and $1 \leq r < \frac{M}{2}$. Then

$$\frac{1}{(q, q^{M-1}; q^M)_L} - \frac{1}{(q^r, q^{M-r}; q^M)_L} \geq 0$$

if and only if $r \not\mid (M - r)$.

One can apply Berkovich-Garvan’s result to solve similar “ Ehrenpreis Problems” for Kanade-Russell’s “mod 9 identities” in [10].

The first result in this paper extends the above in a way that is independent of the modulus. We will use this in Section 3 to solve an “Ehrenpreis Problem” for conjectural product-sum identities of Kanade-Russell in [11].

Theorem 1. Let $a, b, c$ and $M$ be integers satisfying $1 < a < b < c$ and $1 + c = a + b$. Then if $a \not\mid b$,

$$\frac{1}{(q, q^c; q^M)_L} - \frac{1}{(q^a, q^b; q^M)_L} \geq 0 \quad \text{for any } L \geq 0.$$ 

Note that we do not necessarily assume $a, b, c \leq M$. Translated into a partition inequality, Theorem 1.1 says that there are more partitions of $n$ into parts of the forms $Mj + 1$ and $Mj + c$ than there are partitions of $n$ into parts of the forms $Mj + a$ and $Mj + b$, where throughout $1 \leq j \leq L$.

Partition inequalities with a fixed number of parts were considered by McLaughlin in [13]. Answering two of McLaughlin’s questions, we give combinatorial proofs of finite analogues of Theorems 7 and 8 from [13].

Theorem 2. Let $a, b$ and $M$ be integers satisfying $1 \leq a < b < \frac{M}{2}$ and gcd$(b, M) = 1$. Define $c(m, n)$ by

$$\frac{1}{(zq^a, zq^{M-a}; q^M)_L(1 - q^{LM+a})} - \frac{1}{(zq^b, zq^{M-b}; q^M)_L} =: \sum_{m,n \geq 0} c(m, n)z^mq^n.$$ 

Then for any $L, n \geq 0$, we have $c(m, nM) \geq 0$. If in addition $M$ is even and $a$ is odd, then we also have $c(m, nM + \frac{M}{2}) \geq 0$ for every $n \geq 0$.

Note that we do not necessarily make the assumption gcd$(a, M) = 1$ that is in [13]. While these partition inequalities hold only for $n$ in certain residue classes (mod $M$), Theorem 2 is a strengthening of Theorem 1 for these $n$. The following is a distinct parts analogue.
Theorem 3. Let $a, b$ and $M$ be integers satisfying $1 \leq a < b < \frac{M}{2}$ and $\gcd(b, M) = 1$. Define $d(m, n)$ by

$$\left(-zq^a, -zq^{M-a}; q^M\right)_L (1 + zq^{LM+a})\left(-zq^b, -zq^{M-b}; q^M\right)_L =: \sum_{m,n \geq 0} d(m,n)z^m q^n.$$ 

Then for any $L,n \geq 0$, we have $d(m,nM) \geq 0$. If in addition $M$ is even and $a$ is odd, then we also have $d\left(m,nM + \frac{M}{2}\right) \geq 0$ for every $n \geq 0$.

Remark. Taking the limit as $L \to \infty$ in Theorems 2 and 3 recovers McLaughlin’s original partition inequalities.

In Section 2, we begin by reviewing the $M$-modular diagram of a partition. Then we provide combinatorial proofs of Theorems 1-3. In Section 3, we apply Theorem 1 and Andrews’ Method of Anti-Telescoping (see [1]) to give a solution to an “Ehrenpreis Problem” for recently conjectured sum-product identities of Kanade-Russell [11]. Our concluding remarks in Section 4 ask for Andrews-Baxter style “motivated proofs” of these conjectured identities.

2. Combinatorial Proofs of Theorems 1-3

2.1. Notation

The $M$-modular diagram of a partition $\lambda = \{\lambda_1, \ldots, \lambda_\ell\}$ is a modification of the Ferrer’s diagram, wherein each $\lambda_j$ is first written as $Mq + r$ for $0 \leq r < M$, and then is represented as a row of $q^M$’s and a single $r$ at the end of the row. These $r$’s we will refer to as ends or $r$-ends. For example, the 10-modular diagram of $\lambda = (53^2, 46, 36, 16, 11, 1)$ has three 6-ends, two 3-ends and two 1-ends:

$$\begin{array}{ccccccc}
10 & 10 & 10 & 10 & 10 & 3 \\
10 & 10 & 10 & 10 & 10 & 3 \\
10 & 10 & 10 & 10 & 6 \\
10 & 10 & 10 & 6 \\
10 & 6 \\
10 & 1 \\
1 &
\end{array}$$

We will also speak of attaching and removing a column from an $M$-modular diagram. These operations are best defined with an example:
We shall only attach or remove columns consisting entirely of $M$’s, and it is easy to see that these operations preserve $M$-modular diagrams.

2.2. Proof of Theorem 1

We provide a combinatorial proof via injection that is nearly identical to that of Theorem 5.1 in [4], but we highlight a technical difference that arises in the general version. In keeping with [4], we let $\nu_j = \nu_j(\lambda)$ denote the number of parts of $\lambda$ congruent to $j \pmod{M}$. (The modulus never varies and will be clear from context.)

Proof. First let $L = 1$. We will prove the general case as a consequence of this one. For each $n$, we seek an injection

$$\varphi_1 : \{(a^k, b^\ell) \vdash n : k, \ell \geq 0\} \hookrightarrow \{(1^k, c^\ell) \vdash n : k, \ell \geq 0\}.$$ 

Let $d := \gcd(a, b)$. Explicitly, $\varphi_1$ is as follows:

$$\varphi_1 (a^k, b^\ell) = \begin{cases} 
(1^{\ell+a(k-\ell)}, c^\ell) & \text{if } k \geq \ell, \\
(1^{k+b(\ell-k)}, c^k) & \text{if } \ell > k \text{ and } \frac{a}{d} \nmid (\ell - k), \\
(1^{k+1+b(\ell-k-1)-a}, c^{k+1}) & \text{if } \ell > k \text{ and } \frac{a}{d} \mid (\ell - k). 
\end{cases}$$

This definition can be motivated by noting that each pre-image consists either of $k$ pairs $(a, b)$ and $k - \ell$ excess $a$’s, or of $\ell$ pairs $(a, b)$ and $\ell - k$ excess $b$’s. (There can also be no excess.) The pairs are mapped as $(a, b) \mapsto (1, c)$. The excess $a$’s or $b$’s are treated by the following cases.
Case 1. For the $k - \ell$ excess $a$’s, $(a) \mapsto (1^a)$.

Case 2. For the $\ell - k$ excess $b$’s, $(b) \mapsto (1^b)$.

Case 3. For all but the last two excess $b$’s, $(b) \mapsto (1^b)$. For the last two $b$’s, $(b^2) \mapsto (1^{b-a+1}, c)$.

Note that in Case 3 there are at least two excess $b$’s, for if not, $\frac{c}{a} = 1$ and then $a \mid b$, a contradiction. Also, by hypothesis, $b > \frac{c}{a}$, so that $2b > c$.

Let $(1^{\nu_1}, c^{\nu_c})$ be a partition in the image of $\varphi_1$. The cases are separated as follows:

Case 1. $a \mid (\nu_1 - \nu_c)$,

Case 2. $a \nmid (\nu_1 - \nu_c)$ and $b \mid (\nu_1 - \nu_c)$,

Case 3. $\nu_1 - \nu_c \equiv -b \pmod{a}$ and $\nu_1 - \nu_c \equiv -a \pmod{b}$.

This concludes the proof for $L = 1$.

Now let $L \geq 2$. Again we define an injection

$$\varphi_L : \{\lambda \vdash n : \lambda_j \in \{a, b, \ldots, LM + a, LM + b\}\}$$

$$\mapsto \{\lambda \vdash n : \lambda_j \in \{1, c, \ldots, LM + 1, LM + c\}\}.$$

Let $\lambda$ be a partition in the left set. Then $\lambda$ consists of the triple

$$\left(\lambda_{(a)}, \lambda_{(b)}, (a^k, b^\ell)\right),$$

where $\lambda_{(a)}$ is the $M$-modular diagram obtained by subtracting $a$ from every part of the form $Mj + a$; $\lambda_{(b)}$ is defined similarly. We apply $\varphi_1$ to $(a^k, b^\ell)$ and reattach the 1-ends and $c$-ends based on the case into which $(a^k, b^\ell)$ falls.

Case 1: $k \geq \ell$. Attach the 1-ends to $\lambda_{(a)}$ and the $c$-ends to $\lambda_{(b)}$. The map $\varphi_1$ guarantees exactly $\#\lambda_{(b)}$ $c$-ends. Likewise, there are at least as many 1-ends as there are parts of $\lambda_{(a)}$; any excess 1’s are attached as parts to $\lambda_{(a)}$. The required image of $\lambda$ is then the union of these two partitions.

Cases 2 and 3: $\ell > k$. Attach the 1-ends to $\lambda_{(b)}$ and the $c$-ends to $\lambda_{(a)}$ as before. $\varphi_1$ guarantees at least $\#\lambda_{(a)}$ $c$-ends. In Case 2 we are guaranteed at least $\#\lambda_{(a)}$ 1-ends because $b > 1$ implies

$$k + b(\ell - k) > \ell.$$ 

In Case 3, $\frac{a}{b} > 1$ implies $\ell - k > 1$, so

$$k + 1 + b(\ell - k - 1) - a = \ell + (b - 1)(\ell - k - 1) - a \geq \ell,$$
and we are guaranteed at least \( \#\lambda(a) \) 1-ends.

Given the image of \( \lambda \), we may clearly recover \( \lambda(a) \) and \( \lambda(b) \) based on its 1-ends and c-ends and the fact that \( \varphi_1 \) is an injection. Thus, \( \varphi_L \) is an injection. \( \blacksquare \)

**Remark.** The condition \( a \nmid b \) in Theorem 1 is necessary to avoid cases like

\[
\frac{1}{(q, q^2; q^6)_L} - \frac{1}{(q^2, q^4; q^6)_L},
\]

in which the coefficient of \( q^4 \) is \(-1\).

**Remark.** If we had copied the proof of Theorem 5.1 in [4] exactly, then the conditions \( \frac{a}{d} \parallel \) and \( \frac{b}{d} \parallel \) would be replaced by \( \frac{a}{d} \parallel \) and \( \frac{b}{d} \parallel \). But this is not an injection because Case 2 is only correctly separated from the other two when \( \gcd(a, b) = 1 \). For example, this direct version of Berkovich-Garvan’s map gives:

\[
\begin{align*}
4^7, 6^4 & \rightarrow (1^6, 9^4), \quad \text{instead of our} \quad 4^7, 6^4 & \rightarrow \begin{cases} 1^16, 9^4 \\ 4^4, 6^6 & \rightarrow \begin{cases} 1^17, 9^5 \\
\end{cases}
\end{align*}
\]

In the first example, the partitions fall into cases 1 and 2. The second example corrects the overlap and places the partitions into cases 1 and 3.

We demonstrate the injection of Theorem 1 with an example.

**Example 1.** Here, \( (n, M, L, a, b, c) = (52, 10, 2, 4, 6, 9) \). Numbers above arrows indicate the case into which a pre-image falls.

\[
\begin{array}{ccc|ccc}
16^3, 4 & \rightarrow & 113, 92, 1 & 142, 6^4 & \rightarrow & 192, 9, 1^{15} \\
16^2, 14, 6 & \rightarrow & 19, 112, 9, 12 & 14^2, 6^2, 4^3 & \rightarrow & 112, 9^2, 1^{12} \\
16^2, 6^2, 4^2 & \rightarrow & 112, 9^3, 1^3 & 14^2, 4^6 & \rightarrow & 11^2, 1^{30} \\
16^2, 4^5 & \rightarrow & 19^2, 1^{14} & 16, 6^4, 4^2 & \rightarrow & 19, 11^2, 1^{13} \\
16, 14^2, 4^2 & \rightarrow & 19, 11^2, 1^{11} & 14, 6^3, 4^5 & \rightarrow & 11, 9^3, 1^{14} \\
16, 14, 6^3, 4 & \rightarrow & 19, 11, 9^2, 1^4 & 14, 6, 4^7 & \rightarrow & 11, 9, 1^{32} \\
16, 14, 6^4, 4^4 & \rightarrow & 19, 11, 9, 1^{13} & 6^8 & \rightarrow & 9, 1^{43} \\
16, 6^6 & \rightarrow & 11, 1^{41} & 6^6, 4^4 & \rightarrow & 9^5, 1^{17} \\
16, 6^4, 4^3 & \rightarrow & 11, 9^4, 1^5 & 6^4, 4^7 & \rightarrow & 9^4, 1^{16} \\
16, 6^2, 4^6 & \rightarrow & 19, 9^2, 1^{15} & 6^2, 4^{10} & \rightarrow & 9^2, 1^{34} \\
16, 4^9 & \rightarrow & 19, 1^{53} & 4^{13} & \rightarrow & 1^{52} \\
14^3, 6, 4 & \rightarrow & 11^3, 9, 1^{10} & & & \\
\end{array}
\]
2.3. Proofs of Theorems 2 and 3

We begin by recalling the main steps in McLaughlin’s proof of Theorem 7 from [13]; our proof is based on a combinatorial reading. First, Cauchy’s Theorem ([3], Th. 2.1) is used with some algebraic manipulation to write, for fixed $m,$

$$
\sum_{n \geq 0} c(m, n)q^n = \sum_{0 \leq k < \frac{m}{M}} \frac{q^{kM}}{(q^{M}; q^{M})_{m-k}(q^{M}; q^{M})_k}
\times \left( q^{a(m-2k)} + q^{(M-a)(m-2k)} - q^{b(m-2k)} - q^{(M-b)(m-2k)} \right).
$$

It then happens that the factor in parentheses is equal to

$$
q^{a(m-2k)} \left( 1 - q^{(b-a)(m-2k)} \right) \left( 1 - q^{(M-b-a)(m-2k)} \right).
$$

But the conditions on $a$, $b$ and $M$ that lead to the condition $M \mid (m-2k)$ in the sum imply that both factors above are canceled in $q^{a(m-2k)}$. This gives nonnegativity.

The key steps in the proof are the decomposition of the sum over $k$ and the nonnegativity of

$$
\frac{(1 - q^r)(1 - q^s)}{(q; q)_n} \quad \text{for } 1 \leq r < s \leq n.
$$

Both of these have simple combinatorial explanations, which we employ with $M$-modular diagrams to piece together a proof of Theorem 2. Our proof naturally leads to the finite versions with any $L \geq 1$ instead of $\infty$. The proof of Theorem 3 is then a slight modification.

**Proof of Theorem 2.** Let $\mathcal{P}(n, m, j, A)$ denote the set of partitions of $n$ into $m$ parts congruent to $\pm j$ modulo $M$ such that the largest part is at most $A$. (We have suppressed the modulus $M$ from the notation.) Let $\mathcal{P}_k(n, m, j, A)$ be the subset of partitions $\lambda \in \mathcal{P}(n, m, j, A)$ with either $\nu_j(\lambda) = k$ or $\nu_{M-j}(\lambda) = k$.

Clearly, we have the disjoint union $\mathcal{P}(n, m, j, A) = \bigsqcup_{0 \leq k \leq \frac{m}{M}} \mathcal{P}_k(n, m, j, A).$ Thus, to show

$$
\mathcal{P}(nM, m, b, LM - b) \hookrightarrow \mathcal{P}(nM, m, a, LM + a),
$$

we may provide injections

$$
\varphi_k : \mathcal{P}_k(nM, m, b, LM - b) \hookrightarrow \mathcal{P}_k(nM, m, a, LM + a)
$$

for each $k \in [0, \frac{m}{M}]$.

Each $\lambda \in \mathcal{P}(nM, m, b, LM - b)$ consists of a triple

$$
(\lambda(\omega), \lambda(M - b), (b^n, (M - b)^{M - b})),
$$
where $\lambda_{(b)}$ is the $M$-modular diagram with $\nu_b$ nonnegative parts created by removing the $b$-ends. The $M$-modular diagram $\lambda_{(M-b)}$ is defined analogously by removing the $(M-b)$-ends.

When $k = \frac{m}{2}$, we simply map

$$\varphi_k \left( \lambda_{(b)}, \lambda_{(M-b)}, (b \frac{m}{2}, (M-b) \frac{m}{2}) \right) := \left( \lambda_{(b)}, \lambda_{(M-b)}, (a \frac{m}{2}, (M-a) \frac{m}{2}) \right).$$

The required partition is then obtained by reattaching the $a$-ends to $\lambda_{(b)}$ and reattaching the $(M-a)$-ends to $\lambda_{(M-b)}$.

Now assume $k < \frac{m}{2}$. Note that

$$0 \equiv nM \equiv b\nu_b(\lambda) - b\nu_{M-b}(\lambda) \pmod{M},$$

which implies $\nu_b(\lambda) - \nu_{M-b}(\lambda) \equiv 0 \pmod{M}$ because $\gcd(b, M) = 1$. Thus, we assume without loss of generality that $M \mid (m - 2k)$.

Let $y := \frac{(b-a)(m-2k)}{M}$ and $z := \frac{(M-b-a)(m-2k)}{M}$. These are positive integers.

**Case 1**: $\nu_{M-b}(\lambda) = k$. There are $k$ pairs of $(b, M-b)$ and $m-2k$ excess $b$'s. We map

$$\varphi_k \left( \lambda_{(b)}, \lambda_{(M-b)}, (b^{m-k}, (M-b)^k) \right) := \left( \lambda_{(b)} \cup \left[ \begin{array}{c} M \\ \vdots \\ y \text{ rows} \\ M \end{array} \right], \lambda_{(M-b)}, (a^{m-k}, (M-a)^k) \right)$$

$$= : \left( \lambda'_{(b)}, \lambda_{(M-b)}, (a^{m-k}, (M-a)^k) \right).$$

Here $\lambda'_{(b)}$ is the $M$-modular diagram formed by attaching the above column to $\lambda_{(b)}$.

Note that $a < b < M$ implies $0 < y < m-k$, so that $\lambda'_{(b)}$ is still an $M$-modular diagram with $m-k$ nonnegative parts.

To obtain the required partition, attach the $a$-ends to $\lambda'_{(b)}$ and the $(M-a)$-ends to $\lambda_{(M-b)}$. It is evident that there are $m$ parts. Size is preserved, as

$$|\lambda'_{(b)}| + |\lambda_{(M-b)}| + (m-k)a + k(M-a)$$

$$= |\lambda_{(b)}| + My + |\lambda_{(M-b)}| + a(m-2k) + kM$$

$$= |\lambda_{(b)}| + (b-a)(m-2k) + |\lambda_{(M-b)}| + a(m-2k) + kM$$

$$= |\lambda_{(b)}| + |\lambda_{(M-b)}| + b(m-2k) + kM$$

$$= |\lambda|.$$
Case 2a: \( \nu_b(\lambda) = k \) and \( \lambda_{(M-b)} \) does not contain a column of height \( y \).\(^1\) There are \( k \) pairs of \( (b, M - b) \) and \( m - 2k \) excess \( (M - b) \)'s. We map

\[
\varphi_k \left( \lambda(b), \lambda_{(M-b)}, (b^k, (M-b)^{m-k}) \right) := \left( \lambda(b), \lambda_{(M-b)} \cup \begin{bmatrix} M \\ \vdots \\ M \end{bmatrix} \right)_{x \text{ rows}},
\]

\[
=: \left( \lambda(b), \lambda'_{(M-b)}, (a^{m-k}, (M-a)^k) \right),
\]

where \( \lambda'_{(M-b)} \) is defined by attaching the above column. Note again that \( b, a < \frac{M}{2} \) implies \( 0 < z < m - k \), so that \( \lambda'_{(M-b)} \) is still an \( M \)-modular diagram with \( m - k \) nonnegative parts. Furthermore, \( b - a \neq M - b - a \), so \( \lambda'_{(M-b)} \) still does not contain a column of height \( y \).

To obtain the required partition, attach the \( a \)-ends to \( \lambda'_{(M-b)} \) and the \( (M-a) \)-ends to \( \lambda(b) \). It is evident that there are \( m \) parts. Size is preserved, as

\[
|\lambda(b)| + |\lambda'_{(M-b)}| + (m-k) a + k(M-a) \\
= |\lambda(b)| + |\lambda_{(M-b)}| + a(m-2k) + kM \\
= |\lambda(b)| + |\lambda_{(M-b)}| + (M-b-a)(m-2k) + a(m-2k) + kM \\
= |\lambda(b)| + |\lambda_{(M-b)}| + (M-b)(m-2k) + kM \\
= |\lambda|.
\]

Moreover, it is clear that the operations are reversible, so that, within Case 2a, \( \varphi_k \) is an injection.

Case 2b: \( \nu_b(\lambda) = k \) and \( \lambda_{(M-b)} \) contains a column of height \( y \).\(^2\) In this case we send

\[
(\lambda(b), \lambda_{(M-b)}, (b^k, (M-b)^{m-k})) \mapsto \left( \lambda(b), \lambda_{(M-b)} \setminus \begin{bmatrix} M \\ \vdots \\ M \end{bmatrix} \right)_{y \text{ rows}},
\]

\[
=: \left( \lambda(b), \lambda'_{(M-b)}, (a^k, (M-a)^{m-k}) \right),
\]

where \( \lambda'_{(M-b)} \) is defined by removing the above column. As before, we still may consider \( \lambda'_{(M-b)} \) an \( M \)-modular diagram with \( m - k \) nonnegative parts.

\(^1\) Or equivalently, the \( y \)-th part of \( \lambda_{(M-b)} \) equals the \( (y + 1) \)-st part.

\(^2\) Or equivalently, the \( y \)-th part of \( \lambda_{(M-b)} \) is strictly greater than the \( (y + 1) \)-st part.
To obtain the required partition, attach the $a$-ends to $\lambda_{(b)}$ and the $(M-a)$-ends to $\lambda'_{(M-b)}$. It is evident that there are $m$ parts. Size is preserved, as

$$|\lambda_{(b)}| + |\lambda'_{(M-b)}| + ka + (m - k)(M - a)$$

$$= |\lambda_{(b)}| + |\lambda'_{(M-b)}| - My + kM + (M - a)(m - 2k)$$

$$= |\lambda_{(b)}| + |\lambda'_{(M-b)}| - (b - a)(m - 2k) + kM + (M - a)(m - 2k)$$

$$= |\lambda_{(b)}| + |\lambda'_{(M-b)}| + (M - b)(m - 2k) + kM$$

$$= |\lambda|.$$  

Moreover, it is clear that the operations are reversible, so that, within Case 2b, $\varphi_k$ is an injection.

Let $(\lambda_{(a)}, \lambda_{(M-a)}, a^\nu, (M - a)^{\nu_{M-a}})$ lie in the image of $\varphi_k$. Then cases are separated as follows.

**Case 1:** $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ contains a column of height $y$.

**Case 2a:** $\nu_a > \nu_{M-a}$ and $\lambda_{(a)}$ does not contain a column of height $y$.

**Case 2b:** $\nu_a < \nu_{M-a}$.

Finally, note that in each case $\varphi_k$ adds at most $M$ to the largest part of what becomes $\lambda_{(a)}$, so indeed $\varphi_k$ maps $P_k(nM, m, b, LM - b)$ into $P_k(nM, m, a, LM + a)$ as required. This completes the proof of the first statement.

When $M$ is even and $a$ is odd, we can use exactly the same injections, assuming because of (2) that $m - 2k \equiv \frac{M}{2} \pmod{M}$. We note that $\text{gcd}(b, M) = 1$ implies that $b$ is also odd, so $y$ and $z$ are still integers.

**Remark.** We note that the extra factor $\frac{1}{(1 - q^{m+1})}$ in the left term of Theorem 2 is necessary. For example, in

$$\frac{1}{(zq^2, zq^3; q^7)_2} - \frac{1}{(zq^3, zq^4; q^7)_2},$$

the coefficients of $z^7q^{70}$, $z^{13}q^{70}$, $z^{15}q^{70}$, and $z^{18}q^{70}$ are all negative.

The proof of Theorem 3 is similar, but now cases are determined by columns that occur twice.

**Proof of Theorem 3.** We define injections $\varphi'_k$ to be the same as $\varphi_k$, except that in Cases 2a and 2b we condition on whether or not a partition contains two columns of height $y$. This ensures that $\varphi'_k$ preserves distinct parts partitions:

**Case 1:** Note that $\lambda_{(b)}$ is a distinct parts partition into $m - k$ nonnegative parts (so 0 occurs at most once). As such, $\lambda_{(b)}$ must contain a column of height $y$. (Recall
that $y < m - k.$) Attaching another such column means that $\lambda'_b(M)$ still has distinct nonnegative parts. Attaching the ends as above also preserves distinct parts.

**Case 2a:** Again attaching the column to $\lambda(M-b)$ preserves distinct parts because $z < m - k$. The fact that $M - b - a \neq b - a$ implies that $\lambda'_b(M)$ still does not contain two columns of height $y$.

**Case 2b:** Since $\lambda(M-b)$ contains two columns of height $y$, removing one such column preserves distinct parts.

Cases are separated as follows.

**Case 1:** $\nu_a > \nu_{M-a}$ and $\lambda(a)$ contains two columns of height $y$.

**Case 2a:** $\nu_a > \nu_{M-a}$ and $\lambda(a)$ does not contain two columns of height $y$.

**Case 2b:** $\nu_a < \nu_{M-a}$.

This concludes the proof.  

**Remark.** Unlike in Theorem 2, it appears that the extra factor $(1 + q^{LM+a})$ in the left term of Theorem 3 is often not needed for nonnegativity. A computational search up to $M \leq 12$, $L \leq 20$ and $nM \leq 250$ reveals that for

$$\sum_{m,n \geq 0} d'(m,n)z^mq^a := (-zq^a, -zq^{M-a}; q^M)_L - (-zq^b, -zq^{M-b}; q^M)_L,$$

we have some $d'(m,nM) < 0$ only when $(a,b,M) = (1,2,5)$.

In fact, we can condition on more than just 2 columns to prove the following new result.

**Proposition 1.** Let $d \geq 0$, $1 \leq a < b < M/2$ and $\gcd(b,M) = 1$. Let $p(d)(n,m,j,A)$ denote the number of partitions of $n$ into $m$ parts congruent to $\pm j \pmod{M}$, whose parts are at most $A$ such that the gap between successive parts is greater than $dM$. Then for all $n,m \geq 0$,

$$p(d)(nM,m,a,LM+a) \geq p(d)(nM,m,b,LM-b).$$

If in addition $a$ is odd, then we also have

$$p(d)\left( nM + \frac{M}{2},m,a,LM+a \right) \geq p(d)\left( nM + \frac{M}{2},m,b,LM-b \right).$$

Substituting $d = 0$ and $d = 1$ gives Theorems 2 and 3 respectively.
Proof. Let \( \lambda = (\lambda(b), \lambda(M - b), (b^m, (M - b)^{rM - b})) \) be a partition counted by \( p^{(d)}(nM, m, b, LM - b) \). Then the \( M \) modular diagrams \( \lambda(b) \) and \( \lambda(M - b) \) are partitions into nonnegative multiples on \( M \) such that the difference in successive parts is at least \( (d + 1)M \). Our injections \( \varphi^{(d)}_k \) are the same as before, except that we condition in cases 2 or 3 on whether or not \( \lambda(M - b) \) contains \( d + 2 \) columns of height \( y \).

\[ \Box \]

3. Applications to Kanade-Russell’s Conjectures

In [11], Kanade and Russell conjectured several new Rogers-Ramanujan-type product-sum identities—three arising from the theory of affine Lie algebras, and several companions. Bringmann, Jennings-Shaffer and Mahlburg were able to prove many of these [6], and they reduced the sum-sides of the four conjectures below from triple series to a single series. Here, \( kR_j \) is Identity \( j \) in [11], and \( H_j(x) \) is the sum side as denoted in [6].

\[ \begin{align*}
\mathcal{K}R_4 : \quad H_4(1) & = \frac{1}{(q, q^4, q^5, q^9, q^{11}; q^{12})_{\infty}}, \\
\mathcal{K}R_{4a} : \quad H_5(1) & = \frac{1}{(q^4, q^7, q^9, q^{12})_{\infty}}, \\
\mathcal{K}R_6 : \quad H_6(1) & = \frac{1}{(q, q^3, q^7, q^{11}; q^{12})_{\infty}}, \\
\mathcal{K}R_{6a} : \quad H_9(1) & = \frac{1}{(q^3, q^4, q^7, q^{11}; q^{12})_{\infty}}.
\end{align*} \]

The pairs of sum-sides, \( (H_4(1), H_5(1)) \) and \( (H_6(1), H_9(1)) \), are composed of two generating functions for partitions that satisfy the same set of gap conditions, but \( H_5 \) and \( H_9 \) have an additional condition on the smallest part (see [11]). Hence, as with the Rogers-Ramanujan sum-sides, we have the inequalities

\[ H_4(1) - H_5(1) \geq 0 \quad \text{and} \quad H_6(1) - H_9(1) \geq 0, \]

which, if the conjectures are true, imply the following result.

**Proposition 2.** The following inequalities hold.

\[ \frac{1}{(q, q^4, q^5, q^9, q^{11}; q^{12})_{\infty}} - \frac{1}{(q, q^5, q^7, q^9, q^{12})_{\infty}} \geq 0, \quad (3) \]

\[ \frac{1}{(q^3, q^4, q^7, q^{11}; q^{12})_{\infty}} - \frac{1}{(q^3, q^5, q^7, q^{11}; q^{12})_{\infty}} \geq 0. \quad (4) \]
Proof. (4) is an immediate consequence of Theorem 1, since for every $L \geq 0$,

$$\frac{1}{(q, q^8; q^{12})_L} - \frac{1}{(q^4, q^8; q^{12})_L} \geq 0.$$  

Multiplying both sides by the positive series $\frac{1}{(q, q^4, q^{11}; q^{12})_\infty}$ and taking the limit as $L \to \infty$ finishes the proof of (4).

Andrews’ Anti-teleseoping Method [1] works seamlessly to show (3). Define

$$P(j) := (q, q^4, q^{11}; q^{12})_j \quad \text{and} \quad Q(j) := (q, q^7, q^8; q^{12})_j,$$

and note that the following implies (3):

$$\frac{1}{P(L)} - \frac{1}{Q(L)} \geq 0 \quad \text{for all } L \geq 0. \quad \text{(5)}$$

Now we write

$$\frac{1}{P(L)} - \frac{1}{Q(L)} = \frac{1}{Q(L)} \left( \frac{Q(L)}{P(L)} - 1 \right)$$

$$= \frac{1}{Q(L)} \sum_{j=1}^{L} \left( \frac{Q(j)}{P(j)} - \frac{Q(j-1)}{P(j-1)} \right)$$

$$= \sum_{j=1}^{L} \frac{1}{Q(j-1)P(j)} \left( \frac{Q(j)}{Q(j-1)} - \frac{P(j)}{P(j-1)} \right),$$

whose $j$-th term is

$$\frac{(1 - q^{12j-11})q^{12(j-1)}}{(q^{12j-11}, q^{12j-5}, q^{12j-4}, q^{12})_L \cdot q^{-j}(q^4, q^8, q^{11}; q^{12})_j} \times (-q^7 - q^8 + q^4 + q^{11})$$

$$= \frac{(1 - q^{12j-11})q^{12(j-1)}}{(q^{12j-11}, q^{12j-5}, q^{12j-4}, q^{12})_L \cdot q^{-j}(q^4, q^8, q^{11}; q^{12})_j} \times q^4(1 - q^3)(1 - q^4). \quad \text{(6)}$$

The terms $1 - q^4$ and $(1 - q^{12j-11})$ are cancelled in the denominator, and we can write $\frac{1-q^3}{1-\frac{1}{q}} = 1 + q + q^2$. Hence, (6) is nonnegative for every $j$, proving (5) and then (3). \hfill \square

Another pair of identities in [11] with an Ehrenpreis Problem set-up is the following.

\[
\begin{align*}
\mathcal{K}\mathcal{R}_5 : \quad H_6(1) &= \frac{1}{(q^2; q^4)_\infty} \prod_{n \geq 0} \left( 1 + q^{4n+1} + q^{2(4n+1)} \right), \\
\mathcal{K}\mathcal{R}_{5a} : \quad H_7(1) &= \frac{1}{(q^2; q^3)_\infty} \prod_{n \geq 0} \left( 1 + q^{4n+3} + q^{2(4n+3)} \right)
\end{align*}
\]
Both identities were proved in [6], and there is an obvious injection proving
\[
\frac{1}{(q^2;q^4)_\infty} \prod_{n \geq 0} \left(1 + q^{4n+1} + q^{2(4n+1)}\right) - \frac{1}{(q^2;q^4)_\infty} \prod_{n \geq 0} \left(1 + q^{4n+3} + q^{2(4n+3)}\right) \geq 0,
\]
namely, sending each \((4n + 3)\) to the pair \((4n + 1, 2)\).

Finally, we discuss the Ehrenpreis problems among \(\mathcal{K}_1\), \(\mathcal{K}_2\) and \(\mathcal{K}_3\). These were proved in [6], and their respective sum-sides were denoted \(H_1(x)\), \(H_2(x)\) and \(H_3(x)\). Using the methods of [11], we have found slightly different conditions for the partitions enumerated on the sum-side:

1. No consecutive parts are allowed.
2. Odd parts do not repeat.
3. Even parts appear at most twice.
4. We have \((\lambda_j, \lambda_{j+1}, \lambda_{j+2}) \notin \{(2k, 2k, 2k), (2k, 2k, 2k+2), (2k+1, 2k+3, 2k+5), (2k-2, 2k, 2k)\}\) for any \(j\) and \(k\).

Note that our fourth condition is changed slightly from Kanade and Russell’s in [11], page 5. The sum-side of \(\mathcal{K}_2\) has the further restriction that the part 1 may not appear, and in the sum-side of \(\mathcal{K}_3\), the parts 1, 2 and 3 may not appear. Hence, \(H_1(1) \geq H_2(1) \geq H_3(1)\) and it follows from Theorem 1.1 of [6] that
\[
\frac{1}{(q^4, q^8, q^{10}, q^{12})_\infty} \leq \frac{(-q^3, -q^9, q^{12})_\infty}{(q^2, q^4, q^8, q^{10}, q^{12})_\infty} \leq \frac{1}{(q^4, q^8, q^7, q^8, q^{12})_\infty}.
\]

The inequality between the far left and right products is a consequence of Theorem 5.3 of [4], but a direct proof of the other two inequalities remains open.

4. Concluding Remarks

As we pointed out in the introduction, (1) was the start of Andrews-Baxter’s “motivated” proof of the Rogers-Ramanujan identities [2]. They defined \(G_1 := (q, q^4; q^5)_\infty^{-1}\) and \(G_2 := (q^2, q^3; q^5)_\infty^{-1}\), and then recursively
\[
G_i := \frac{G_{i-2} - G_{i-1}}{q^{i-2}}, \quad \text{for } i \geq 3.
\]

They then observed computationally that \(G_i = 1 + \sum_{n \geq 1} q^{i \cdot n} \geq 0\). Thus, as \(i \to \infty\) the coefficient of \(q^n\) in \(G_i\) is eventually 0. This was their “Empirical Hypothesis,” and proving it leads easily to a new proof of the Rogers-Ramanujan identities.

---

\(^3\)As in [11], we have written parts in increasing order.
Note that, starting from the sum-sides of $G_1$ and $G_2$, the recursive definition (7) and the Empirical Hypothesis are completely natural. For example, if $\mathcal{RR}$ denotes the set of gap-2 partitions, then by the Rogers-Ramanujan Identities,

$$G_1 - G_2 = \sum_{\lambda \in \mathcal{RR}} q^{\lambda_1} = q \left( 1 + \sum_{\lambda \in \mathcal{RR}} q^{\lambda} \right),$$

and so

$$G_2 - G_3 = \sum_{\lambda \in \mathcal{RR}} q^{\lambda_2} = q^2 \left( 1 + \sum_{\lambda \in \mathcal{RR}} q^{\lambda} \right),$$

and so on.

For $\mathcal{RR}_4$, $\mathcal{RR}_{4a}$, $\mathcal{RR}_6$ and $\mathcal{RR}_{6a}$, we can expect the more complicated conditions on the sum-sides to lead to more complicated recurrences. For example, the recurrence below was shown for $\mathcal{RR}_4$ ([6], equation 4.2).

$$H_4(x) = (1 + xq)H_4(xq^2) + xq^2(1 + xq^3)H_4(xq^4) + x^2q^6(1 - xq^4)H_4(xq^6).$$

Combinatorial proofs of the above and the similar recurrences in [6] may give insight into an “Empirical Hypothesis” for $\mathcal{RR}_4$, $\mathcal{RR}_{4a}$, $\mathcal{RR}_6$ and $\mathcal{RR}_{6a}$. Indeed, the techniques for “motivated proofs” have been expanded to accommodate identities with gap-conditions more complicated than those of $\mathcal{RR}$, notably in [7], [9] and [12]. Perhaps further developments will give an answer to the following question: Do there exist “motivated proofs” of $\mathcal{RR}_4$, $\mathcal{RR}_{4a}$, $\mathcal{RR}_6$ and $\mathcal{RR}_{6a}$?

This would be especially interesting, since to our knowledge there have not yet been “motivated proofs” featuring asymmetric products.

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References


